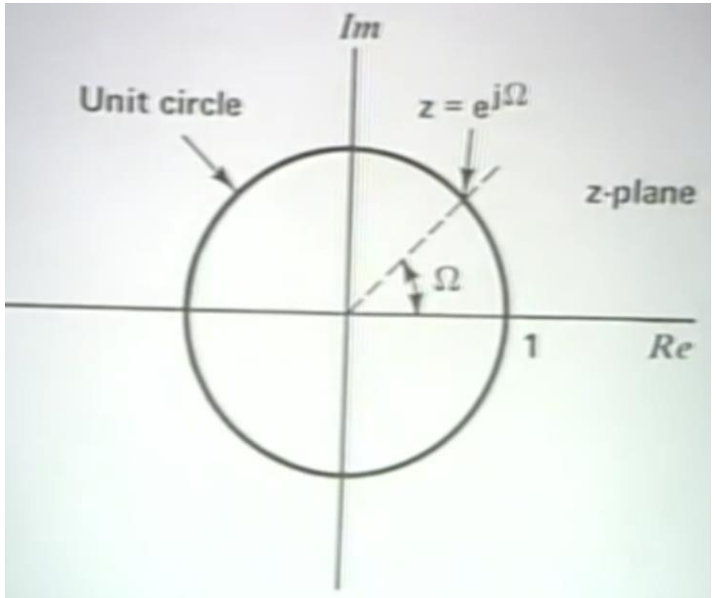
Chapter 5

The Z Transform

1. Definition
2. Properties of the z-Transform
3. Inverse z-Transform
4. Solution of Difference Equations Using the z-Transform

From DFT to Z-transform



Generalizing DFT → Z-transform

Eigen function
Unit circle in the complex space

$e^{j\Omega n}$ → $z = re^{j\Omega}$

Add module r
All the complex space

$$X(\Omega) = \sum_{n=-\infty}^{n=+\infty} x[n]\, e^{-j\Omega n}$$

← $z = e^{j\Omega} (r = 1)$

$$X(z) = \sum_{n=-\infty}^{n=+\infty} x[n]\, z^{-n}$$

input: z^n → Output (convolution): $\sum_{k=-\infty}^{+\infty} h[n]\, z^{n-k}$

→ $z^n \rightarrow z^n \underbrace{\sum_{k=-\infty}^{+\infty} h[n]\, z^{-k}}_{H(z)}$

→ input: z^n → Output: $H(z)\, z^n$

→ Z-transform: $H[z] = \sum_{n=-\infty}^{n=+\infty} h[n]\, z^{-n}$

$$X(re^{j\Omega}) = \sum_{n=-\infty}^{n=+\infty} x[n]\, \underbrace{(re^{j\Omega})^{-n}}_{r^{-n}e^{-j\Omega n}}$$

→ $= \sum_{n=-\infty}^{n=+\infty} x[n]\, r^{-n} e^{-j\Omega n}$ (Fourier transform of $x[n]r^{-n}$)

→ $X(z) = Z[x[n]] = DFT[x[n]r^{-n}]$

Z - Transform

Introduction

- The *Z-transform* plays the same role in the analysis of *discrete-time* signals and LTI systems as the *Laplace transform* does in the *continuous-time* signals and LTI systems.

- It offers the techniques for digital filter design and frequency analysis of digital signals.

Definition of z-transform:

The z-transform of the discrete-time $\boldsymbol{x}[\boldsymbol{n}]$ is given by:

$$x[n] \xrightarrow{\mathcal{Z}} X(z)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

Where z is a complex variable

For a ***causal*** sequence:

$x(n) = 0 \; for \; n < 0$

$$X(z) = Z(x(n)) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$$= x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

All the values of z that make the summation to exist form a ***region of convergence*** (***ROC***) .

Example 1

Problem:

Given the sequence, $x(n) = u(n)$, find the z-transform of $x(n)$

Solution:

From the definition of the z-transform:

$$X(z) = \sum_{n=0}^{\infty} u(n) z^{-n} = \sum_{n=0}^{\infty} \left(z^{-1}\right)^n = 1 + \left(z^{-1}\right) + \left(z^{-1}\right)^2 + \cdots$$

we know, $1 + r + r^2 + \cdots = \dfrac{1}{1-r}$ when $|r| < 1.$

ROC: Region of Convergence (values of z for the convergence)

Therefore, $X(z) = \dfrac{1}{1-z^{-1}} = \dfrac{1}{1-\dfrac{1}{z}} = \dfrac{z}{z-1}$

When, $|z^{-1}| < 1 \Rightarrow |z| > 1$

Example 2

Problem:

Consider the exponential sequence, $x(n) = a^n \, u(n)$, find the z-transform of $x(n)$.

Solution:

From the definition of the z-transform

$$X(z) = \sum_{n=0}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = 1 + \left(az^{-1}\right) + \left(az^{-1}\right)^2 + \cdots$$

Since this is a geometric series Therefore,

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z - a}$$

that will converge for, $X(z) = \frac{z}{z-a}$, for $|z| > |a|$

Region of Convergence

Z-Transform Table

Line No.	$x(n), n\geq 0$	z-Transform $X(z)$	Region of Convergence
1	$x(n)$	$\sum_{n=0}^{\infty} x(n)z^{-n}$	
2	$\delta(n)$	1	$\lvert z\rvert > 0$
3	$au(n)$	$\frac{az}{z-1}$	$\lvert z\rvert > 1$
4	$nu(n)$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
5	$n^2u(n)$	$\frac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
6	$a^n u(n)$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
7	$e^{-na}u(n)$	$\frac{z}{(z-e^{-a})}$	$\lvert z\rvert > e^{-a}$
8	$na^n u(n)$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
9	$\sin(an)u(n)$	$\frac{z\sin(a)}{z^2-2z\cos(a)+1}$	$\lvert z\rvert > 1$
10	$\cos(an)u(n)$	$\frac{z[z-\cos(a)]}{z^2-2z\cos(a)+1}$	$\lvert z\rvert > 1$
11	$a^n\sin(bn)u(n)$	$\frac{[a\sin(b)]z}{z^2-[2a\cos(b)]z+a^2}$	$\lvert z\rvert > \lvert a\rvert$
12	$a^n\cos(bn)u(n)$	$\frac{z[z-a\cos(b)]}{z^2-[2a\cos(b)]z+a^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
13	$e^{-an}\sin(bn)u(n)$	$\frac{[e^{-a}\sin(b)]z}{z^2-[2e^{-a}\cos(b)]z+e^{-2a}}$	$\lvert z\rvert > e^{-a}$
14	$e^{-an}\cos(bn)u(n)$	$\frac{z[z-e^{-a}\cos(b)]}{z^2-[2e^{-a}\cos(b)]z+e^{-2a}}$	$\lvert z\rvert > e^{-a}$

$$\frac{Az}{z-P}+\frac{A^*z}{z-P^*}$$

$2|A||P|^n\cos(n\theta+\varphi)u(n)$ where P and A are complex constants defined by $P = |P|\angle\theta,\ A = |A|\angle\varphi$

Example 3

Problem:

Find the z-transform for each of the following sequences:

a. $x(n) = 10\sin(0.25\pi n)u(n)$ b. $x(n) = e^{-0.1n}\cos(0.25\pi n)u(n)$

Solution:

a. From line 9 in the Table: $X(z) = 10Z(\sin(0.2\pi n)u(n))$

$$= \frac{10\sin(0.25\pi)z}{z^2 - 2z\cos(0.25\pi) + 1} = \frac{7.07z}{z^2 - 1.414z + 1}$$

b. From line 14 in the Table: $X(z) = Z(e^{-0.1n}\cos(0.25\pi n)u(n)) = \frac{z(z - e^{-0.1}\cos(0.25\pi))}{z^2 - 2e^{-0.1}\cos(0.25\pi)z + e^{-0.2}}$

$$= \frac{z(z - 0.6397)}{z^2 - 1.2794z + 0.8187}$$

Z-Transform Properties (1)

Linearity: $$Z(ax_1(n) + bx_2(n)) = aZ(x_1(n)) + bZ(x_2(n))$$

$x_1(n)$ and $x_2(n)$ denote the sampled sequences, ***a*** and ***b*** are the arbitrary constants.

Example 4

Problem:

Find the z-transform of $x(n) = u(n) - (0.5)^n u(n)$

Solution:

Applying the linearity of the z-transform $X(z) = Z(x(n)) = Z(u(n)) - Z(0.5^n(n))$

Using z-transform Table

$$Z(u(n)) = \frac{z}{z-1}$$ Line 3

$$Z(0.5^n u(n)) = \frac{z}{z-0.5}$$ Line 6

Therefore, we get, $X(z) = \frac{z}{z-1} - \frac{z}{z-0.5}$

Z-Transform Properties (2)

Shift Theorem: $Z(x(n-m)) = z^{-m}X(z)$

$x(n) \xrightarrow{\text{Z-Transform}} X(z)$

Verification: $Z(x(n-m)) = \sum_{n=0}^{\infty} x(n-m)z^{-n}$

$$= x(-m)z^{-0} + \cdots + x(-1)z^{-(m-1)} + x(0)z^{-m} + x(1)z^{-m-1} + \ldots$$

Since $x(n)$ is assumed to be causal: $x(-m) = x(-m+1) = \cdots = x(-1) = 0$

Then we achieve, $Z(x(n-m)) = x(0)z^{-m} + x(1)z^{-m-1} + x(2)z^{-m-2} + \ldots$

Factoring z^{-m} from Equation we get,

$$Z(x(n-m)) = z^{-m}\left(x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots\right) = z^{-m}X(z)$$

Example 5

Problem:

Determine the z-transform of $y(n) = (0.5)^{(n-5)} \cdot u(n-5)$

where $u(n-5) = 1$ for $n \geq 5$ and $u(n-5) = 0$ for $n<5$.

Solution:

Using shift theorem,

$$Y(z) = Z\left[(0.5)^{n-5}u(n-5)\right] = z^{-5}Z\left[(0.5)^{n}u(n)\right]$$

Using z-transform table, line 6:

$$Y(z) = z^{-5} \cdot \frac{z}{z-0.5} = \frac{z^{-4}}{z-0.5}$$

Z-Transform Properties (3)

Convolution

In time domain, $x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{\infty} x_1(n-k)x_2(k)$ eq.(1)

In Z-transform domain, $X(z) = X_1(z)X_2(z)$ Here, $X(z) = Z(x(n)), X_1(z) = Z(x_1(n))$, and $X_2(z) = Z(x_2(n))$.

Verification:

$x(n)$ from $eq.(1)$

Taking the z-transform of eq.(1) $X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} x_1(n-k)x_2(k)z^{-n}$

$z^{-n} = z^{-k}z^{-(n-k)}$

$\Rightarrow X(z) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} x_2(k)z^{-k}x_1(n-k)z^{-(n-k)} \Rightarrow X(z) = \sum_{k=0}^{\infty} x_2(k)z^{-k} \sum_{n=0}^{\infty} x_1(n-k)z^{-(n-k)}$

let $m = n - k$:

$\Rightarrow X(z) = \sum_{k=0}^{\infty} x_2(k)z^{-k} \sum_{m=0}^{\infty} x_1(m)z^{-m} \Rightarrow X(z) = X_2(z)X_1(z) = X_1(z)X_2(z)$

Example 6

Problem:

Given the sequences,

$$x_1(n) = 3\delta(n) + 2\delta(n-1)$$

$$x_2(n) = 2\delta(n) - \delta(n-1)$$

Find the z-transform of the convolution.

Solution:

Applying the z-transform of the two sequences,

$$Z[x_1(n)] \Rightarrow X_1(z) = 3 + 2z^{-1}$$

$$Z[x_2(n)] \Rightarrow X_2(z) = 2 - z^{-1}$$

Therefore we get,

$$X(z) = X_1(z)X_2(z) = (3 + 2z^{-1})(2 - z^{-1})$$

$$= 6 + z^{-1} - 2z^{-2}$$

Inverse z-Transform: Examples

The inverse z-transform for the function $X(z)$ is defined as: $x(n) = Z^{-1}(X(z))$

Example 7 Find the inverse z-transform of $X(z) = 2 + \frac{4z}{z-1} - \frac{z}{z-0.5}$

Solution We get, $x(n) = 2Z^{-1}(1) + 4Z^{-1}\left(\frac{z}{z-1}\right) - Z^{-1}\left(\frac{z}{z-0.5}\right)$

Using table, $x(n) = 2\delta(n) + 4u(n) - (0.5)^n u(n)$

Example 8 Find the inverse z-transform of $X(z) = \frac{5z}{(z-1)^2} - \frac{2z}{(z-0.5)^2}$

Solution We get, $x(n) = Z^{-1}\left(\frac{5z}{(z-1)^2}\right) - Z^{-1}\left(\frac{2z}{(z-0.5)^2}\right) = 5Z^{-1}\left(\frac{z}{(z-1)^2}\right) - \frac{2}{0.5}Z^{-1}\left(\frac{0.5z}{(z-0.5)^2}\right)$

Using table, $x(n) = 5nu(n) - 4n(0.5)^n u(n)$

Inverse z-Transform: Examples

Example 9 Find the inverse z-transform of $X(z) = \dfrac{10z}{z^2 - z + 1}$

Solution Since, $X(z) = \dfrac{10z}{z^2 - z + 1} = \left(\dfrac{10}{\sin(a)}\right)\dfrac{\sin(a)z}{z^2 - 2z\cos(a) + 1}$ From line 9 in the Table

By coefficient matching, $-2\cos(a) = -1$

Hence, $\cos(a) = 0.5$, and $a = 60^0$. ⇨ $\sin(a) = \sin(60^0) = 0.866$

Therefore, $x(n) = \dfrac{10}{\sin(a)} Z^{-1}\left(\dfrac{\sin(a)z}{z^2 - 2z\cos(a) + 1}\right) = \dfrac{10}{0.866}\sin(n\cdot 60^0) = 11.547\sin(n\cdot 60^0)$

Example 10 Find the inverse z-transform of $X(z) = \dfrac{z^{-4}}{z-1} + z^{-6} + \dfrac{z^{-3}}{z+0.5}$

Solution $x(n) = Z^{-1}\left(z^{-5}\dfrac{z}{z-1}\right) + Z^{-1}\left(z^{-6}\cdot 1\right) + Z^{-1}\left(z^{-4}\dfrac{z}{z+0.5}\right)$

Using Table ⇨ $x(n) = u(n-5) + \delta(n-6) + (-0.5)^{n-4}u(n-4)$

Inverse z-Transform: Using Partial Fraction

Table Partial Fraction(s) and Formulas for Constant(s)

Partial fraction with the first-order real pole: $\frac{R}{z-p}$ $\quad R = (z-p)\frac{X(z)}{z}\Big|_{z=p}$

Partial fraction with the first-order complex poles: $\frac{Az}{(z-P)} + \frac{A^* z}{(z-P^*)}$ $\quad A = (z-P)\frac{X(z)}{z}\Big|_{z=P}$

P^* = complex conjugate of P A^* = complex conjugate of A

Partial fraction with mth-order real poles:

$$\frac{R_m}{(z-p)} + \frac{R_{m-1}}{(z-p)^2} + \cdots + \frac{R_1}{(z-p)^m} \quad R_k = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left((z-p)^m \frac{X(z)}{z}\right)\Big|_{z=p}$$

k from m to 1

Inverse z-Transform: Using Partial Fraction

Example 11

Problem: Find the inverse z-transform of $X(z) = \dfrac{1}{(1 - z^{-1})(1 - 0.5z^{-1})}$

Solution: First eliminate the negative power of z.

$$X(z) = \frac{z^2}{z^2(1 - z^{-1})(1 - 0.5z^{-1})} = \frac{z^2}{(z-1)(z-0.5)}$$

Dividing both sides by z, $\dfrac{X(z)}{z} = \dfrac{z}{(z-1)(z-0.5)} = \dfrac{A}{(z-1)} + \dfrac{B}{(z-0.5)}$

Finding the constants:

$$A = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{z}{(z-0.5)}\bigg|_{z=1} = 2$$

$$B = (z-0.5)\frac{X(z)}{z}\bigg|_{z=0.5} = \frac{z}{(z-1)}\bigg|_{z=0.5} = -1$$

$$\frac{X(z)}{z} = \frac{2}{(z-1)} + \frac{-1}{(z-0.5)}$$

multiplying by z

$$X(z) = \frac{2z}{(z-1)} + \frac{-z}{(z-0.5)}$$

Therefore, inverse z-transform is: $x(n) = 2u(n) - (0.5)^n u(n)$

Inverse z-Transform: Using Partial Fraction

Problem: Find $y(n)$ if $$Y(z) = \frac{z^2(z+1)}{(z-1)(z^2-z+0.5)}$$ **Example 12**

Solution:

Dividing Y(z) by z, $$\frac{Y(z)}{z} = \frac{z(z+1)}{(z-1)(z^2-z+0.5)}$$

Applying the partial fraction expansion, ➡ $$\frac{Y(z)}{z} = \frac{B}{z-1} + \frac{A}{(z-0.5-j0.5)} + \frac{A^*}{(z-0.5+j0.5)}$$

We first find B: $$B = (z-1)\frac{Y(z)}{z}\Big|_{z=1} = \frac{z(z+1)}{(z^2-z+0.5)}\Big|_{z=1} = \frac{1\times(1+1)}{(1^2-1+0.5)} = 4$$

Next find A: $$A = (z-0.5-j0.5)\frac{Y(z)}{z}\Big|_{z=0.5+j0.5} = \frac{z(z+1)}{(z-1)(z-0.5+j0.5)}\Big|_{z=0.5+j0.5}$$

$$= \frac{(0.5+j0.5)(0.5+j0.5+1)}{(0.5+j0.5-1)(0.5+j0.5-0.5+j0.5)} = \frac{(0.5+j0.5)(1.5+j0.5)}{(-0.5+j0.5)j}$$

Example 12 -contd.

Using the polar form, $A = \frac{(0.707\angle 45^\circ)(1.58114\angle 18.43^\circ)}{(0.707\angle 135^\circ)(1\angle 90^\circ)} = 1.58114\angle -161.57^\circ$

$$A^* = \bar{A} = 1.58114\angle 161.57^\circ.$$

$$P = 0.5 + 0.5j = |P|\angle\theta = 0.707\angle 45^0 \text{ and } P^* = |P|\angle -\theta = 0.707\angle -45^0$$

Now we have: $Y(z) = \frac{4z}{z-1} + \frac{Az}{(z-P)} + \frac{A^* z}{(z-P^*)}$

Therefore, the inverse z-transform is:

from Line 15 in Table

$$\begin{aligned} y(n) &= 4u(n) + 2|A|(|P|)^n \cos(n\theta + \varphi)u(n) \\ &= 4u(n) + 3.1623(0.7071)^n \cos(45^0 n - 161.57^0)u(n) \end{aligned}$$

Inverse z-Transform: Using Partial Fraction

Problem: Find $x(n)$ if $X(z) = \dfrac{z^2}{(z-1)(z-0.5)^2}$

Example 13

Solution:

Dividing both sides by z: $\dfrac{X(z)}{z} = \dfrac{z}{(z-1)(z-0.5)^2} = \dfrac{A}{z-1} + \dfrac{B}{z-0.5} + \dfrac{C}{(z-0.5)^2}$

Where, $A = (z-1)\dfrac{X(z)}{z}\bigg|_{z=1} = \dfrac{z}{(z-0.5)^2}\bigg|_{z=1} = 4$

Using the formulas for m[th]-order,

$$\frac{R_m}{(z-p)} + \frac{R_{m-1}}{(z-p)^2} + \cdots + \frac{R_1}{(z-p)^m}$$

$$R_k = \frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left((z-p)^m \frac{X(z)}{z}\right)\bigg|_{z=p}$$

$$B = R_2 = \frac{1}{(2-1)!}\frac{d}{dz}\left\{(z-0.5)^2\frac{X(z)}{z}\right\}_{z=0.5}$$

m=2, p=0.5

$$= \frac{d}{dz}\left(\frac{z}{z-1}\right)\bigg|_{z=0.5} = \frac{-1}{(z-1)^2}\bigg|_{z=0.5} = -4$$

Example 13 –contd.

$$C = R_1 = \frac{1}{(1-1)!}\frac{d^0}{dz^0}\left\{(z-0.5)^2\frac{X(z)}{z}\right\}_{z=0.5}$$

$$= \frac{z}{z-1}\bigg|_{z=0.5} = -1$$

Then, $$X(z) = \frac{4z}{z-1} + \frac{-4z}{z-0.5} + \frac{-1z}{(z-0.5)^2}$$

From Table,

$$Z^{-1}\left\{\frac{z}{z-1}\right\} = u(n)$$

$$Z^{-1}\left\{\frac{z}{z-0.5}\right\} = (0.5)^n u(n)$$

$$Z^{-1}\left\{\frac{z}{(z-0.5)^2}\right\} = 2n(0.5)^n u(n)$$

Finally we get, $$x(n) = 4u(n) - 4(0.5)^n u(n) - 2n(0.5)^n u(n)$$

Partial Fraction Expansion Using MATLAB

Problem: Find the partial expansion of $X(z) = \dfrac{1}{(1-z^{-1})(1-0.5z^{-1})}$

Example 14

Solution:

The denominator polynomial can be found using MATLAB:

```
» conv([1 −1],[1 −0.5])
D =
1.0000 −1.5000 0.5000
```

$(c_0 + c_1 z^{-1} + c_2 z^{-2})$ ⇓ $[c_0\ \ c_1\ \ c_2]$

Therefore, $X(z) = \dfrac{1}{(1-z^{-1})(1-0.5z^{-1})} = \dfrac{1}{1-1.5z^{-1}+0.5^{-2}} = \dfrac{z^2}{z^2-1.5z+0.5}$

and $\dfrac{X(z)}{z} = \dfrac{z}{z^2-1.5z+0.5}$

$$\frac{B(z)}{A(z)} = \frac{r_1}{z-p_1} + \frac{r_2}{z-p_2} + \cdots + k_0 + k_1 z^{-1} + \cdots$$

$[\mathbf{R}, \mathbf{P}, \mathbf{K}] = \mathbf{residue}(\mathbf{B}, \mathbf{A})$

MATLAB performs the partial fraction expansion

```
» [R,P,K]=residue([1 0],[1 −1.5 0.5])

R =  2        P = 1.0000      K = []
    −1            0.5000
```

residues — poles — direct term

The solution is: $X(z) = \dfrac{2z}{z-1} - \dfrac{z}{z-0.5}$

Partial Fraction Expansion Using MATLAB

Problem: Find the partial expansion of $Y(z) = \frac{z^2(z+1)}{(z-1)(z^2-z+0.5)}$ **Example 15**

Solution:

```
» N=conv([1 0 0],[1 1])
N =
1 1 0 0
» D=conv([1 −1], [1 −1 0.5])
D =
1.0000 −2.0000 1.5000 −0.5000
```

$$Y(z) = \frac{z^2(z+1)}{(z-1)(z^2-z+0.5)} = \frac{z^3+z^2}{z^3-2z^2+1.5z-0.5}$$

```
» [R,P,K]=residue([1 1 0],[1 −2 1.5 −0.5])
R =
4.0000
−1.5000 − 0.5000i
−1.5000 + 0.5000i
P =
1.0000
0.5000 + 0.5000i
0.5000 − 0.5000i
K =
[]
```

$\frac{Y(z)}{z}$

$$Y(z) = \frac{Bz}{z-p_1} + \frac{Az}{z-p} + \frac{A^* z}{z-p^*}$$

where $B = 4,\ p_1 = 1,$

$A = -1.5 - 0.5j, \quad p = 0.5 + 0.5j,$

$A^* = -1.5 + 0.5j, \quad$ and $p = 0.5 - 0.5j$

Partial Fraction Expansion Using MATLAB

Problem: Find the partial expansion of $X(z) = \dfrac{z^2}{(z-1)(z-0.5)^2}$ **Example 16**

Solution:

```
» D=conv(conv([1 -1], [1 -0.5]),[1 -0.5])
D =
1.0000 -2.0000 1.2500 -0.2500
```

Then $X(z) = \dfrac{z^2}{(z-1)(z-0.5)^2} = \dfrac{z^2}{z^3 - 2z^2 + 1.25z - 0.25}$ ⇒ $\dfrac{X(z)}{z} = \dfrac{z}{z^3 - 2z^2 + 1.25z - 0.25}$

```
» [R,P,K]=residue([1 0],[1 -2 1.25 -0.25])
R =
4.0000
-4.0000
-1.0000
P =
1.0000
0.5000
0.5000
K =
[]
```

⇒ $X(z) = \dfrac{4z}{z-1} - \dfrac{4z}{z-0.5} - \dfrac{z}{(z-0.5)^2}$

Difference Equation Using Z-Transform

- The procedure to solve difference equation using Z-Transform:

  1. Apply the z-transform to the difference equation.
  2. Substitute the initial conditions.
  3. Solve for the difference equation in the z-transform domain.
  4. Find the solution in the time domain by applying the inverse z-transform.

Example 17

Problem: Solve the difference equation when the initial condition is $y(-1) = 1$.

$$y(n) - 0.5y(n-1) = 5(0.2)^n u(n)$$

Solution:

We have

$$Z(y(n-1)) = \sum_{n=0}^{\infty} y(n-1)z^{-n}$$
$$= y(-1) + y(0)z^{-1} + y(1)z^{-2} + \cdots$$
$$= y(-1) + z^{-1}(y(0) + y(1)z^{-1} + y(2)z^{-2} + \cdots) \Rightarrow Z(y(n-1)) = y(-1) + z^{-1}Y(z)$$

Taking z-transform on both sides: $Y(z) - 0.5(y(-1) + z^{-1}Y(z)) = 5Z(0.2^n u(n))$

Substituting the initial condition and z-transform on right hand side using Table:

$$Y(z) - 0.5(1 + z^{-1}Y(z)) = 5z/(z-0.2)$$

Arranging Y(z) on left hand side: $Y(z) - 0.5z^{-1}Y(z) = 0.5 + 5z/(z-0.2)$

$$\Rightarrow Y(z) = \frac{(5.5z-0.1)}{(1-0.5z^{-1})(z-0.2)} = \frac{z(5.5z-0.1)}{(z-0.5)(z-0.2)} \Rightarrow \frac{Y(z)}{z} = \frac{5.5z-0.1}{(z-0.5)(z-0.2)} = \frac{A}{z-0.5} + \frac{B}{z-0.2}$$

Example 17 –contd.

Solving for A and B:

$$A = (z-0.5)\frac{Y(z)}{z}\bigg|_{z=0.5} = \frac{5.5z-0.1}{z-0.2}\bigg|_{z=0.5} = \frac{5.5\times 0.5-0.1}{0.5-0.2} = 8.8333$$

$$B = (z-0.2)\frac{Y(z)}{z}\bigg|_{z=0.2} = \frac{5.5z-0.1}{z-0.5}\bigg|_{z=0.2} = \frac{5.5\times 0.2-0.1}{0.2-0.5} = -3.3333$$

Therefore, $$Y(z) = \frac{8.8333z}{(z-0.5)} + \frac{-3.3333z}{(z-0.2)}$$

Taking inverse z-transform, we get the solution:

$$\boxed{y(n) = 8.3333(0.5)^n u(n) - 3.3333(0.2)^n u(n)}$$

Example 18

Problem:

A DSP system is described by the following differential equation with zero initial condition:

$$y(n) + 0.1y(n-1) - 0.2y(n-2) = x(n) + x(n-1)$$

a. Determine the impulse response $y(n)$ due to the impulse sequence $x(n) = \delta(n)$.

b. Determine the system response $y(n)$ due to the unit step function excitation, where $u(n) = 1 \; for \; n \geq 0$

Solution:

a. Applying the z-transform on both sides: $Y(z) + 0.1Y(z)z^{-1} - 0.2Y(z)z^{-2} = X(z) + X(z)z^{-1}$

Applying $X(z) = Z(\delta(n)) = 1$ On right side

$$\Rightarrow \quad Y(z)(1 + 0.1z^{-1} - 0.2z^{-2}) = 1(1 + z^{-1})$$

$$\Rightarrow \quad Y(z) = \frac{1 + z^{-1}}{1 + 0.1z^{-1} - 0.2z^{-2}}$$

Example 18 -contd.

We multiply the numerator and denominator by z^2

$$Y(z) = \frac{z^2 + z}{z^2 + 0.1z - 0.2} = \frac{z(z+1)}{(z-0.4)(z+0.5)}$$

Using the partial fraction expansion

$$\frac{Y(z)}{z} = \frac{z+1}{(z-0.4)(z+0.5)} = \frac{A}{z-0.4} + \frac{B}{z+0.5}$$

Solving for A and B:

$$A = (z-0.4)\frac{Y(z)}{z}\bigg|_{z=0.4} = \frac{z+1}{z+0.5}\bigg|_{z=0.4} = \frac{0.4+1}{0.4+0.5} = 1.5556$$

$$B = (z+0.5)\frac{Y(z)}{z}\bigg|_{z=-0.5} = \frac{z+1}{z-0.4}\bigg|_{z=-0.5} = \frac{-0.5+1}{-0.5-0.4} = -0.5556$$

Therefore,

$$Y(z) = \frac{1.5556z}{(z-0.4)} + \frac{-0.5556z}{(z+0.5)}$$

Hence the impulse response:

$$\boxed{y(n) = 1.5556(0.4)^n u(n) - 0.5556(-0.5)^n u(n)}$$

Example 18 –contd.

b. The input is step unit function: $x(n) = u(n)$

Corresponding z-transform: $X(z) = \dfrac{z}{z-1}$

Notice that $Y(z) + 0.1\,Y(z)z^{-1} - 0.2\,Y(z)z^{-2} = X(z) + X(z)z^{-1}$ [Slide 27]

Then the z-transform of the output sequence $y(n)$,

$$Y(z) = \left(\frac{z}{z-1}\right)\left(\frac{1+z^{-1}}{1+0.1z^{-1}-0.2z^{-2}}\right) = \frac{z^2(z+1)}{(z-1)(z-0.4)(z+0.5)}$$

Using the partial fraction expansion $Y(z) = \dfrac{2.2222z}{z-1} + \dfrac{-1.0370z}{z-0.4} + \dfrac{-0.1852z}{z+0.5}$

and the system response is found by using Table:

$$\boxed{y(n) = 2.2222u(n) - 1.0370(0.4)^n u(n) - 0.1852(-0.5)^n u(n)}$$