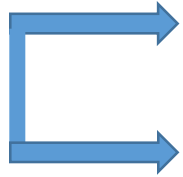


# Mathematical Models of Systems

- Development of mathematical models from schematics of physical systems.

by applying the fundamental physical laws of science and engineering.

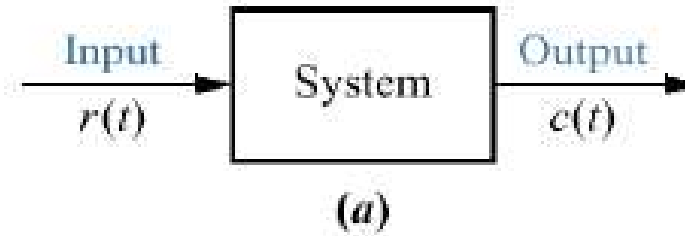
- Two Methods:



*Transfer functions* in the frequency domain,

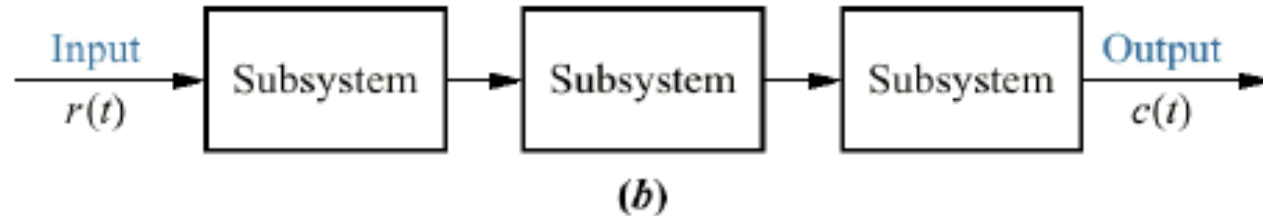
*State equations* in the time domain.

a. Block diagram representation of a system;



From mathematical model equations we will obtain the relationship between the system's output and input.

b. block diagram Representation of an interconnection of subsystems



Note: The input,  $r(t)$ , stands for *reference input*.  
The output,  $c(t)$ , stands for *controlled variable*.

- Transfer function* (mathematical function), is inside each block.

# Modeling in Frequency Domain

## Laplace Transform Review

- A system represented by a differential equation is difficult to model as a block diagram.
- A differential equation can describe the relationship between the input and output of a system.
- By using *Laplace transform* we can represent the input, output, and system as separate entities.

$$A(s)Y(s) = B(s)U(s)$$

**Laplace transform can be defined as:**

$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st} dt$$

Where  $s = \sigma + j\omega$ , a complex variable

**Inverse Laplace transform:**

$$\mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma-j\omega}^{\sigma+j\omega} F(s)e^{st} ds = f(t)u(t)$$

$$\text{Where, } \begin{aligned} u(t) &= 1 & t > 0 \\ &= 0 & t < 0 \end{aligned}$$

Multiplication of  $f(t)$  by  $u(t)$  yields a time function that is zero for  $t < 0$ .

# Laplace Transform Table


**Table 2.1**  
Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at}u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2 + \omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2 + \omega^2}$

**Problem:** Find the Laplace transform of

$$f(t) = Ae^{-at}u(t)$$

**Solution:**

$$\begin{aligned} F(s) &= \int_0^{\infty} f(t)e^{-st} dt = \int_0^{\infty} Ae^{-at}e^{-st} dt \\ &= A \int_0^{\infty} e^{-(s+a)t} dt = -\frac{A}{s+a} e^{-(s+a)t} \Big|_{t=0}^{\infty} \\ &= \frac{A}{s+a} \end{aligned}$$


# Laplace Transform Theorems

Table 2.2

Item no.	Theorem	Name
1.	$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st} dt$	Definition
2.	$\mathcal{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathcal{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathcal{L}[e^{-at}f(t)] = F(s + a)$	Frequency shift theorem
5.	$\mathcal{L}[f(t - T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - \sum_{k=1}^n s^{n-k} f^{(k-1)}(0-)$	Differentiation theorem
10.	$\mathcal{L}\left[\int_{0-}^t f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s \rightarrow 0} sF(s)$	Final value theorem <sup>1</sup>
12.	$f(0+) = \lim_{s \rightarrow \infty} sF(s)$	Initial value theorem <sup>2</sup>

<sup>1</sup>For this theorem to yield correct finite results, all roots of the denominator of  $F(s)$  must have negative real parts, and no more than one can be at the origin.

<sup>2</sup>For this theorem to be valid,  $f(t)$  must be continuous or have a step discontinuity at  $t = 0$  (that is, no impulses or their derivatives at  $t = 0$ ).

## Inverse Laplace Transform

**Problem:** Find inverse Laplace Transform of

$$F_1(s) = \frac{1}{(s+3)^2}$$

**Solution:**

From item 3 and 5 of Table 2.1,

$$f_1(t) = e^{-3t}tu(t)$$

Frequency shift theorem item 4 of Table 2.2,

$$\mathcal{L}[e^{-at}f(t)] = F(s+a).$$

# Inverse Laplace: Partial-Fraction Expansion

*A partial-fraction expansion:* transform of a complicated function to a sum of simpler terms for which we know the Laplace transform of each term.

## Case 1. (Roots of the Denominator of F(s) Are Real and Distinct)

**Problem:** 
$$F_1(s) = \frac{s^3 + 4s^2 + 6s + 5}{s^2 + 3s + 2} = \frac{N(s)}{D(s)}$$

The order of N(s) > order of D(s) we must perform the division until we obtain a remainder whose numerator is of order less than its denominator

**Solution:**

$$\begin{array}{r}
 s + 1 \\
 s^2 + 3s + 2 \overline{) s^3 + 4s^2 + 6s + 5} \\
 \underline{-(s^3 + 3s^2 + 2s)} \phantom{+ 5} \\
 s^2 + 4s + 5 \\
 \underline{-(s^2 + 3s + 2)} \\
 s + 3
 \end{array}$$

Partial-Fraction Expansion

$$F(s) = \frac{s + 3}{(s+1)(s+2)} = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)}$$

To find  $K_1$ , multiply by  $(s+1)$ . Thus

$$\frac{s + 3}{(s+2)} = K_1 + \frac{(s+1)K_2}{(s+2)}$$

Letting  $s = -1$ ,  $K_1 = 2$

Similarly,  $K_2 = -2$

$$F_1(s) = s + 1 + \frac{s + 3}{s^2 + 3s + 2}$$

Taking the inverse Laplace transform,  
(Table 2.2 Item 7)

$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + \mathfrak{F}^{-1} \left[ \frac{s + 3}{(s+1)(s+2)} \right]$$

$$f(t) = (2e^{-t} - 2e^{-2t})u(t)$$

**Final solution:** 
$$f_1(t) = \frac{d\delta(t)}{dt} + \delta(t) + (2e^{-t} - 2e^{-2t})u(t)$$

# Laplace Transform Solution of a Differential Equation

**Problem:** Solve for  $y(t)$ , if all initial conditions are zero.

$$\frac{d^2 y}{dt^2} + 12 \frac{dy}{dt} + 32y = 32u(t)$$

**Solution:** The Laplace transform is,

(Table 2.2 Item 8)

$$\longrightarrow s^2 Y(s) + 12sY(s) + 32Y(s) = \frac{32}{s}$$

$$Y(s) = \frac{32}{s(s^2 + 12s + 32)} = \frac{32}{s(s+4)(s+8)}$$

$$= \frac{K_1}{s} + \frac{K_2}{s+4} + \frac{K_3}{s+8}$$



$$K_1 = \left. \frac{32}{(s+4)(s+8)} \right|_{s \rightarrow 0} = 1$$

$$K_2 = \left. \frac{32}{s(s+8)} \right|_{s \rightarrow -4} = -2$$

$$K_3 = \left. \frac{32}{s(s+4)} \right|_{s \rightarrow -8} = 1$$

Taking inverse Laplace transform, we get

$$y(t) = (1 - 2e^{-4t} + e^{-8t})u(t)$$

inverse Laplace transform



$$\text{Hence, } Y(s) = \frac{1}{s} - \frac{2}{s+4} + \frac{1}{s+8}$$

# Inverse Laplace: Partial-Fraction Expansion

## Case 2. (Roots of the Denominator of $F(s)$ Are Real and Repeated)

**Problem:** Find inverse Laplace transform of

$$F(s) = \frac{2}{(s+1)(s+2)^2} \quad (1)$$

### Solution:

We can write the partial-fraction expansion as a sum of terms

$$F(s) = \frac{K_1}{(s+1)} + \frac{K_2}{(s+2)^2} + \frac{K_3}{(s+2)} \quad (2)$$

reduced  
multiplicity

$$K_1 = \left. \frac{2}{(s+2)^2} \right|_{s \rightarrow -1} = 2$$

To find  $K_2$ , multiply (1) = (2) by  $(s+2)^2$

$$\frac{2}{(s+1)} = (s+2)^2 \frac{K_1}{(s+1)} + K_2 + (s+2)K_3 \quad (3)$$

Letting  $s \rightarrow -2$ , we obtain  $K_2 = -2$

To find  $K_3$ , differentiate (3) w.r.t.  $s$ :

$$\frac{-2}{(s+1)^2} = \frac{(s+2)s}{(s+1)^2} K_1 + K_3$$

Letting  $s \rightarrow -2$ , we obtain  $K_3 = -2$

Therefore, inverse Laplace transform is:

$$f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t}$$

For repeated roots with multiplicity  $r$ , we have

$$F(s) = \frac{K_1}{(s+p_1)^r} + \frac{K_2}{(s+p_1)^{r-1}} + \dots + \frac{K_r}{(s+p_1)}$$

$$K_i = \frac{1}{(i-1)!} \left. \frac{d^{i-1} F_1(s)}{ds^{i-1}} \right|_{s \rightarrow -p_1} \quad i = 1, 2, \dots, r;$$

$$F_1(s) = (s+p_1)^r F(s)$$

# Inverse Laplace: Partial-Fraction Expansion

**Case 3.** (Roots of the Denominator of  $F(s)$  Are Complex or Imaginary)

$$F(s) = \frac{3}{s(s^2 + 2s + 5)}$$

**Problem:** Find inverse Laplace transform of

**Solution:**

This function can be expanded in the following form:

$$\frac{3}{s(s^2 + 2s + 5)} = \frac{K_1}{s} + \frac{K_2s + K_3}{s^2 + 2s + 5} \quad (1)$$

$K_1$  is found in the usual way:  $\frac{3}{s^2 + 2s + 5} \Big|_{s \rightarrow 0} = \frac{3}{5} \equiv K_1$

To find  $K_2$  and  $K_3$  :

Multiply (1) by  $s(s^2 + 2s + 5)$ , and put  $K_1 = 3/5$

$$\Rightarrow 3 = \frac{3}{5}(s^2 + 2s + 5) + K_2s^2 + K_3s$$

$$\Rightarrow 3 = \left(K_2 + \frac{3}{5}\right)s^2 + \left(K_3 + \frac{6}{5}\right)s + 3$$

Balancing coefficients(matching)

$$K_2 + \frac{3}{5} = 0$$

$$K_3 + \frac{6}{5} = 0$$

Hence,

$$\Rightarrow K_2 = -\frac{3}{5}, \text{ and } K_3 = -\frac{6}{5}$$

$$F(s) = \frac{3/5}{s} - \frac{3}{5} \frac{s+2}{s^2 + 2s + 5}$$

Using Item 7 in Table 2.1 and Items 2 and 4 in Table 2.2, we get

$$\Im[Ae^{-at} \cos \omega t] = \frac{A(s+a)}{(s+a)^2 + \omega^2}$$

$$\Im[Be^{-at} \sin \omega t] = \frac{B\omega}{(s+a)^2 + \omega^2}$$

Adding,  $\Im[Ae^{-at} \cos \omega t + Be^{-at} \sin \omega t] = \frac{A(s+a) + B\omega}{(s+a)^2 + \omega^2}$

We have,  $s^2 + 2s + 5 = s^2 + 2s + 1 + 4 = (s+1)^2 + 2^2$

$$\Rightarrow a = 1 \text{ and } \omega = 2$$

$$\Rightarrow F(s) = \frac{3/5}{s} - \frac{3}{5} \frac{(s+1) + (1/2)(2)}{(s+1)^2 + 2^2}$$

$$f(t) = \frac{3}{5} - \frac{3}{5}e^{-t} \left( \cos 2t + \frac{1}{2} \sin 2t \right)$$

# Transfer Function

- A *Transfer Function* is the ratio of the output of a system to the input of a system. It allows us to algebraically combine mathematical representations of subsystems to yield a total system representation.



- General  $n^{\text{th}}$  order, linear time-invariant differential equation:

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \dots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \dots + b_0 r(t) \quad \text{c: output, r: input}$$

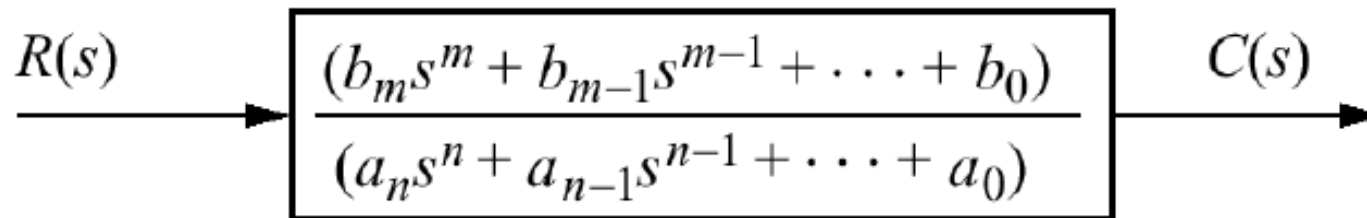
Taking Laplace transform,

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \dots + a_0 C(s) + [\text{Initial condition is zero}]$$

$$= b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + \dots + b_0 R(s) + [\text{Initial condition is zero}]$$

Transfer function:

$$\frac{C(s)}{R(s)} = G(s) = \frac{(b_m s^m + b_{m-1} s^{m-1} + \dots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \dots + a_0)}$$



Block diagram of a transfer function

We can find the output

$$C(s) = G(s)R(s)$$

# Transfer Function for a Differential Equation

**Problem 1:** Find the transfer function represented by  $\frac{dc(t)}{dt} + 2c(t) = r(t)$

**Solution:**

Taking Laplace transform and assuming zero initial conditions, we have

$$sC(s) + 2C(s) = R(s)$$

Transfer function,  $G(s)$ ,  $\Rightarrow G(s) = \frac{C(s)}{R(s)} = \frac{1}{s+2}$

---

**Problem 2:** Find the transfer function represented by  $\frac{d^3c}{dt^3} + 3\frac{d^2c}{dt^2} + 7\frac{dc}{dt} + 5c = \frac{d^2r}{dt^2} + 4\frac{dr}{dt} + 3r$ .

**Solution:**

$$s^3C(s) + 3s^2C(s) + 7sC(s) + 5C(s) = s^2R(s) + 4sR(s) + 3R(s)$$
$$\Rightarrow G(s) = \frac{C(s)}{R(s)} = \frac{s^2 + 4s + 3}{s^3 + 3s^2 + 7s + 5}$$

# Problem Solving

**Problem:** Find the ramp response for a system whose transfer function is

$$G(s) = \frac{s}{(s+4)(s+8)}$$

**Solution:**

The input (ramp)

$$C(s) = R(s)G(s) = \frac{1}{s^2} \times \frac{s}{(s+4)(s+8)} = \frac{1}{s(s+4)(s+8)}$$

$$\Rightarrow C(s) = \frac{A}{s} + \frac{B}{(s+4)} + \frac{C}{(s+8)}$$

$$A = \left. \frac{1}{(s+4)(s+8)} \right|_{s \rightarrow 0} = \frac{1}{32}$$

$$B = \left. \frac{1}{s(s+8)} \right|_{s \rightarrow -4} = -\frac{1}{16}$$

$$C = \left. \frac{1}{s(s+4)} \right|_{s \rightarrow -8} = \frac{1}{32}$$

Hence,

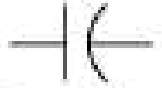


$$c(t) = \frac{1}{32} - \frac{1}{16}e^{-4t} + \frac{1}{32}e^{-8t}$$

# Electric Network Transfer Functions

Apply the transfer function to the mathematical modeling of electronic circuits including passive networks and O-Amp circuits.

**Table 2.3**

Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

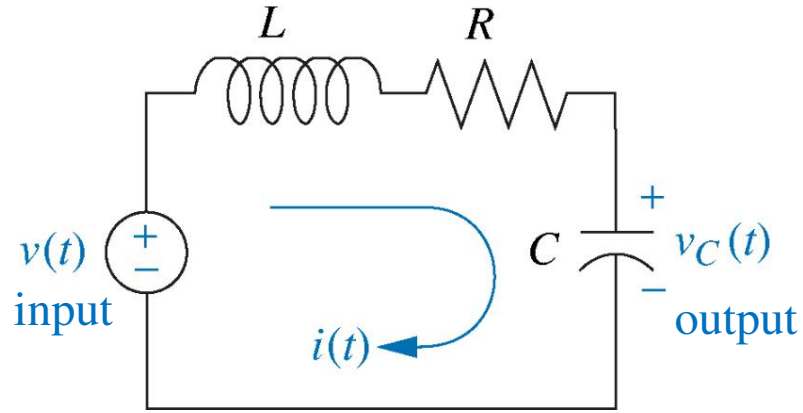
 Capacitor	$v(t) = \frac{1}{C} \int_0^t i(\tau) d\tau$	$i(t) = C \frac{dv(t)}{dt}$	$v(t) = \frac{1}{C} q(t)$	$\frac{1}{Cs}$	$Cs$
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R} v(t)$	$v(t) = R \frac{dq(t)}{dt}$	$R$	$\frac{1}{R} = G$
 Inductor	$v(t) = L \frac{di(t)}{dt}$	$i(t) = \frac{1}{L} \int_0^t v(\tau) d\tau$	$v(t) = L \frac{d^2 q(t)}{dt^2}$	$Ls$	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book:  $v(t) = V$  (volts),  $i(t) = A$  (amps),  $q(t) = Q$  (coulombs),  $C = F$  (farads),  $R = \Omega$  (ohms),  $G = \bar{U}$  (mhos),  $L = H$  (henries).

# Transfer Function: Single Loop

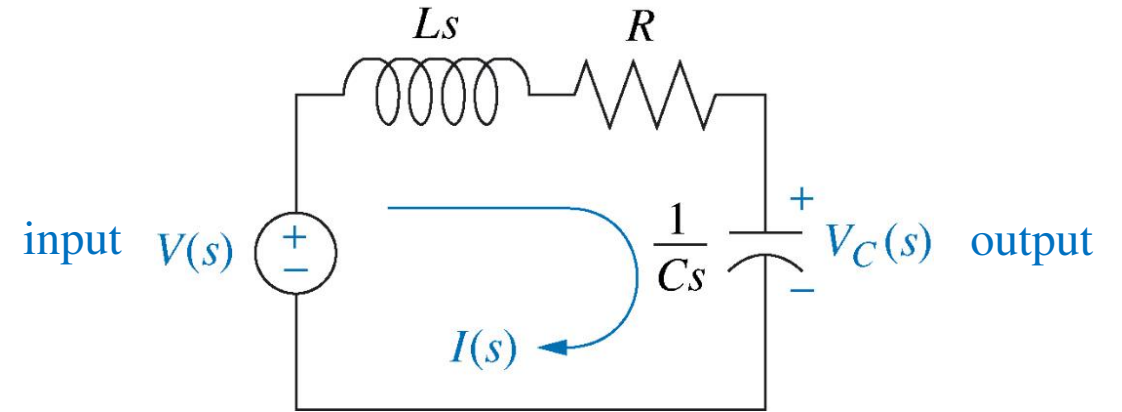
1

**Problem:** Find the transfer function relating capacitor voltage,  $V_C(s)$ , to input voltage,  $V(s)$ .



RLC network

Laplace-transform



Laplace-transformed network

Summing the voltages around the loop,

$$L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_0^t i(\tau) d\tau = V(t)$$

take the Laplace transform

$$V_C(s) = \frac{1}{Cs} I(s) \quad V_L(s) = Ls I(s) \quad V_R(s) = R I(s)$$

Capacitor

Inductor

Resistor

$$\left( Ls + R + \frac{1}{Cs} \right) I(s) = V(s) \Rightarrow \frac{I(s)}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}}$$

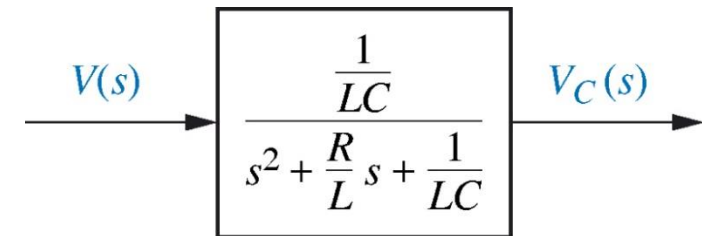
$$\text{We know, } V_C(s) = I(s) \frac{1}{Cs} \Rightarrow \frac{V_C(s)}{V(s)} = \frac{I(s)}{V(s)} \times \frac{1}{Cs}$$

$$\Rightarrow V_C(s) = \frac{V(s)}{Ls + R + \frac{1}{Cs}} \times \frac{1}{Cs}$$

$$\Rightarrow \frac{V_C(s)}{V(s)} = \frac{1}{Ls + R + \frac{1}{Cs}} \times \frac{1}{Cs} = \frac{Cs}{s^2 LC + sRC + 1} \times \frac{1}{Cs}$$

$$= \frac{1}{s^2 LC + sRC + 1}$$

$$= \frac{1/LC}{s^2 + s \frac{R}{L} + \frac{1}{LC}}$$



# Transfer Function: Single Node

2

Transfer functions also can be obtained using Kirchhoff's current law and summing currents flowing from nodes. currents leaving the node are positive and currents entering the node are negative.

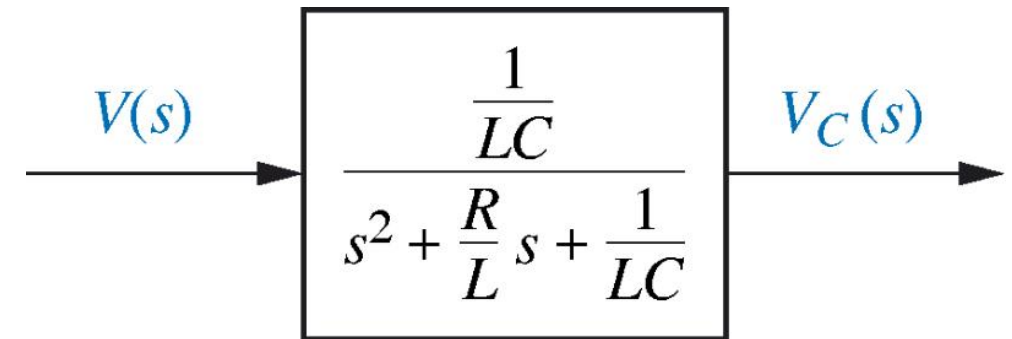
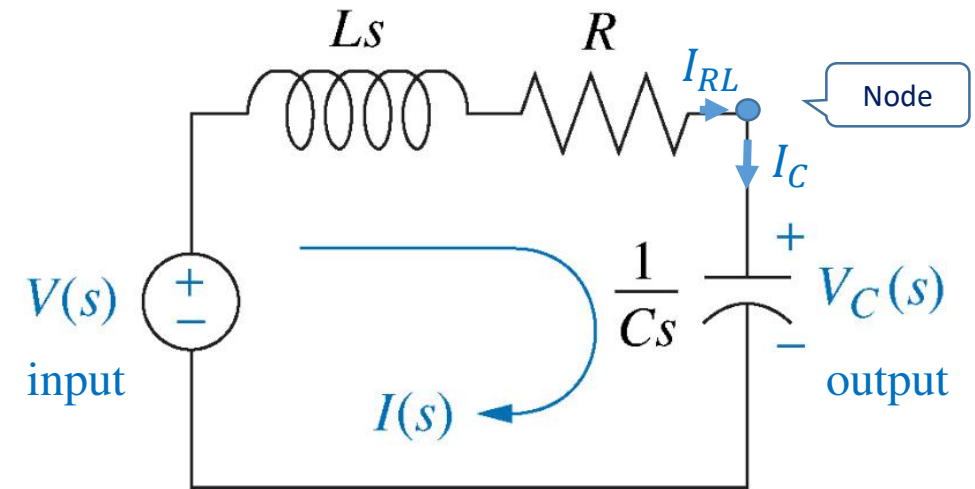
$$\sum I_{in} = \sum I_{out} \quad \Rightarrow \quad I_c(s) = I_{RL}(s) \quad \text{Same current}$$

$$\begin{aligned} \Rightarrow I_c(s) - I_{RL}(s) &= 0 \quad \Rightarrow \quad \frac{V_c}{Z_c} - \frac{V_{RL}}{Z_{RL}} = 0 \\ \Rightarrow \frac{V_c(s)}{\frac{1}{Cs}} + \frac{V_c(s) - V(s)}{R + Ls} &= 0 \quad \text{V}_{RL} = -(V_c - V) \end{aligned}$$

$$\Rightarrow V_c(s) \left( Cs + \frac{1}{R + Ls} \right) = \frac{V(s)}{R + Ls}$$

$$\Rightarrow \frac{V_c(s)}{V(s)} = \frac{1}{s^2 LC + sRC + 1}$$

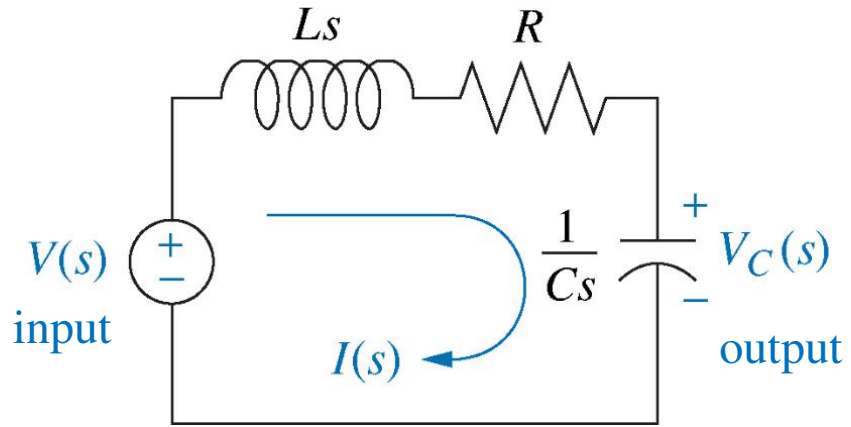
$$= \frac{1/LC}{s^2 + s \frac{R}{L} + \frac{1}{LC}}$$



# Transfer Function: Single Loop via Voltage Division

3

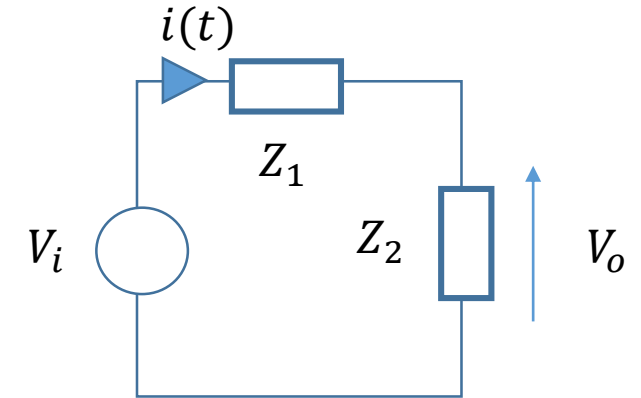
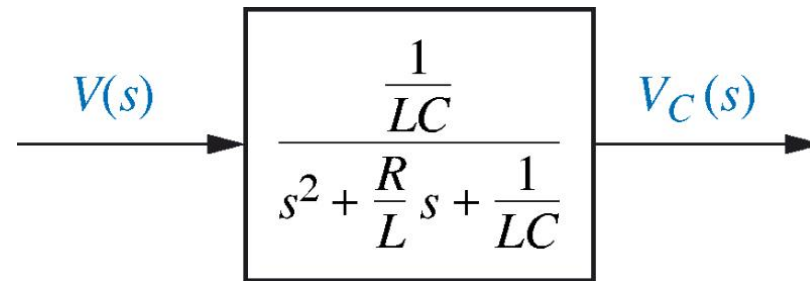
Voltage across capacitor is some proportion of the input voltage.



$$\frac{\text{Impedance of the capacitor}}{\text{Sum of the impedances}}$$

$$V_c(s) = \frac{1/Cs}{\left(Ls + R + \frac{1}{Cs}\right)} V(s)$$

$$\Rightarrow \frac{V_c(s)}{V(s)} = \frac{1/LC}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$



$$V_i = (Z_1 + Z_2) i(t) \quad (1)$$

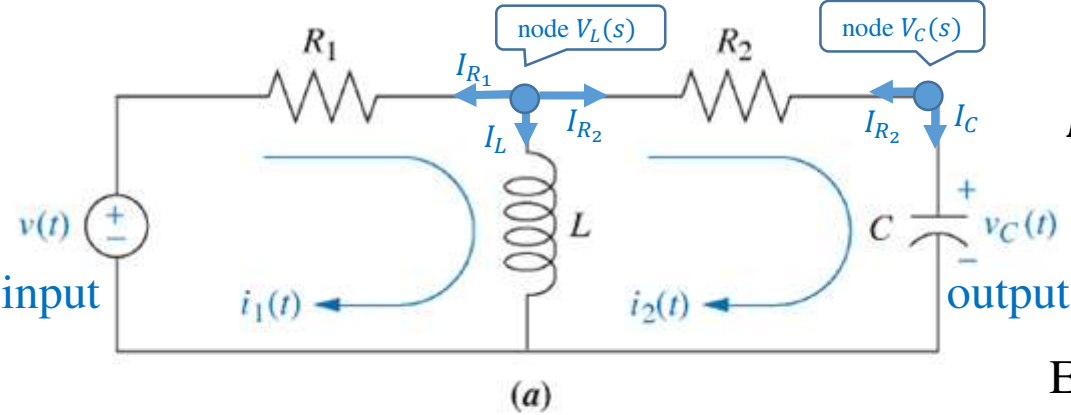
$$V_o = Z_2 i(t) \quad (2)$$

$$\frac{(2)}{(1)} \Rightarrow \frac{V_o}{V_i} = \frac{Z_2}{(Z_1 + Z_2)}$$

Which one is the easiest? Method 1, method 2, or method 3?

# Complex Circuits via Nodal Analysis<sub>1</sub>

Complex electrical networks (those with multiple loops and nodes). We use nodal analysis to find the *transfer function*  $\frac{V_C(s)}{V(s)}$



sum currents at the nodes

$$I_{R_1} + I_L + I_{R_2} = 0 \Rightarrow \frac{V_L(s) - V(s)}{R_1} + \frac{V_L(s)}{Ls} + \frac{V_L(s) - V_C(s)}{R_2} = 0$$

Current from node  $V_L(s)$

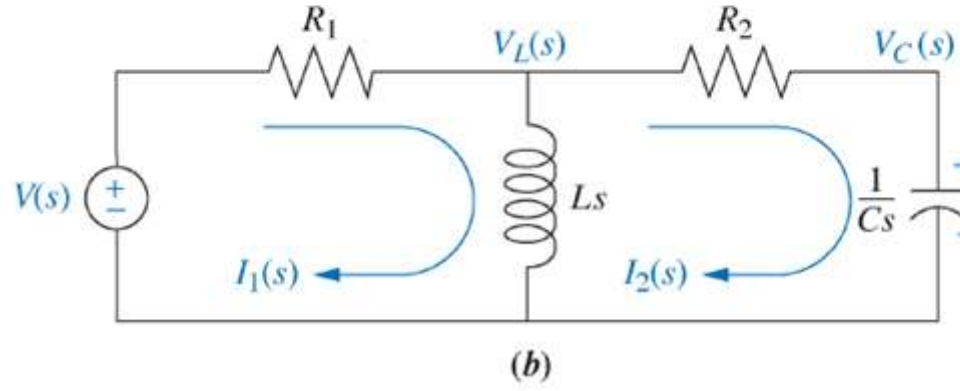
$$\text{And } I_C + I_{R_2} = 0 \Rightarrow CsV_C(s) + \frac{V_C(s) - V_L(s)}{R_2} = 0$$

Current from node  $V_C(s)$

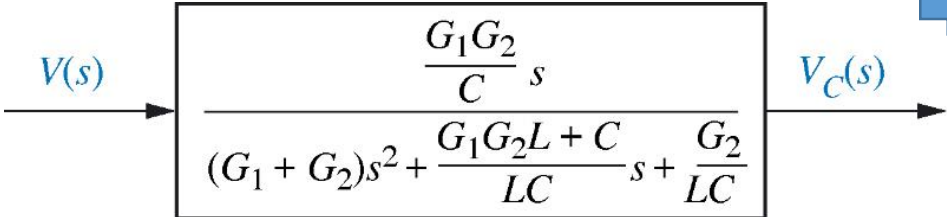
Expressing resistances as conductances,  $G=1/R$

$$\left( G_1 + G_2 + \frac{1}{Ls} \right) V_L(s) - G_2 V_C(s) = V(s) G_1 \quad (1)$$

$$-G_2 V_L(s) + (G_2 + Cs) V_C(s) = 0 \Rightarrow V_L(s) = \frac{G_2 + Cs}{G_2} V_C(s) \quad (2)$$



(2) in (1)  $\Rightarrow \frac{V_C(s)}{V(s)} = \frac{G_1 G_2 L s}{(G_1 + G_2) L C s^2 + (C + G_2(G_1 + G_2)L - G_2^2 L) s + G_2}$



Transfer function:

$$\frac{V_C(s)}{V(s)} = \frac{\frac{G_1 G_2}{C} s}{(G_1 + G_2) s^2 + \frac{G_1 G_2 L + C}{LC} s + \frac{G_2}{LC}}$$

Divide by LC

# Complex Circuits - Mesh Equations via Inspection<sub>2</sub>

For Mesh 1:  $\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around} \\ \text{Mesh 1} \end{bmatrix} \times I_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 1 and mesh2} \end{bmatrix} \times I_2(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 1 and mesh3} \end{bmatrix} \times I_3(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 1} \end{bmatrix}$

For Mesh 2:  $-\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 1 and mesh2} \end{bmatrix} \times I_1(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around} \\ \text{Mesh 2} \end{bmatrix} \times I_2(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 2 and mesh3} \end{bmatrix} \times I_3(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 2} \end{bmatrix}$

For Mesh 3:  $-\begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 1 and mesh3} \end{bmatrix} \times I_1(s) - \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{common to} \\ \text{Mesh 2 and mesh3} \end{bmatrix} \times I_2(s) + \begin{bmatrix} \text{Sum of} \\ \text{impedances} \\ \text{around} \\ \text{Mesh 3} \end{bmatrix} \times I_3(s) = \begin{bmatrix} \text{Sum of applied} \\ \text{voltages around} \\ \text{Mesh 3} \end{bmatrix}$

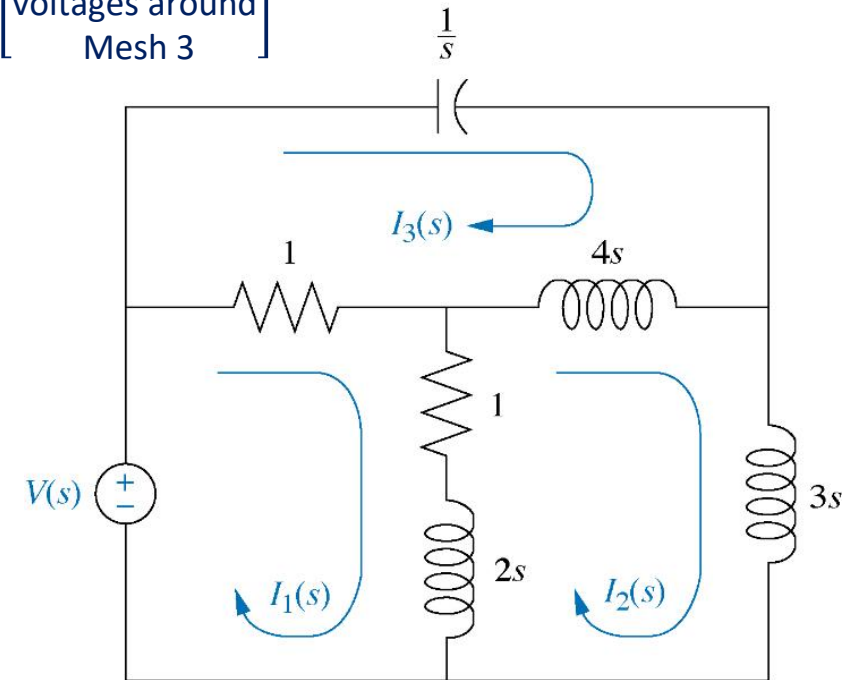
Similarly, Meshes 3, we obtain

For Mesh 1:  $+(2s + 2)I_1(s) - (2s + 1)I_2(s) - I_3(s) = V(s)$

For Mesh 2:  $-(2s + 1)I_1(s) + (9s + 1)I_2(s) - 4sI_3(s) = 0$

For Mesh 3:  $-I_1(s) - 4sI_2(s) + \left(4s + 1 + \frac{1}{s}\right)I_3(s) = 0$

which can be solved simultaneously for any desired transfer function, for example,  $I_3(s)/V(s)$



Three loop electrical network

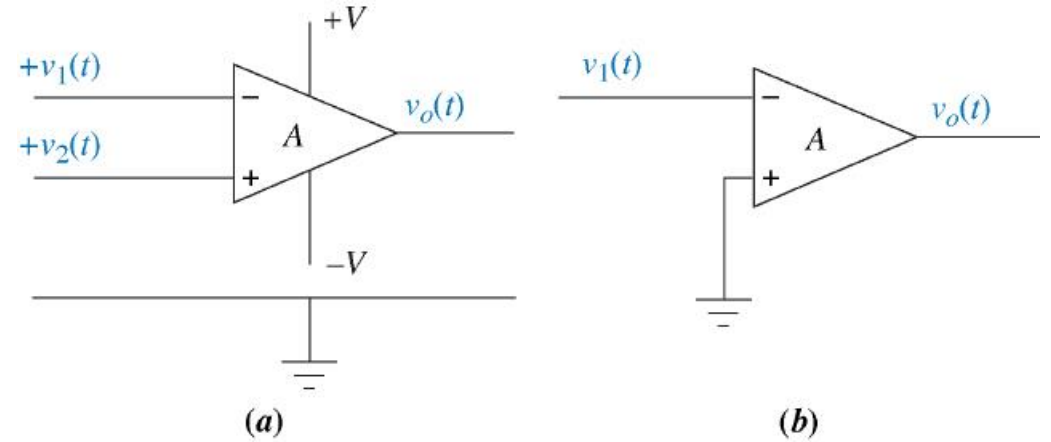
# Operational Amplifier

An *operational amplifier* is an electronic amplifier used as a basic building block to implement transfer functions. It has the following characteristics:

1. Differential input,  $v_2(t) - v_1(t)$
2. High input impedance,  $Z_i = \infty$  (*ideal*)
3. Low output impedance,  $Z_o = 0$  (*ideal*)
4. High constant gain amplification,  $A = \infty$  (*ideal*)

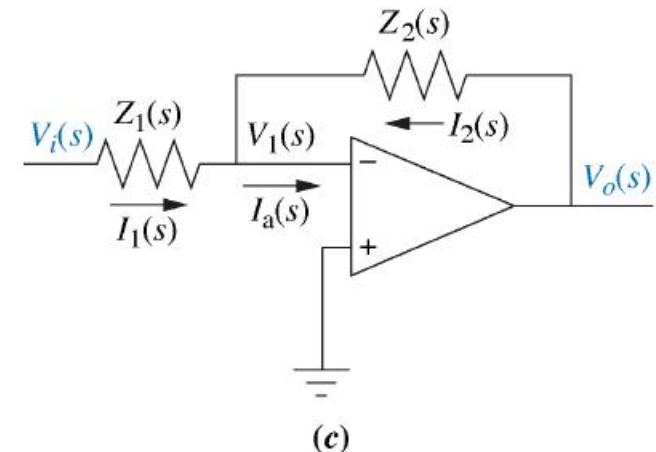
The output,  $v_o(t)$ , is given by:  $v_o(t) = A(v_2(t) - v_1(t))$

- a. Operational amplifier;
- b. schematic for an inverting operational amplifier;
- c. Inverting operational amplifier configured for transfer function realization. Typically, the amplifier gain, A, is omitted.



$I_1(s) = -I_2(s)$ , as  $I_a(s) = 0$ , because of high input impedance

$$\left. \begin{aligned} V_i(s) &= Z_1(s)I_1(s) \\ V_o(s) &= Z_2(s)I_2(s) \\ I_1(s) &= -I_2(s) \end{aligned} \right\} \Rightarrow \boxed{\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)}}$$



# Problem Solving

## Inverting Operational Amplifier

**Problem:** Find the transfer function,  $\frac{V_o(s)}{V_i(s)}$ , for the circuit below.

**Solution:**

For parallel components,  $Z_1(s)$  is the reciprocal of the sum of the admittances.

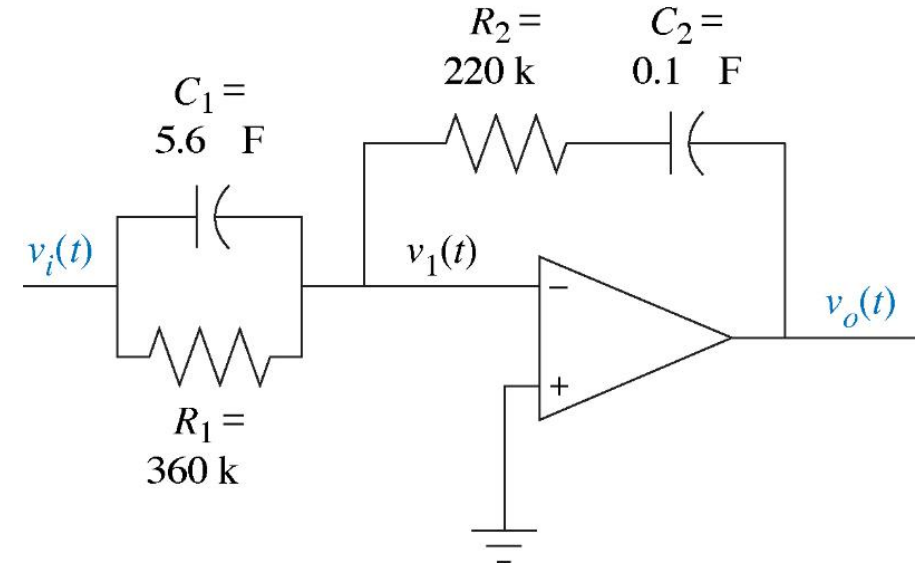
$$Z_1(s) = \frac{1}{C_1 s + \frac{1}{R_1}} = \frac{1}{5.6 \times 10^{-6} s + \frac{1}{360 \times 10^3}}$$

For serial components,  $Z_2(s)$  is the sum of the impedances.

$$Z_2(s) = R_2 + \frac{1}{C_2 s} = 220 \times 10^3 + \frac{10^7}{s}$$



$$\frac{V_o(s)}{V_i(s)} = -\frac{Z_2(s)}{Z_1(s)} = -1.232 \times \frac{s^2 + 45.95s + 22.55}{s}$$



# Non-inverting Operational Amplifier

Using voltage division,

$$V_0(s) = [Z_1(s) + Z_2(s)] I(s)$$

$$V_1(s) = Z_1(s) I(s)$$

$$\Rightarrow V_1(s) = \frac{Z_1(s)}{Z_1(s) + Z_2(s)} V_o(s)$$

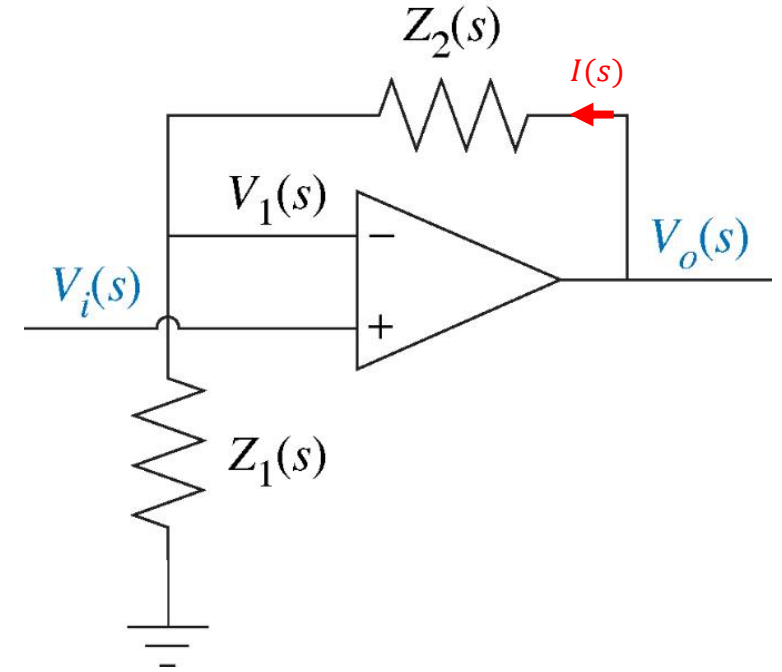
$$\text{We have: } V_o(s) = A(V_i(s) - V_1(s)) \Rightarrow V_o(s) = A \left[ V_i(s) - \frac{Z_1(s)}{Z_1(s) + Z_2(s)} V_o(s) \right]$$

$$\Rightarrow V_o(s) \left[ 1 + A \frac{Z_1(s)}{Z_1(s) + Z_2(s)} \right] = A V_i(s)$$

$$\Rightarrow \frac{V_o(s)}{V_i(s)} = \frac{A}{1 + AZ_1(s)/(Z_1(s) + Z_2(s))}$$

For large A, we disregard '1' in the denominator.

$$\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}$$



# Problem Solving

## Non-Inverting Operational Amplifier

**PROBLEM:** Find the transfer function,  $V_o(s)/V_i(s)$ , for the Non-inverting operational amplifier circuit

**SOLUTION:**

We find each of the impedance functions,

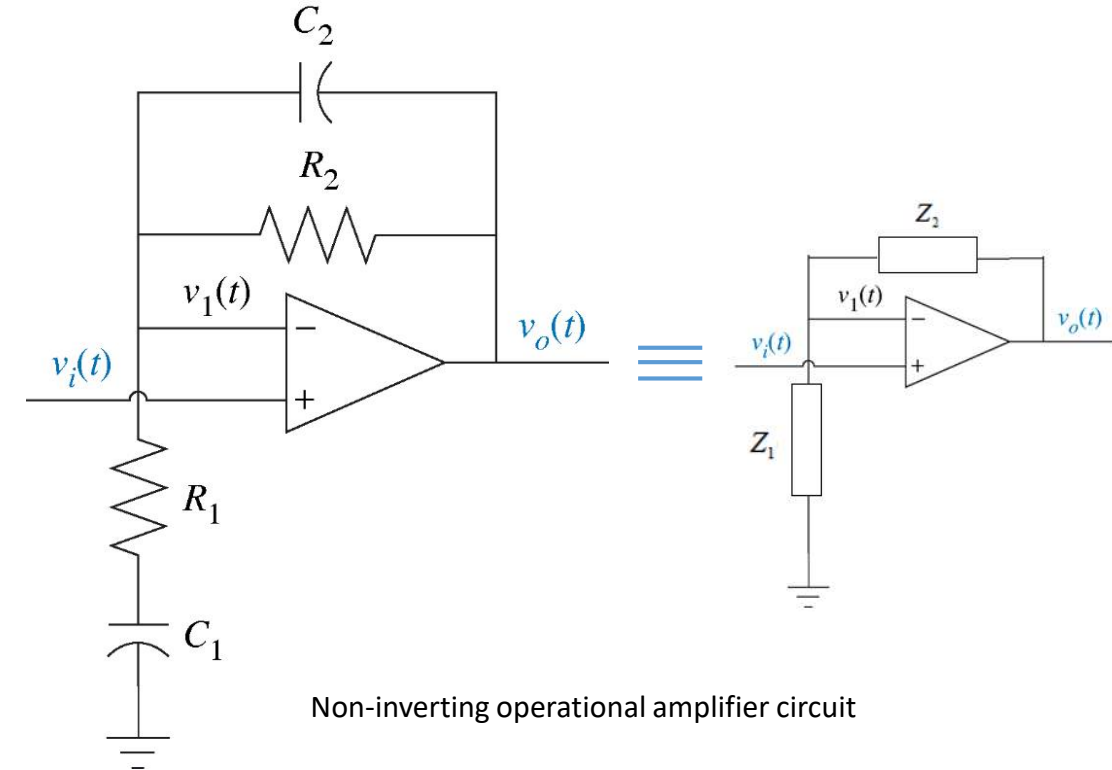
$$Z_1(s) = R_1 + \frac{1}{C_1s} = \frac{R_1C_1s + 1}{C_1s} \quad \text{and} \quad Z_2(s) = \frac{1}{C_2s + \frac{1}{R_2}} = \frac{R_2}{R_2C_2s + 1}$$

Now use the following equation:  $\frac{V_o(s)}{V_i(s)} = \frac{Z_1(s) + Z_2(s)}{Z_1(s)}$

$$\frac{V_o(s)}{V_i(s)} = \left[ \frac{R_1C_1s + 1}{C_1s} + \frac{R_2}{R_2C_2s + 1} \right] \frac{C_1s}{R_1C_1s + 1} = 1 + \frac{R_2C_1s}{(R_2C_2s + 1)(R_1C_1s + 1)}$$

➔ 
$$\frac{V_o(s)}{V_i(s)} = 1 + \frac{R_2C_1s}{R_1R_2C_1C_2s^2 + R_2C_2s + R_1C_1s + 1}$$

Substituting yields 
$$\frac{V_o(s)}{V_i(s)} = \frac{C_2C_1R_2R_1s^2 + (C_2R_2 + C_1R_2 + C_1R_1)s + 1}{C_2C_1R_2R_1s^2 + (C_2R_2 + C_1R_1)s + 1}$$

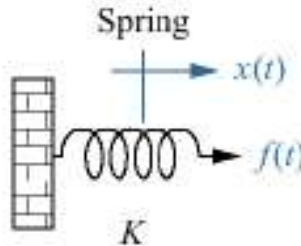
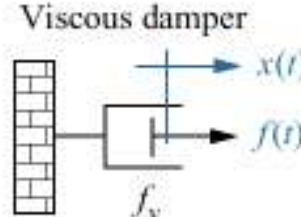
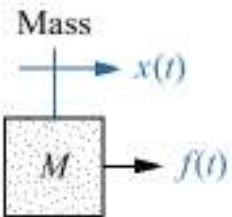


# Translational Mechanical System Transfer Functions

*Mechanical systems* (like electrical networks) have three passive linear components: **Spring** and the **mass** (energy-storage elements); and **viscous damper** (dissipates energy).

**Table 2.4**

Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
 <p>Spring</p>	$f(t) = K \int_0^t v(\tau) d\tau$	$f(t) = Kx(t)$	$K$
 <p>Viscous damper</p>	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
 <p>Mass</p>	$f(t) = M \frac{dv(t)}{dt}$	$f(t) = M \frac{d^2 x(t)}{dt^2}$	$Ms^2$

$K$ : Spring constant

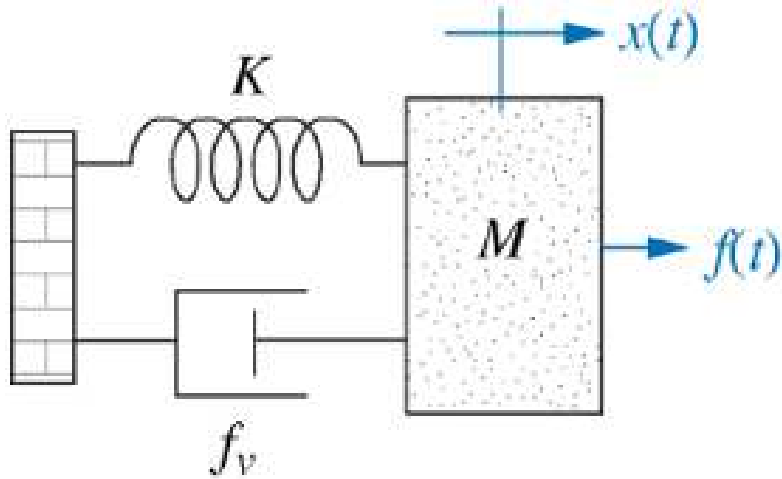
$f_v$ : Coefficient of viscous friction

$M$ : Coefficient of mass

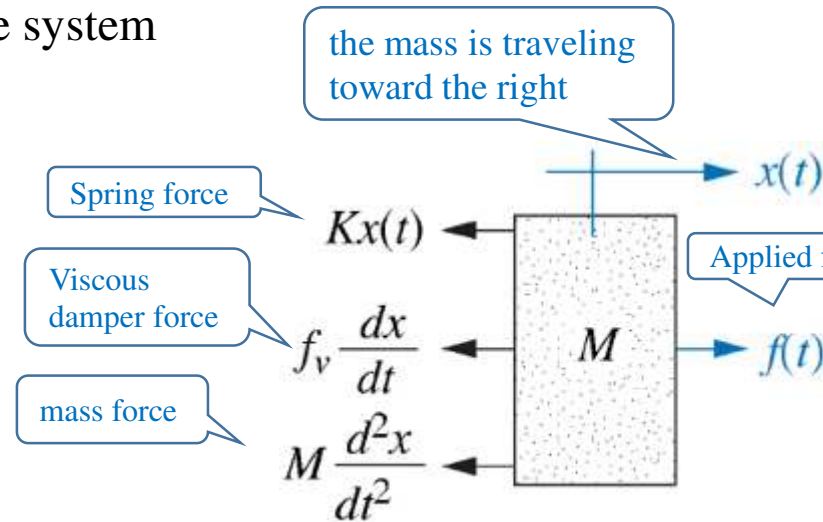
Note: The following set of symbols and units is used throughout this book:  $f(t) = \text{N}$  (newtons),  $x(t) = \text{m}$  (meters),  $v(t) = \text{m/s}$  (meters/second),  $K = \text{N/m}$  (newtons/meter),  $f_v = \text{N-s/m}$  (newton-seconds/meter),  $M = \text{kg}$  (kilograms = newton-seconds<sup>2</sup>/meter).

# Transfer Functions: One Degree of Freedom

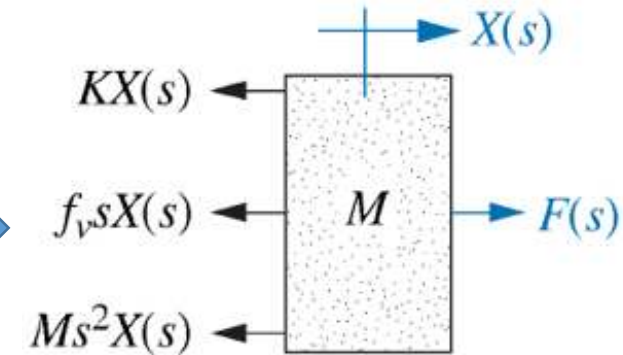
Find the transfer function,  $X(s)/F(s)$ , for the system



Mass, spring, and damper system



Free-body diagram of mass, spring, and damper system;



Transformed free body diagram

- All the forces impede (obstruct and block) the motion and act to oppose the applied force.

Differential equation of motion (Newton's law)

$$M \frac{d^2x(t)}{dt^2} + f_v \frac{dx(t)}{dt} + Kx(t) = f(t)$$

Sum of impedances  $\times X(s) =$  Sum of applied forces (zero initial conditions)

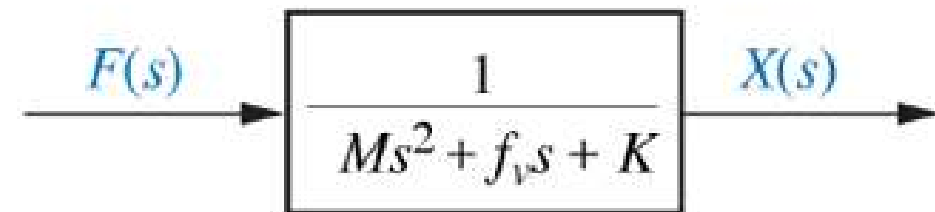
LT  $\rightarrow$

$$(K + f_v s + Ms^2) \times X(s) = F(s)$$

Solving for the transfer function yields

$$G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + f_v s + K}$$

$\rightarrow$

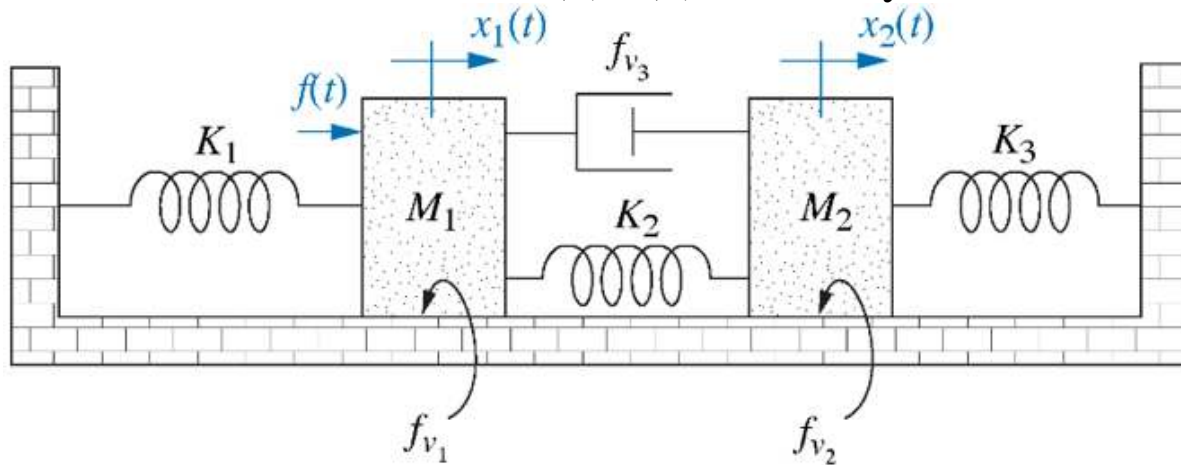


Transfer function

# Transfer Functions: Two Degrees of Freedom

Number of differential equations required to describe the system is equal to the number of *linearly independent* motions (*degrees of freedom*).

- Find the transfer function,  $X_2(s)/F(s)$ , for the system



Two-degrees-of-freedom translational mechanical system

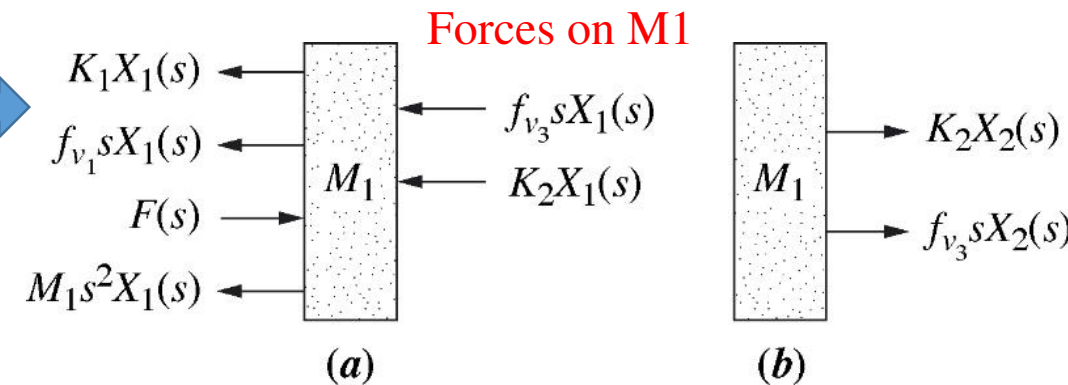
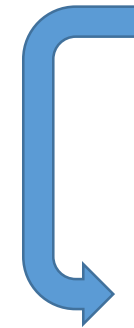
*Two-degrees-of-freedom* : since Each mass can be moved in the horizontal direction while the other is held still.

The Laplace transform of the equation of motion of M1

$$[M_1s^2 + (f_{v1} + f_{v3})s + (K_1 + K_2)]X_1(s) - (f_{v3}s + K_2)X_2(s) = F(s) \quad (1)$$

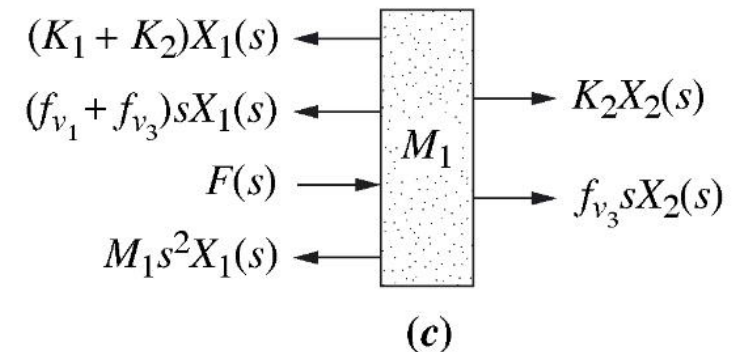
$$A X_1(s) - B X_2(s) = F \quad (1)$$

- Forces on M1 due only to motion of M1
- forces on M1 due only to motion of M2
- all forces on M1



hold M2 and move M1

hold M1 and move M2



total force on M1  
(superposition or sum)

# Transfer Functions: Two Degrees of Freedom Continued

The Laplace transform of the equation of motion of M2

$$-(f_{v3}s + K_2)X_1(s) + [M_2s^2 + (f_{v2} + f_{v3})s + (K_2 + K_3)]X_2(s) = 0$$

$$-C X_1(s) + DX_2(s) = 0 \quad \Rightarrow \quad X_1(s) = \frac{D}{C} X_2(s) \quad (2)$$

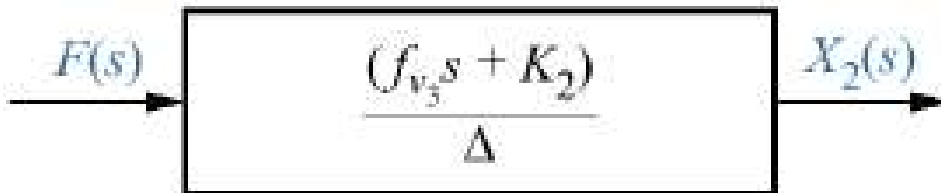
$(2) \text{ in } (1) \Rightarrow A \frac{D}{C} X_2(s) - BX_2(s) = F \quad \Rightarrow \quad \frac{X_2(s)}{F(s)} = \frac{C}{AD - CB}$

Transfer function:  $\frac{X_2(s)}{F(s)} = G(s) = \frac{(f_{v3}s + K_2)}{\Delta}$

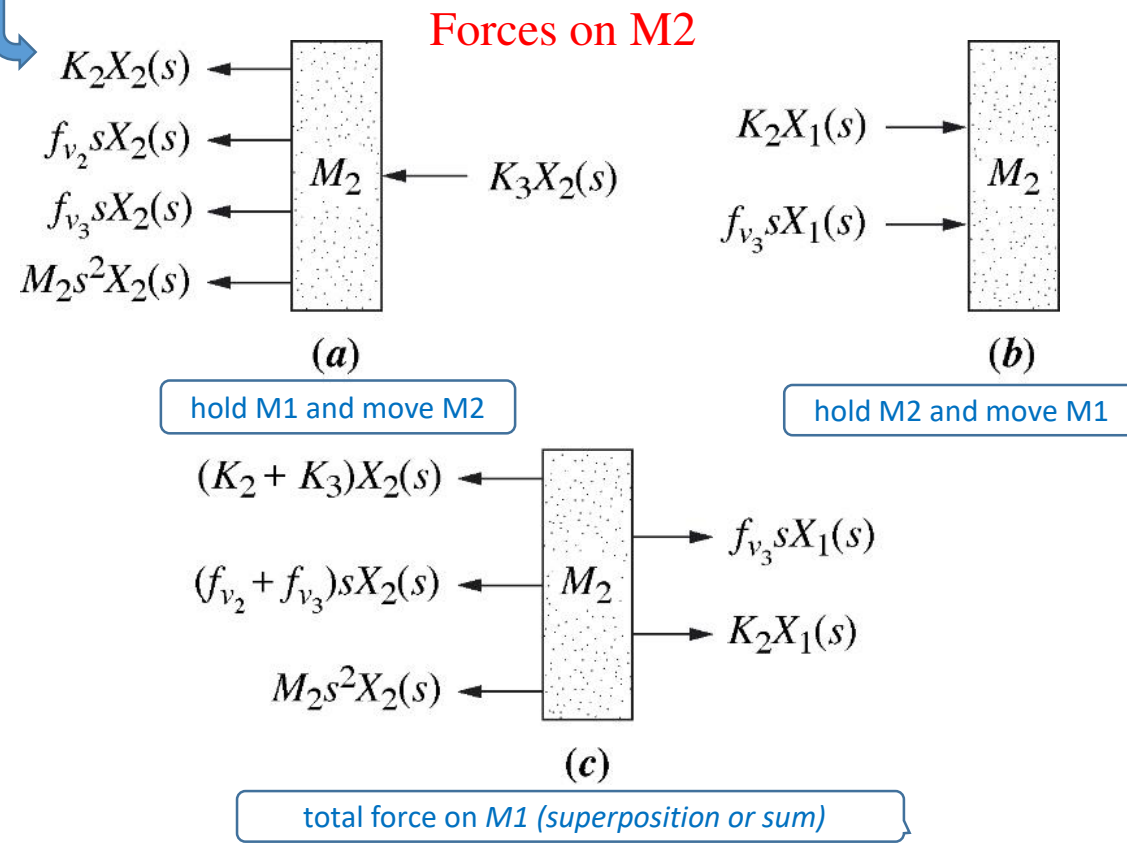
Determinant

where

$$\Delta = \begin{vmatrix} [M_1s^2 + (f_{v1} + f_{v3})s + (K_1 + K_2)] & -(f_{v3}s + K_2) \\ -(f_{v3}s + K_2) & [M_2s^2 + (f_{v2} + f_{v3})s + (K_2 + K_3)] \end{vmatrix}$$

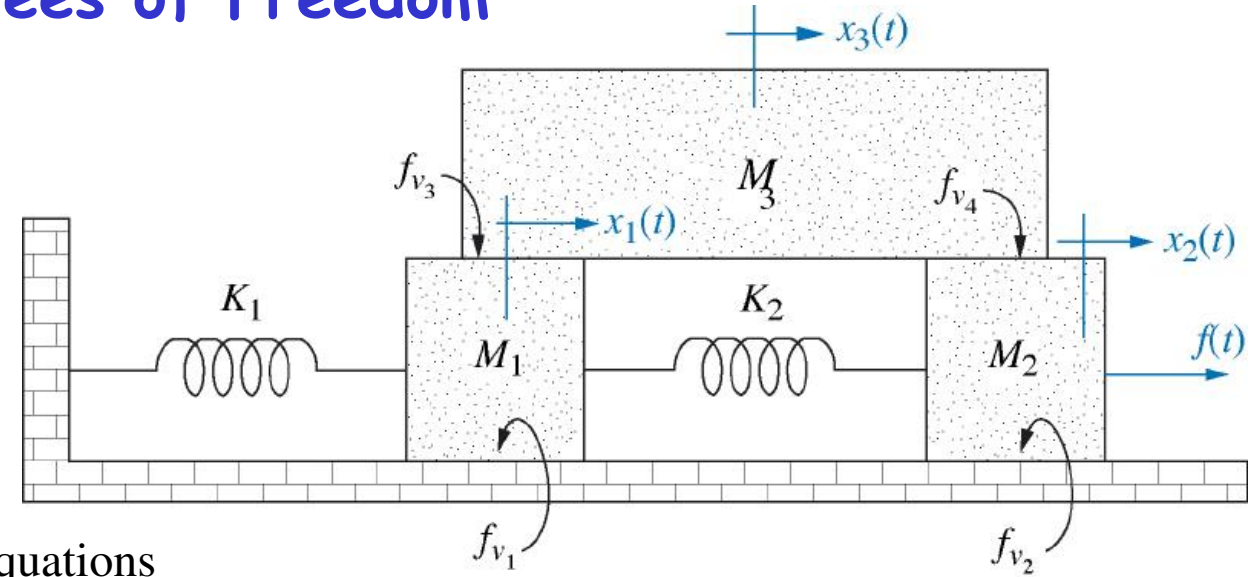


- a. Forces on M2 due only to motion of M2;
- b. forces on M2 due only to motion of M1;
- c. all forces on M2



# Transfer Functions: Three Degrees of Freedom

- Write, the equations of motion for the mechanical network
- The system has three degrees of freedom, since each of the three masses can be moved independently while the others are held still.
- The form of the equations will be similar to electrical mesh equations



For M1:

$$\left[ \begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{connected} \\ \text{to the motion} \\ \text{at } x_1 \end{array} \right] X_1(s) - \left[ \begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_2 \end{array} \right] X_2(s) - \left[ \begin{array}{c} \text{Sum of} \\ \text{impedances} \\ \text{between} \\ x_1 \text{ and } x_3 \end{array} \right] X_3(s) = \left[ \begin{array}{c} \text{Sum of} \\ \text{applied forces} \\ \text{at } x_1 \end{array} \right]$$

Similarly, for M2 and M3, we obtain

$$\begin{aligned} \mathbf{M}_1: & [M_1 s^2 + (f_{v1} + f_{v3})s + (K_1 + K_2)]X_1(s) - K_2 X_2(s) - f_{v3} s X_3(s) = 0 \\ \mathbf{M}_2: & -K_2 X_1(s) + [M_2 s^2 + (f_{v2} + f_{v4})s + K_2]X_2(s) - f_{v4} s X_3(s) = F(s) \\ \mathbf{M}_3: & -f_{v3} s X_1(s) - f_{v4} s X_2(s) + [M_3 s^2 + (f_{v3} + f_{v4})s]X_3(s) = 0 \end{aligned}$$

# Nonlinearity

Linear systems have two properties: (1) *additivity*, and (2) *homogeneity*.

1. *Additivity (superposition)*:

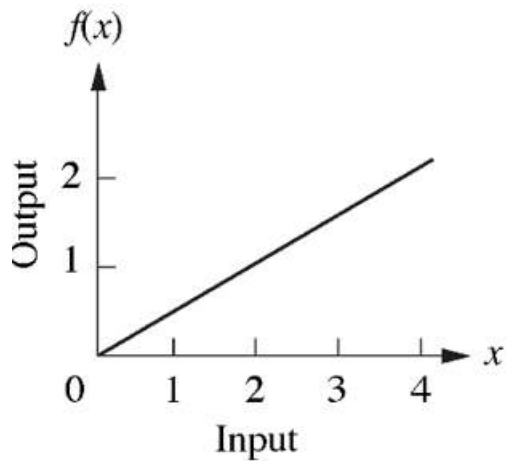
If  $r_1(t) \rightarrow c_1(t)$  and  $r_2(t) \rightarrow c_2(t)$ , then  $r_1(t) + r_2(t) \rightarrow c_1(t) + c_2(t)$

$r(t)$ : Input

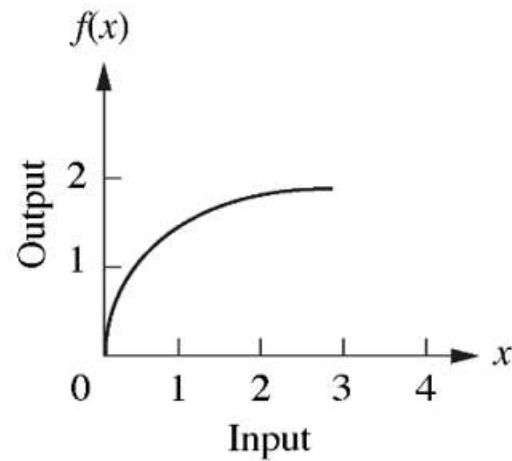
$c(t)$ : Output

2. *Homogeneity*:

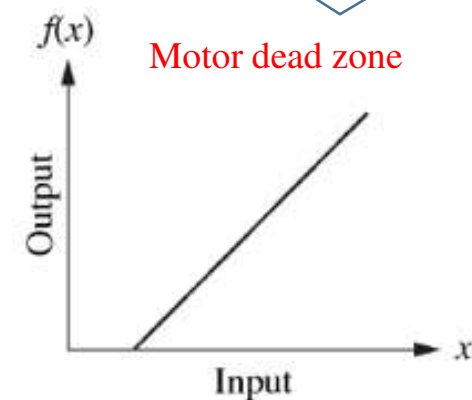
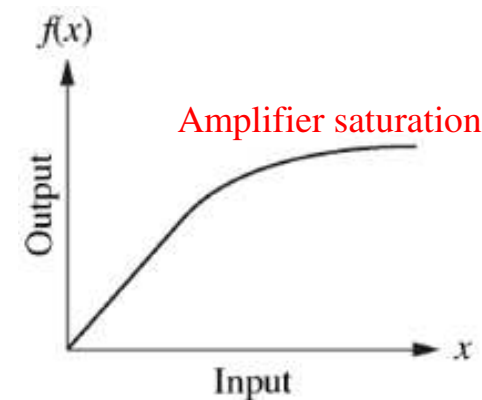
If  $r_1(t) \rightarrow c_1(t)$ , then  $A r_1(t) \rightarrow A c_1(t)$



Linear system



Nonlinear system



motor does not respond at very low input voltages due to frictional forces exhibits a nonlinearity called *dead zone*

Some physical nonlinearities

# Linearization

- If the system is nonlinear, we must linearize the system before we can find the transfer function.
- Making linear approximation to a nonlinear system.
- For a nonlinear system operating at point A:  $[x_0, f(x_0)]$

*Linear approximation :*

related by the Slope  $m_A$  (line)  
of the curve at the point A

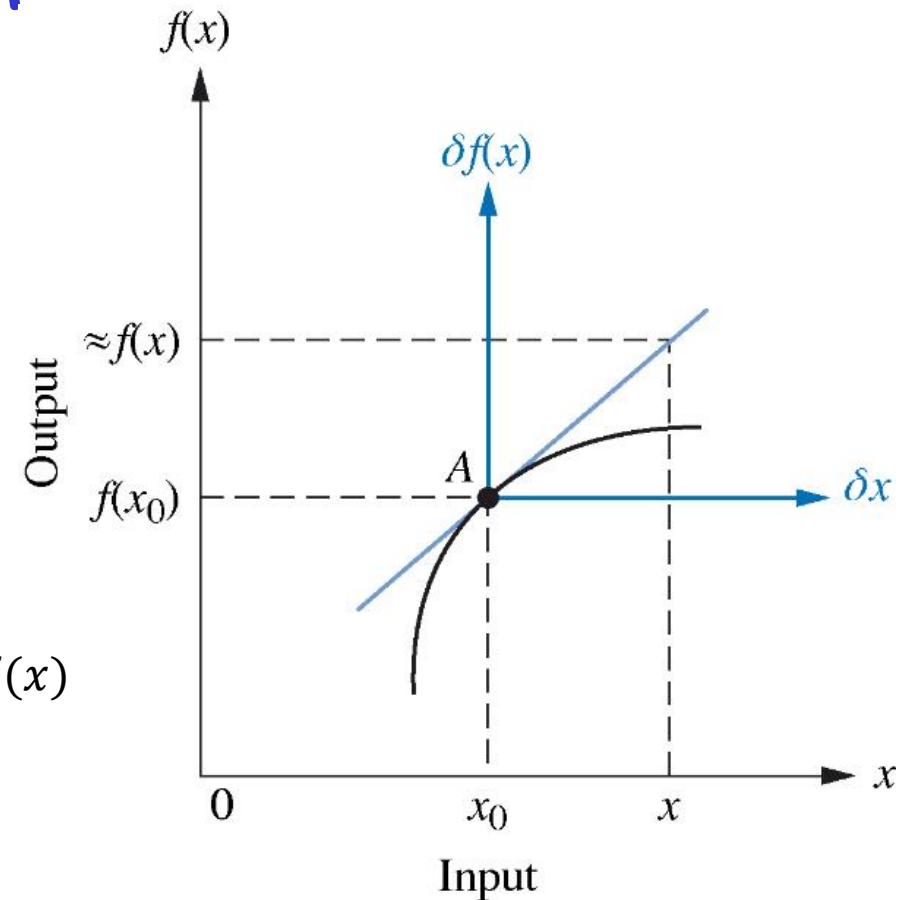
small changes in the input  $\delta x$   $\longrightarrow$  small changes in the output  $\delta f(x)$

Thus,  $f(x) - f(x_0) \approx m_A(x - x_0)$

$\longrightarrow \delta f(x) \approx m_A \delta x$  Derivative of  $f(x)$  at  $x = x_0$

$\longrightarrow f(x) \approx f(x_0) + m_A(x - x_0) = f(x_0) + m_A \delta x$

$\longrightarrow \boxed{f(x) \approx f(x_0) + m_A \delta x}$



Linearization about a point A

# Linearizing a Function

**Problem:** Linearize  $f(x) = 5 \cos(x)$  about  $x = \pi/2$ .

**Solution:**

We first find that the derivative of  $f(x)$  at  $x = \pi/2$

$$\left. \frac{df}{dx} \right|_{x=\pi/2} = -5 \sin x \Big|_{x=\pi/2} = -5$$

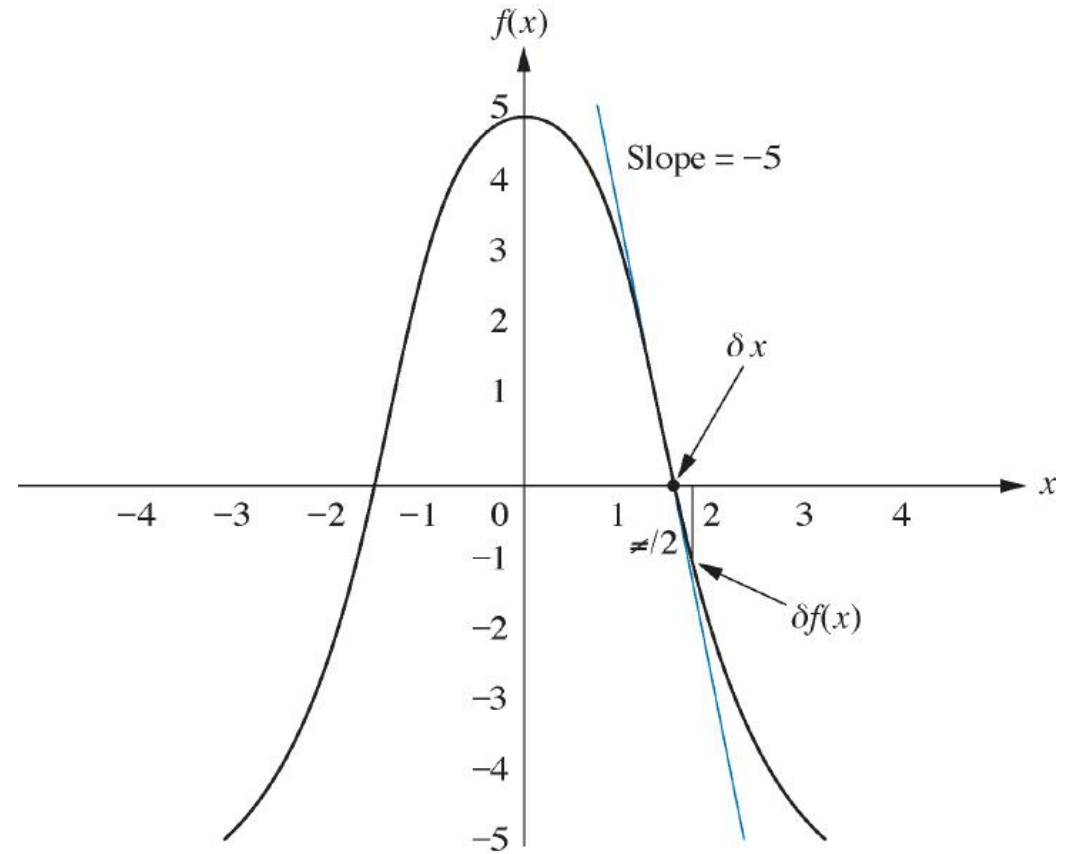
Slope at  $x = \frac{\pi}{2}$

Also

$$f(x_0) = f(\pi/2) = 5 \cos(\pi/2) = 0$$

the system can be represented as

$$f(x) = -5 \delta x \text{ for small excursions of } x \text{ about } \pi/2$$



# Modeling in The Time Domain

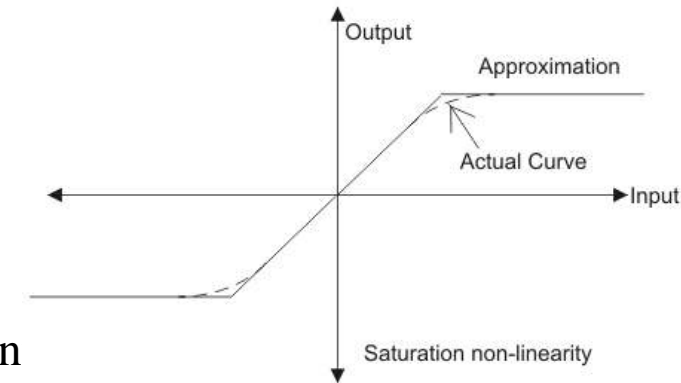
## State-space Method

Two approaches are available for the analysis and design of feedback control systems.

### 1. Frequency domain approach (classical approach):

based on converting a system's differential equation to a transfer function.

- **Advantage:** rapidly providing stability and transient response information. Thus we can immediately see the effects of varying system parameters.
- **Disadvantage:** limited application. It can be applied only to linear, time-invariant systems or systems that can be approximated as such.

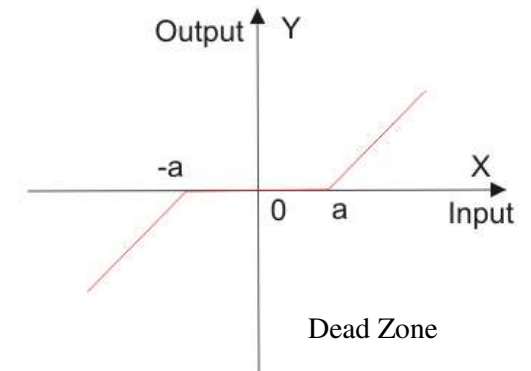


### 2. State-space approach (time domain / modern approach):

a sudden, forceful backward movement

Can be used:

- To represent non-linear systems that have backlash, saturation, dead zone.
- It can handle systems with nonzero initial conditions.
- Multiple-inputs, multiple-outputs systems can easily be represented.
- Many commercial software packages are available.



Many calculation is needed before actual realization.

# RL Network: State-Space Representation

The *state-space* approach for representing physical systems (state equations and the output equations are a viable (feasible) representation of the system.).

1. Select a *state variable* (possible system variable) : say  $i(t)$ .

2. Write differential equation (in terms of the state variable  $i(t)$ ).

$$L \frac{di}{dt} + Ri = v(t)$$

loop equation  
(State Equation)

3. Take Laplace transform:

solve for  $I(s)$ :  $L[sI(s) - i(0)] + RI(s) = V(s) \Rightarrow I(s) = \frac{V(s)}{(Ls + R)} + \frac{L i(0)}{Ls + R}$

If  $v(t) = u(t)$ , then  $V(s) = 1/s$ .  $\Rightarrow I(s) = \frac{1}{s(Ls + R)} + \frac{L i(0)}{Ls + R}$

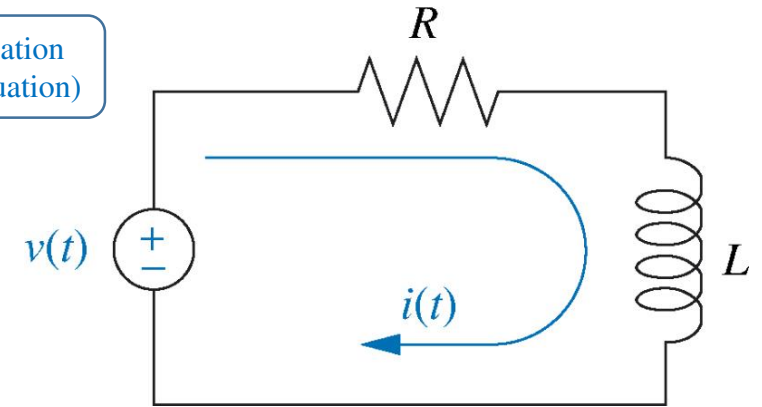
$$\Rightarrow I(s) = \frac{A}{s} + \frac{B}{Ls + R} + \frac{L i(0)}{Ls + R} \Rightarrow I(s) = \frac{1}{R} \left( \frac{1}{s} - \frac{1}{s + \frac{R}{L}} \right) + \frac{i(0)}{s + \frac{R}{L}}$$

Inverse Laplace transform:  $i(t) = \frac{1}{R} (1 - e^{-(R/L)t}) + i(0)e^{-(R/L)t}$

4. Output equations:

*Algebraically* combine the state variables with the system's input and find all of the other system variables for  $t \geq t_0$ .

$$\begin{aligned} v_R(t) &= Ri(t) \\ v_L(t) &= v(t) - Ri(t) \\ \frac{di}{dt} &= \frac{1}{L} [v(t) - Ri(t)] \end{aligned}$$



We can determine the state variable

RL network

If we know **initial condition** of  $i$ ,  $i(0)$ , and input voltage,  $v(t)$ , then we can find the value of any network variables at time  $t \geq t_0$ .

## Self Study

Do the State-space representation of RLC network.

# The General State-Space Representation

## Some Terminology

• *Linear combination:* (of  $n$  variables  $x_i$ )

$$S = K_n x_n + K_{n-1} x_{n-1} + \dots + K_1 x_1$$

none of the variables can be written as a linear combination of the others.

• *Linear independence:*  $S$  is zero if every  $K$  is zero and no  $x$  is zero: variables  $x$  are linearly independent.

• *System variable:* Any variable that responds to an input or initial conditions in a system.

• *State variables:* The smallest set of linearly independent system variables that completely determines (knowing the value at  $t_0$ ) the value of system variables for  $t \geq t_0$

• *State vector:* A vector whose elements are state variables.

• *State space:* The  $n$ -dimensional space whose axes are the state variables.

• *State equations:* A set of  $n$  simultaneous, first-order differential equations with  $n$  variables (state variables).

• *Output equations:* The equation that expresses the output variables of a system as linear combinations of the state variables and the inputs.

# State-space Representation

- A system is represented in *state-space* by the following equations:

$$\begin{cases} \dot{x} = A x + B u & \leftarrow \text{State equation} \\ y = C x + D u & \leftarrow \text{Output equation} \end{cases}$$

$x$ : *state vector*  
 $\dot{x}$ : *derivative of the state vector w.r.t. time*  
 $y$ : *Output vector*  
 $u$ : *input or control vector*  
 $A$ : *system matrix*  
 $B$ : *input matrix*  
 $C$ : *output matrix*  
 $D$ : *feedforward matrix*

- This representation of a system provides *complete knowledge* of all variables of the system at any  $t \geq t_0$

- The choice of state variables:
  - is not unique.
  - minimum number* (equals the order of the differential equation).
  - are *linearly independent*.

## Problem:

Given the following system:

Set the system on the following state-space form:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_1 - x_2 + 5u$$

$$y = x_2$$

## Solution:

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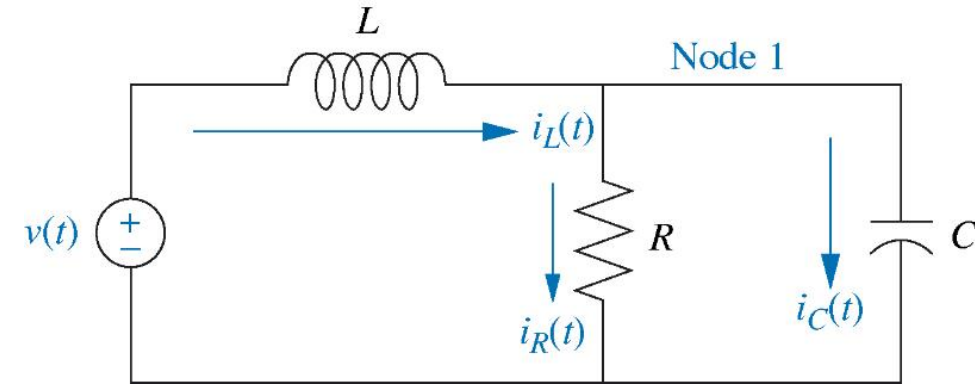
State-space model: 
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 5 \end{bmatrix}}_B u$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_D u$$

# Example-1: State-space Representation

## Problem:

Find a state-state representation of the following electrical network if the *output is  $i_R$*  the current through the resistor, ( *$v(t)$  is the input*).



## Solution:

The following steps will yield a viable representation of the network in state space.

**Step 1:** Label all the branch currents in the network (These include  $i_L$ ,  $i_R$ , and  $i_C$  ).

**Step 2:** Select the state variables (*quantities that are differentiated*  $v_C$  and  $i_L$ , energy-storage elements, the inductor C and the capacitor L) and write derivative equations.

$$C \frac{dv_C}{dt} = i_C, \quad L \frac{di_L}{dt} = v_L$$

We can have  $v_C = i_L$ .

**Step 3:** Express *non-state variables* (right-hand side:  $i_C$  and  $v_L$ ) as a linear combinations of the state variables (differentiated variables:  $v_C$  and  $i_L$ ) and the input,  $v(t)$ .

$$i_C = -i_R + i_L = -\frac{1}{R}v_C + i_L$$

$$v_L = -v_C + v(t)$$

We have  $v_R = v_C$ .

Apply Kirchoff's voltage and current laws, to obtain  $i_C$  and  $v_L$  in terms of the state variables,  $v_C$  and  $i_L$ .

# Example-1: State-space Representation-contd.

**Step 4:** Obtain state equations: (by substituting the values and rearranging)

$$C \frac{dv_C}{dt} = -\frac{1}{R} v_C + i_L, \quad L \frac{di_L}{dt} = -v_C + v(t)$$

$$\Rightarrow \frac{dv_C}{dt} = -\frac{1}{RC} v_C + \frac{1}{C} i_L$$

$$\frac{di_L}{dt} = -\frac{1}{L} v_C + \frac{1}{L} v(t)$$

Matrix Form  $\rightarrow$

$$\begin{cases} \frac{dv_C}{dt} = -\frac{1}{RC} \cdot v_C + \frac{1}{C} \cdot i_L + 0 \cdot v(t) \\ \frac{di_L}{dt} = -\frac{1}{L} \cdot v_C + 0 \cdot i_L + \frac{1}{L} \cdot v(t) \end{cases}$$

State equation

**Step 5:** Find the output equation:  $i_R = \frac{1}{R} v_C$

Matrix Form  $\rightarrow$

$$i_R = \frac{1}{L} \cdot v_C + 0 \cdot i_L$$

output equation

Final result: Convert into vector-matrix form

$$\begin{cases} \dot{x} = A x + B u \\ y = C x + D u \end{cases}$$

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} v(t)$$

$$i_R = \begin{bmatrix} \frac{1}{R} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix}$$

# Example-2: State-space Representation (with a dependent source)

**PROBLEM:** Find the state and output equations for the electrical network shown in Figure.

If the output vector is  $y = [v_{R_2} \quad i_{R_2}]^T$

**Step 1:** Label all the branch currents in the network.

**Step 2:** Select the *state variables* (energy-storage elements: L and C) and write derivative equations (voltage-current relationships).

$$C \frac{dv_C}{dt} = i_C, \quad L \frac{di_L}{dt} = v_L \quad \mathbf{x}_1 = i_L; \quad \mathbf{x}_2 = v_C; \quad \text{the state variables (differentiated variables)}$$

**Step 3:** State equations (we find  $v_L$  and  $i_C$  in terms of the state variables)

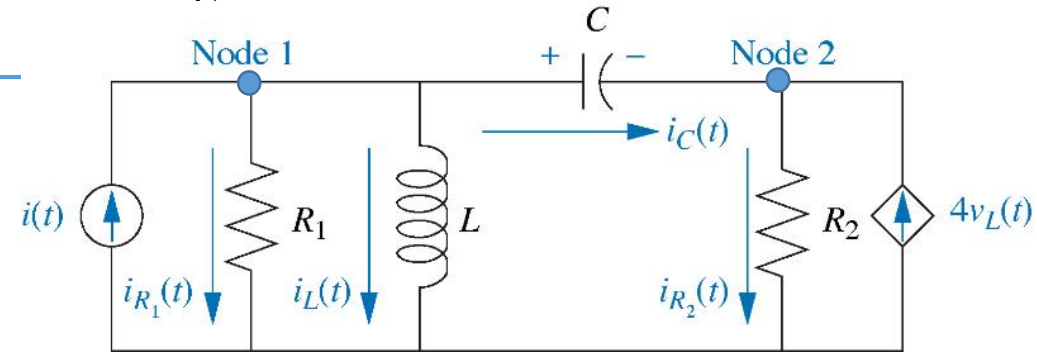
mesh  $LCR_2$   $\longrightarrow$   $v_L = v_C + v_{R_2} = v_C + i_{R_2} R_2$

Node 2  $\longrightarrow$  At node 2,  $i_{R_2} = i_C + 4v_L$ , so we get,

$$v_L = v_C + (i_C + 4v_L) R_2$$

$$\Rightarrow v_L = \frac{1}{1 - 4R_2} (v_C + i_C R_2)$$

$$\Rightarrow (1 - 4R_2)v_L - R_2 i_C = v_C \quad (1)$$



Node 1  $\longrightarrow$

$$i_C = i(t) - i_{R_1} - i_L$$

$$= i(t) - \frac{v_{R_1}}{R_1} - i_L$$

$(v_{R_1} = v_L)$

$$= i(t) - \frac{v_L}{R_1} - i_L$$

$$\Rightarrow -\frac{1}{R_1} v_L - i_C = i_L - i(t) \quad (2)$$

# Example-2: State-space Representation

Solving (1) and (2) simultaneously for  $v_L$  and  $i_C$  yields

$$-\frac{1}{R_1}v_L - i_C = i_L - i(t) \quad \rightarrow \quad i_C = -\frac{1}{R_1}v_L - i_L + i(t) \quad \xrightarrow{(1)} \quad (1 - 4R_2)v_L - R_2\left(-\frac{1}{R_1}v_L - i_L + i(t)\right) = v_C$$

$$\rightarrow \left(1 - 4R_2 + \frac{R_2}{R_1}\right)v_L + R_2i_L - R_2i(t) = v_C \quad \rightarrow \quad v_L = \frac{1}{\Delta}[R_2i_L - v_C - R_2i(t)] \quad \text{with } \Delta = -\left(1 - 4R_2 + \frac{R_2}{R_1}\right)$$

$$\text{and } i_C = \frac{1}{\Delta}\left[(1 - 4R_2)i_L + \frac{1}{R_1}v_C - (1 - 4R_2)i(t)\right]$$

writing the result in vector-matrix form

$$\begin{bmatrix} \dot{i}_L \\ \dot{v}_C \end{bmatrix} = \begin{bmatrix} R_2/(L\Delta) & -1/(L\Delta) \\ (1 - 4R_2)/(C\Delta) & 1/(R_1C\Delta) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} -R_2/(L\Delta) \\ -(1 - 4R_2)/(C\Delta) \end{bmatrix} i(t)$$

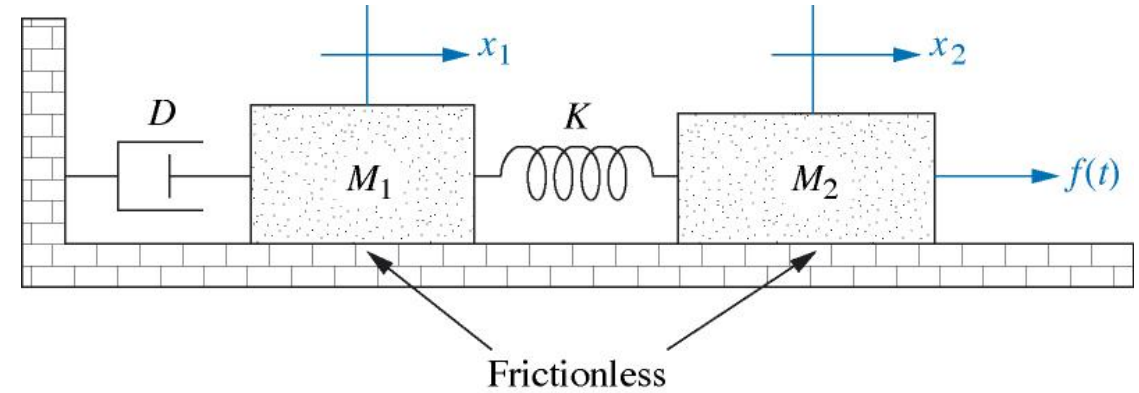
**Step 4:** Output equations  $v_{R2} = -v_C + v_L$ ;  $i_{R2} = i_C + 4v_L$ ;

vector-matrix form, the output equation is

$$\begin{bmatrix} v_{R2} \\ i_{R2} \end{bmatrix} = \begin{bmatrix} R_2/\Delta & -(1 + 1/\Delta) \\ 1/\Delta & (1 - 4R_1)/(\Delta R_1) \end{bmatrix} \begin{bmatrix} i_L \\ v_C \end{bmatrix} + \begin{bmatrix} -R_2/\Delta \\ -1/\Delta \end{bmatrix} i(t)$$

# Example-3: State-space Representation

(Translational Mechanical System)



For M1:  $M_1 s^2 X_1(s) + DsX_1(s) + KX_1(s) - KX_2(s) = 0$

$$\Rightarrow M_1 \frac{d^2 x_1}{dt^2} + D \frac{dx_1}{dt} + Kx_1 - Kx_2 = 0$$

For M2:

$$-KX_1(s) + KX_2(s) + M_2 s^2 X_2(s) = F(s)$$

$$\Rightarrow -Kx_1 + Kx_2 + M_2 \frac{d^2 x_2}{dt^2} = f(t)$$

Let,  $\frac{d^2 x_i}{dt^2} = \frac{dv_i}{dt}$

(acceleration = derivative of velocity)

Select  $x_1, x_2, v_1, v_2$  as *state variables*.

$$\frac{dx_1}{dt} = v_1 \quad \Rightarrow \quad \frac{d^2 x_1}{dt^2} = \frac{dv_1}{dt} = \dot{v}_1$$

$$\frac{dx_2}{dt} = v_2 \quad \Rightarrow \quad \frac{d^2 x_2}{dt^2} = \frac{dv_2}{dt} = \dot{v}_2$$

State equations:

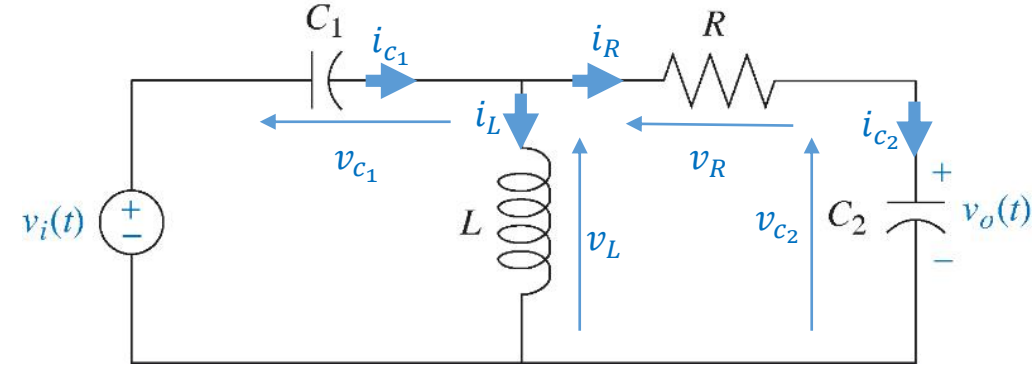
$$\begin{aligned} \frac{dx_1}{dt} &= \quad \quad \quad + v_1 \\ \frac{dv_1}{dt} &= -\frac{K}{M_1} x_1 - \frac{D}{M_1} v_1 + \frac{K}{M_1} x_2 \\ \frac{dx_2}{dt} &= \quad \quad \quad + v_2 \\ \frac{dv_2}{dt} &= +\frac{K}{M_2} x_1 \quad \quad -\frac{K}{M_2} x_2 \quad + \frac{1}{M_2} f(t) \end{aligned}$$

matrix form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{v}_1 \\ \dot{x}_2 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -K/M_1 & -D/M_1 & K/M_1 & 0 \\ 0 & 0 & 0 & 1 \\ K/M_2 & 0 & -K/M_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ v_1 \\ x_2 \\ v_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/M_2 \end{bmatrix} f(t)$$

# Example-4: State-space Representation

**Problem:** Find the state-space representation of the electrical network shown in the figure. The output is  $v_o(t)$ .



**Solution:**

*state variables:*  $v_{c_1}, i_L, v_{c_2}$

The derivative relations (one for each energy-storage element)

$$\begin{cases} C_1 \frac{dv_{c_1}}{dt} = i_{c_1} \\ L \frac{di_L}{dt} = v_L \\ C_2 \frac{dv_{c_2}}{dt} = i_{c_2} \end{cases}$$

Using Kirchhoff's current and voltage laws:

$$\begin{cases} i_{c_1} = i_L + i_R = i_L + \frac{v_R}{R} = i_L + \frac{1}{R}(v_L - v_{c_2}) & \text{Mesh 2} \\ v_L = v_i - v_{c_1} & \text{Mesh 1} \\ i_{c_2} = i_R = \frac{1}{R}(v_L - v_{c_2}) = \frac{1}{R}(v_i - v_{c_1} - v_{c_2}) & \text{Mesh 2, Mesh 1} \end{cases}$$

**Matrix form**

*State vector*

$$\mathbf{x} = \begin{bmatrix} v_{c_1} \\ i_L \\ v_{c_2} \end{bmatrix} \quad \dot{\mathbf{x}} = \begin{bmatrix} -1/RC_1 & 1/C_1 & -1/RC_1 \\ -1/L & 0 & 0 \\ -1/RC_2 & 0 & -1/RC_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1/RC_1 \\ 1/L \\ 1/RC_2 \end{bmatrix} v_i(t)$$

*state-space representation*

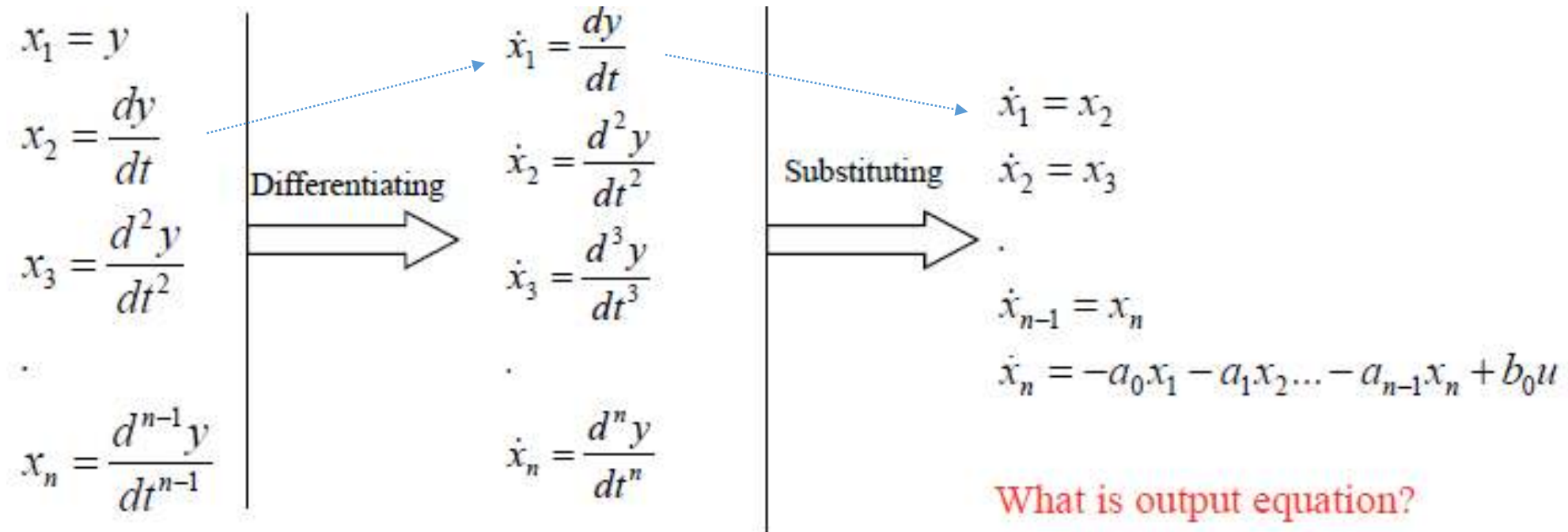
$$\begin{cases} i_{c_1} = -\frac{1}{R}v_{c_1} + i_L - \frac{1}{R}v_{c_2} + \frac{1}{R}v_i \\ v_L = -v_{c_1} + 0i_L + 0v_{c_2} + v_i \\ i_{c_2} = -\frac{1}{R}v_{c_1} + 0i_L - \frac{1}{R}v_{c_2} + \frac{1}{R}v_i \end{cases}$$

# Converting a Transfer Function to State Space

**Phase variables:** A set of state variables where each state variable is defined to be the derivative of the previous state variable.

Consider a differential equation, 
$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = b_0 u$$

Choose the output,  $y(t)$ , and its derivatives as the state variables,  $x_i$ .



# Converting a Transfer Function to State Space

matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 & -a_5 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ b_0 \end{bmatrix}$$

output

$$y = [1 \quad 0 \quad 0 \quad \dots \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}$$

**PROBLEM:** Find the state-space representation in phase-variable form for the transfer function

$$\frac{C(s)}{R(s)} = \frac{24}{(s^3 + 9s^2 + 26s + 24)}$$

**Step 1** Find the associated differential equation  
inverse Laplace transform,

$$(s^3 + 9s^2 + 26s + 24)C(s) = 24R(s)$$

$$\ddot{c} + 9\dot{c} + 26c + 24c = 24r$$

**Step 2** Select the state variables.

$$x_1 = c$$

$$x_2 = \dot{c}$$

$$x_3 = \ddot{c}$$

the state equations.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 24r$$

$$y = c = x_1$$

matrix form,

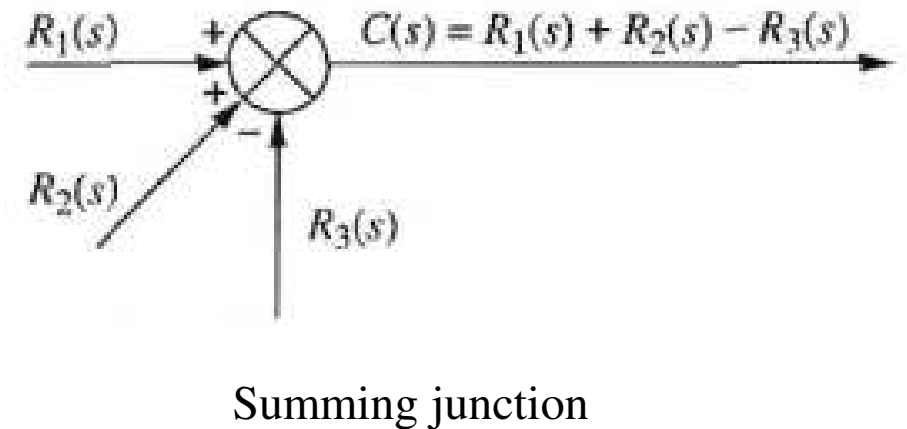
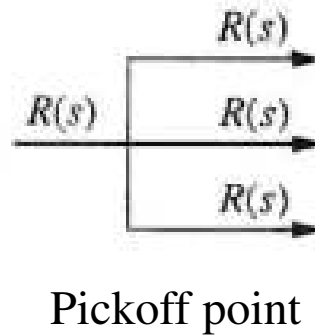
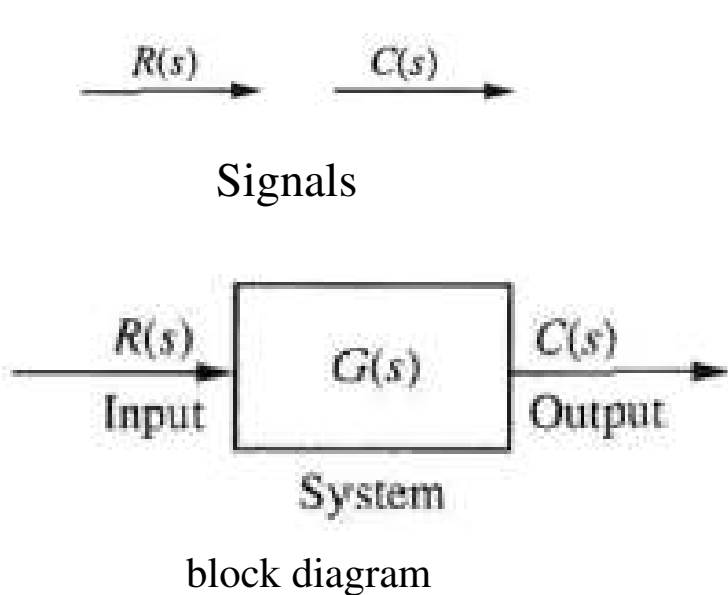


$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r$$

$$y = [1 \quad 0 \quad 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

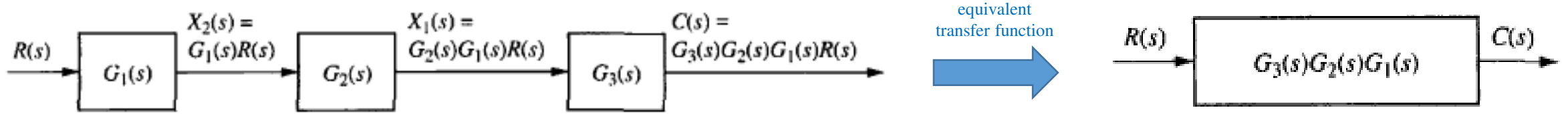
# Block Diagram Reduction

- More complicated systems are represented by the interconnection of many subsystems.
- In order to calculate the transfer function, we want to represent multiple subsystems as a single block.
- A subsystem is represented as a block with an input, an output, and a transfer function.

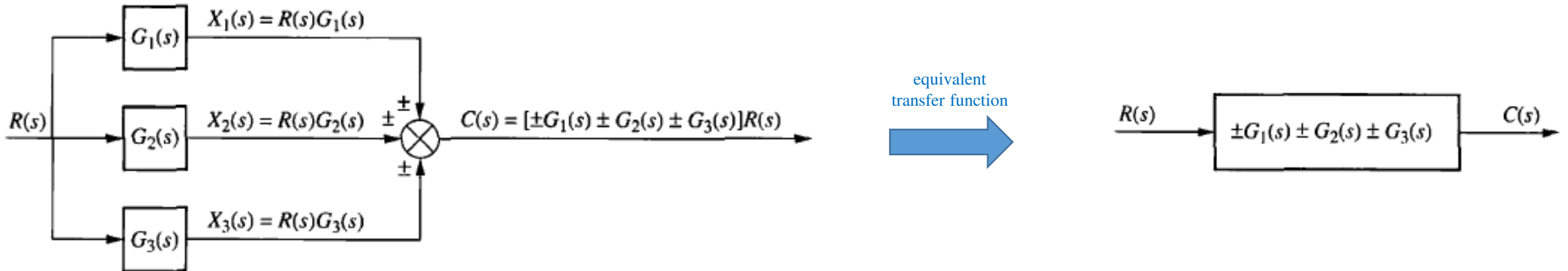


# Reduction of Multiple Subsystems<sub>2</sub>

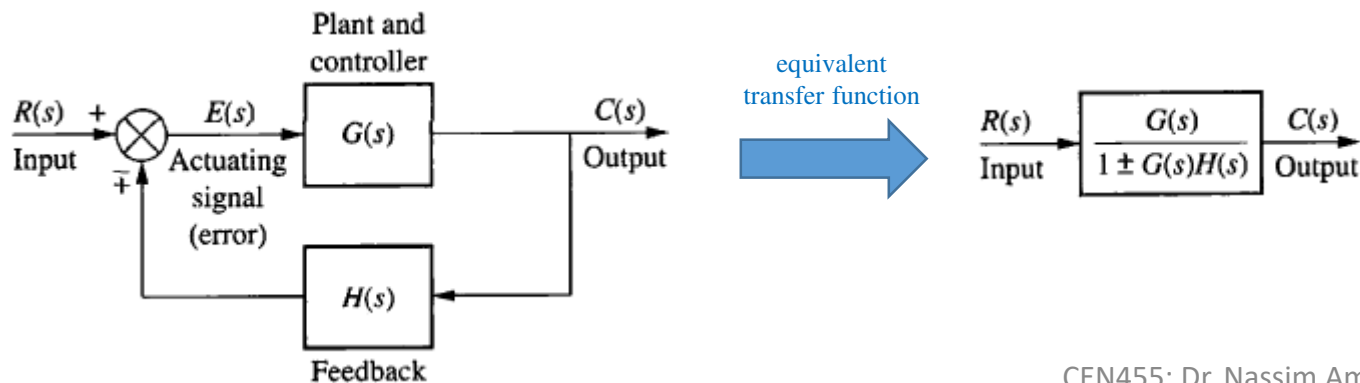
## Cascade Form



## Parallel Form



## Feedback Form



$$C(s) = G(s) E(s)$$

But since  $E(s) = R(s) \mp C(s) H(s)$

$$C(s) = G(s) [R(s) \mp C(s) H(s)]$$

$$C(s) = G(s)R(s) \mp G(s)C(s) H(s)$$

$$[1 \pm G(s) H(s)]C(s) = G(s)R(s)$$

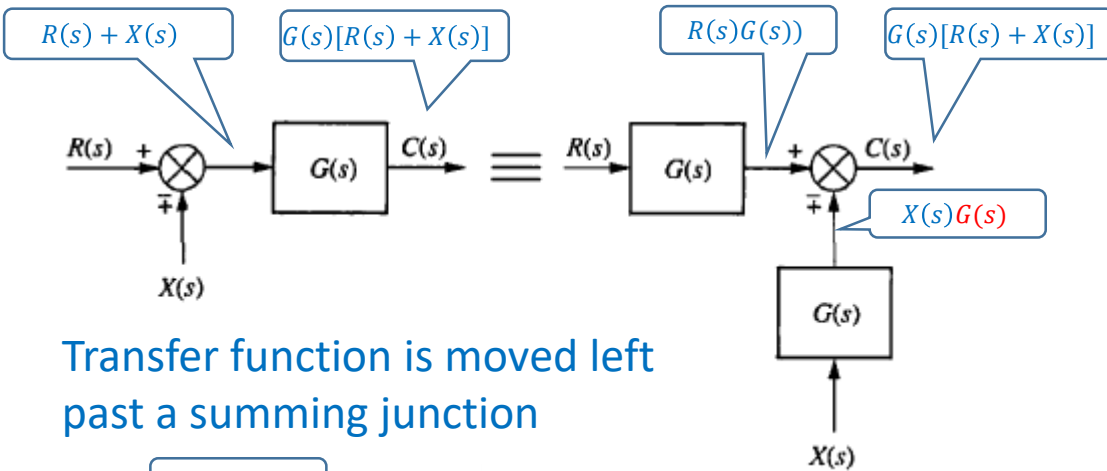
equivalent transfer function  $\rightarrow \frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s) H(s)}$

# Reduction of Multiple Subsystems<sub>3</sub>

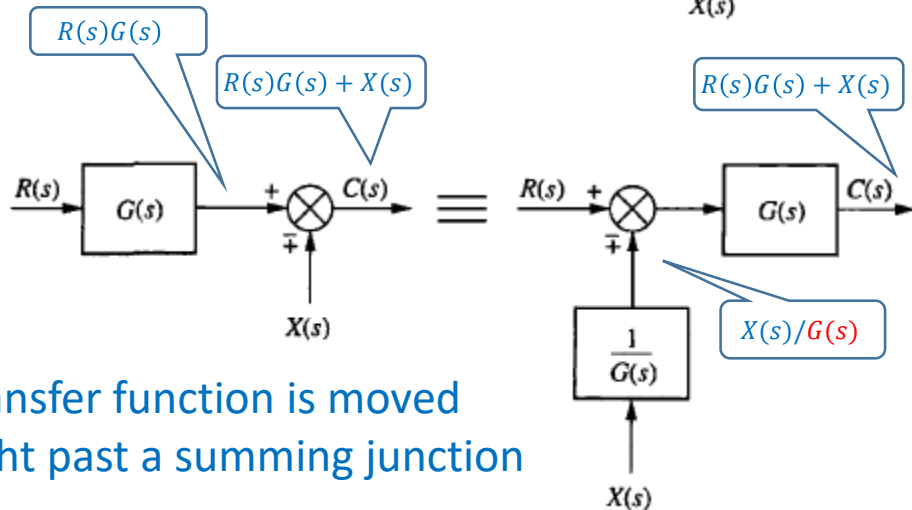
## Moving Blocks to Create Familiar Forms

- Familiar forms (cascade, parallel, and feedback) are not always apparent in a block diagram

### Block diagram algebra for summing junction

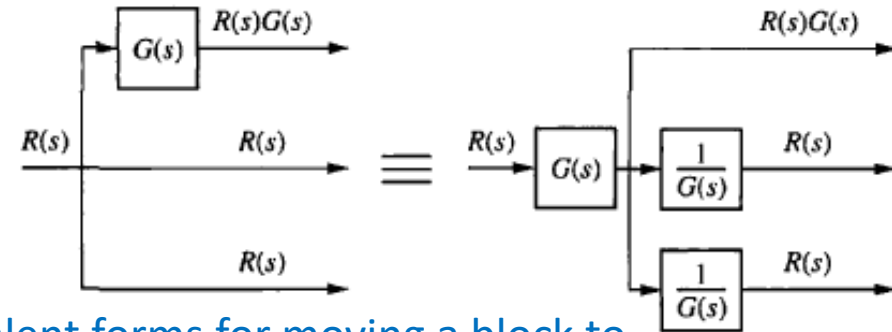


Transfer function is moved left past a summing junction

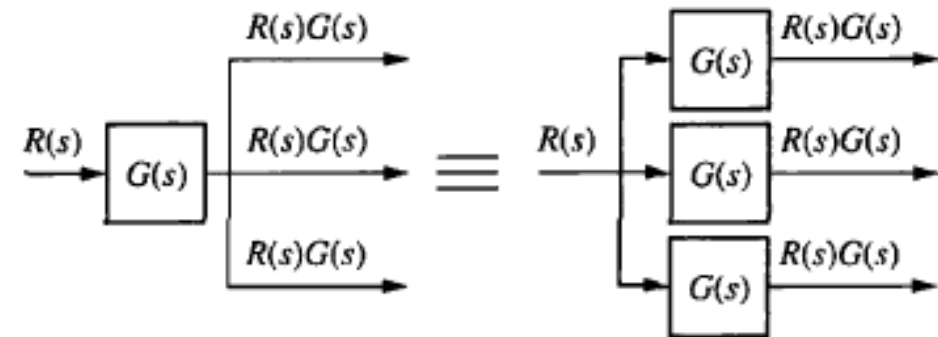


Transfer function is moved right past a summing junction

### Block diagram algebra for pickoff point



Equivalent forms for moving a block to the left past a pickoff point.

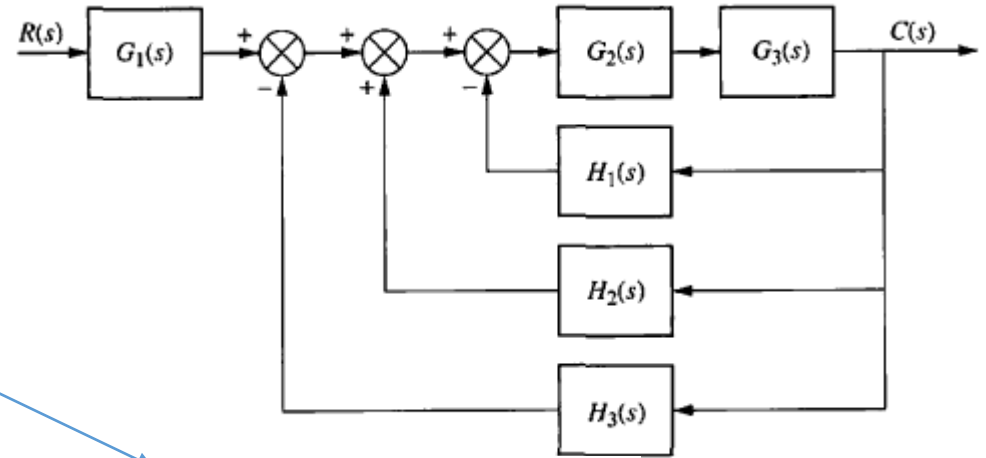


Equivalent forms for moving a block to the right past a pickoff point.

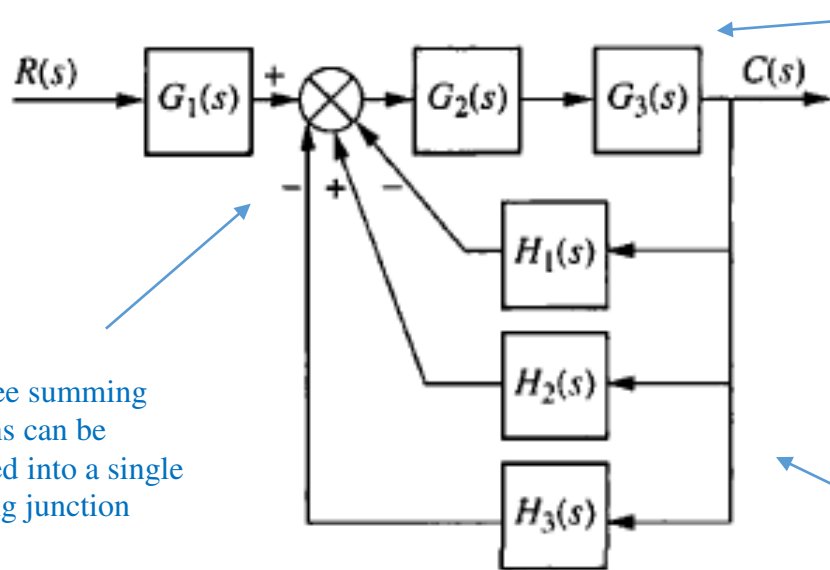
# Example

**Problem:** Reduce the system shown in Figure to a single transfer function.

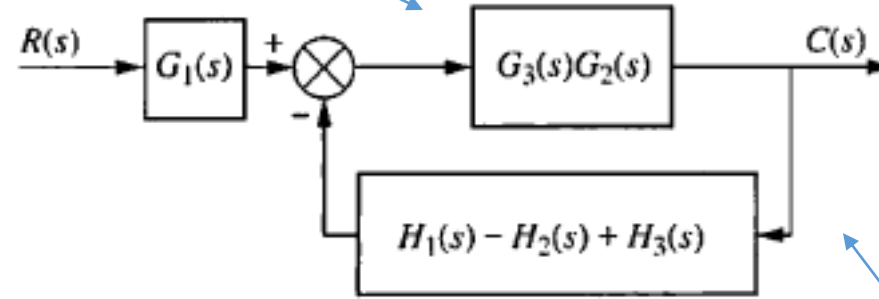
**Solution:**



The three summing junctions can be collapsed into a single summing junction



$G_2(s)$  and  $G_3(s)$  are connected in cascade.



the three feedback functions,  $H_1(s)$ ,  $H_2(s)$ , and  $H_3(s)$  are connected in parallel.

the feedback system is reduced and multiplied by  $G_1(s)$

