Laplace Transform

Pierre-Simon Laplace: French Scholar (1749 – 1827)

Fourier transforms involve purely imaginary complex exponentials: $e^{st}, \ s = j\omega$

Eigenfunction property applies to any complex number s.

Laplace transforms involve complex exponentials: $e^{st}, \ s = \sigma + j\omega$

Laplace Transform: $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$ ⇒ $x(t) \overset{L}{\leftrightarrow} X(s)$

$$X(s)\Big|_{s=j\omega} = F\{x(t)\}$$

Laplace Trans. & Fourier Trans.

$$X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t) e^{-(\sigma + j\omega)t} dt$$

$$= \int_{-\infty}^{\infty} \left[x(t) e^{-\sigma t} \right] e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x'(t) e^{-j\omega t} dt$$

$$= F\{x'(t)\}$$

The Laplace transform is the Fourier transform of the transformed signal $x'(t) = x(t)e^{-\sigma t}$.

Example 9.1

- Consider the signal $x(t) = e^{-at}u(t)$
- The Fourier transform $X(j\omega)$ converges for $a>0$:

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt = \frac{1}{j\omega + a}, \quad a > 0$$

- The Laplace transform is:

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+a)t}dt$$

$$= \int_{0}^{\infty} e^{-(\sigma+a)t}e^{-j\omega t}dt$$

- which is the Fourier Transform of $e^{-(\sigma+a)t}u(t)$

$$X(\sigma + j\omega) = \frac{1}{(\sigma + a) + j\omega}, \quad \sigma + a > 0$$

- Or

$$e^{-at}u(t) \overset{L}{\leftrightarrow} X(s) = \frac{1}{s+a}, \quad \mathrm{Re}\{s\} > -a$$

- If a is negative or zero, the Laplace Transform still exists

Example 9.2

- Consider the signal $x(t) = -e^{-at}u(-t)$
- The Laplace transform is:

$$X(s) = -\int_{-\infty}^{\infty} e^{-at}e^{-st}u(-t)dt$$

$$= -\int_{-\infty}^{0} e^{-(s+a)t}dt$$

$$= \frac{1}{s+a}$$

- Convergence requires that Re{s+a}<0 or Re{s}<-a.
- The Laplace transform expression is identical to Example 9.1 (similar but different signals), however the regions of convergence of s are mutually exclusive (non-intersecting).
- For a Laplace transform, we need both the expression and the Region Of Convergence (ROC).

Fourier transform Convergence

- The Fourier transform does not exist for a fairly wide class of signals, such as the response of an unstable, first order system, the Fourier transform does not exist/converge

- E.g. $x(t) = e^{at}u(t), \quad a>0$

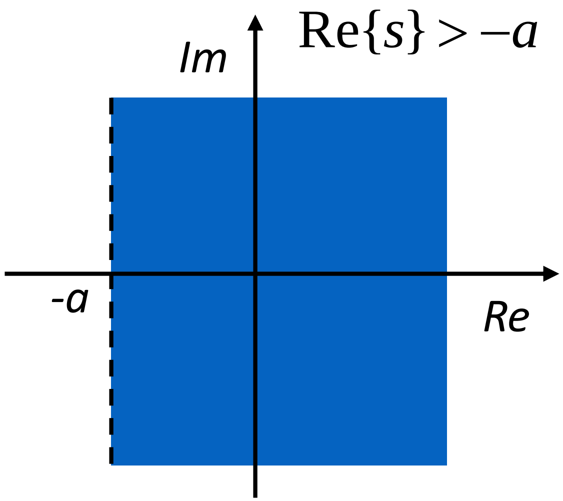
$$X(j\omega) = \int_0^{\infty} e^{at} e^{-j\omega t} dt$$

does not exist (is infinite) because the **signal's energy is infinite**

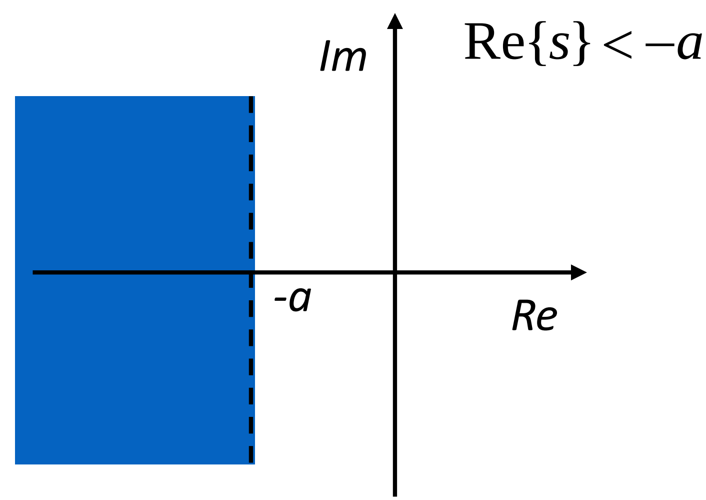
- As the **Dirichlet convergence** conditions say, the Fourier transform exists for most signals with **finite energy**

Region of Convergence (ROC)

- The ROC of the Laplace transform is the set of values for s (=$\sigma+j\omega$) for which the Fourier transform of $x(t)e^{-\sigma t}$ converges (exists).



Example 9.1



Example 9.2

- The shaded regions denote the ROC for the Laplace transform

Example 9.3

- Consider a signal that is the sum of two real exponentials:

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$$

- The Laplace transform is then:

$$X(s) = \int_{-\infty}^{\infty} \left[3e^{-2t}u(t) - 2e^{-t}u(t)\right] e^{-st}\,dt$$

$$= 3\int_{-\infty}^{\infty} e^{-2t}u(t)e^{-st}\,dt - 2\int_{-\infty}^{\infty} e^{-t}u(t)e^{-st}\,dt$$

- Using Example 1, each expression can be evaluated as:

$$X(s) = \frac{3}{s+2} - \frac{2}{s+1}$$

- The ROC associated with these terms are Re{s}>-1 and Re{s}>-2. Therefore, both will converge for Re{s}>-1, and the Laplace transform:

$$X(s) = \frac{s-1}{s^2+3s+2}$$

Poles and Zeros

$$X(s) = \frac{N(s)}{D(s)}$$

Zeros: roots of N(s) → Makes X(s) zero

Poles: roots of D(s) → Makes X(s) infinite

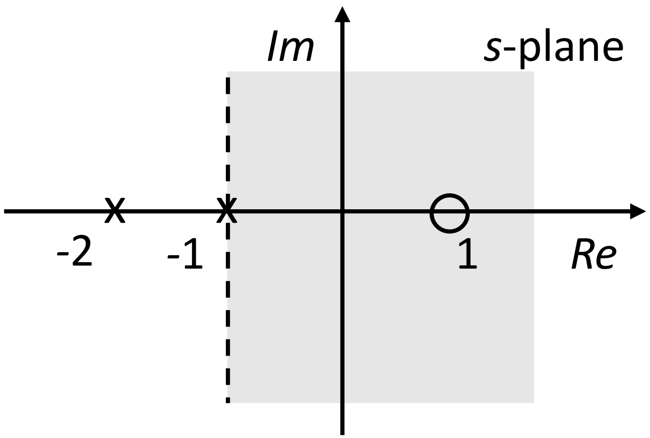
Zero calculation:

s – 1 = 0 → s = 1

Pole calculation:

(s+2)(s+1) = 0 → s = -2, -1

Pole-zero plot



○ – Zero

x – Pole

Laplace transform X(s) is **rational** if it is a ratio of polynomials in the complex variable s.

$$X(s) = \frac{s}{s^2 - 2s + 1}$$ Rational

$$X(s) = \frac{e^t}{s^2 - 2s + 1}$$ Not Rational

Poles and Zeros at Infinity

If the denominator polynomial order is greater than the numerator polynomial order, there are zeros at infinity. The number of zeros at infinity is equal to the number of polynomial order difference.

$$X(s) = \frac{s-1}{s^2+3s+2}$$

⇨ One zero at infinity

If the numerator polynomial order is greater than the denominator polynomial order, there are poles at infinity. The number of poles at infinity is equal to the number of polynomial order difference.

$$X(s) = \frac{(s-1)^2}{(s+1)(s-2)}$$

⇨ Neither poles or zeros are at infinity

Example 9.4

$$x(t) = e^{-2t}u(t) + e^{-t}(\cos 3t)u(t)$$

$$x(t) = \left[e^{-2t} + e^{-t}\frac{e^{3jt} + e^{-3jt}}{2} \right] u(t) = \left[e^{-2t} + \frac{e^{-(1-3j)t} + e^{-(1+3j)t}}{2} \right] u(t)$$

Laplace Transform:

$$X(s) = \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+(1-3j)} + \frac{1}{2}\frac{1}{s+(1+3j)}$$

Re{s} > -2   Re{s} > -1   Re{s} > -1

After combining → Re{s} > -1

$$X(s) = \frac{1}{s+2} + \frac{1}{2}\frac{1}{s+(1-3j)} + \frac{1}{2}\frac{1}{s+(1+3j)}$$

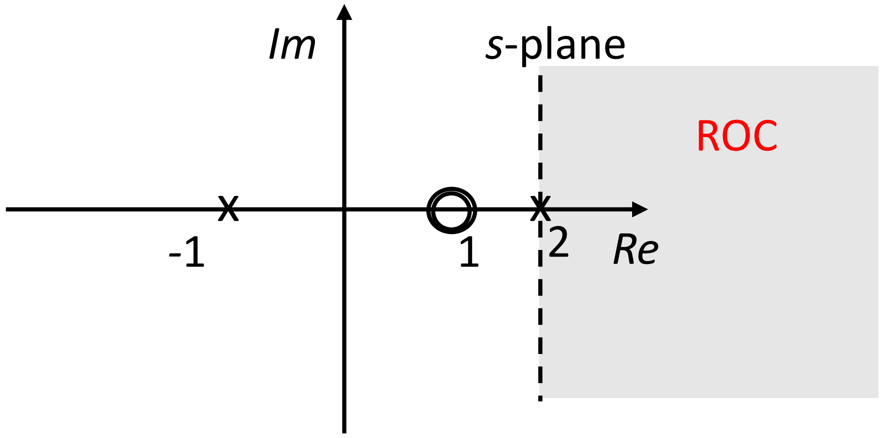
Example 9.5

$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t)$$

$L\{\delta(t)\} = 1,$ ROC = entire s-plane

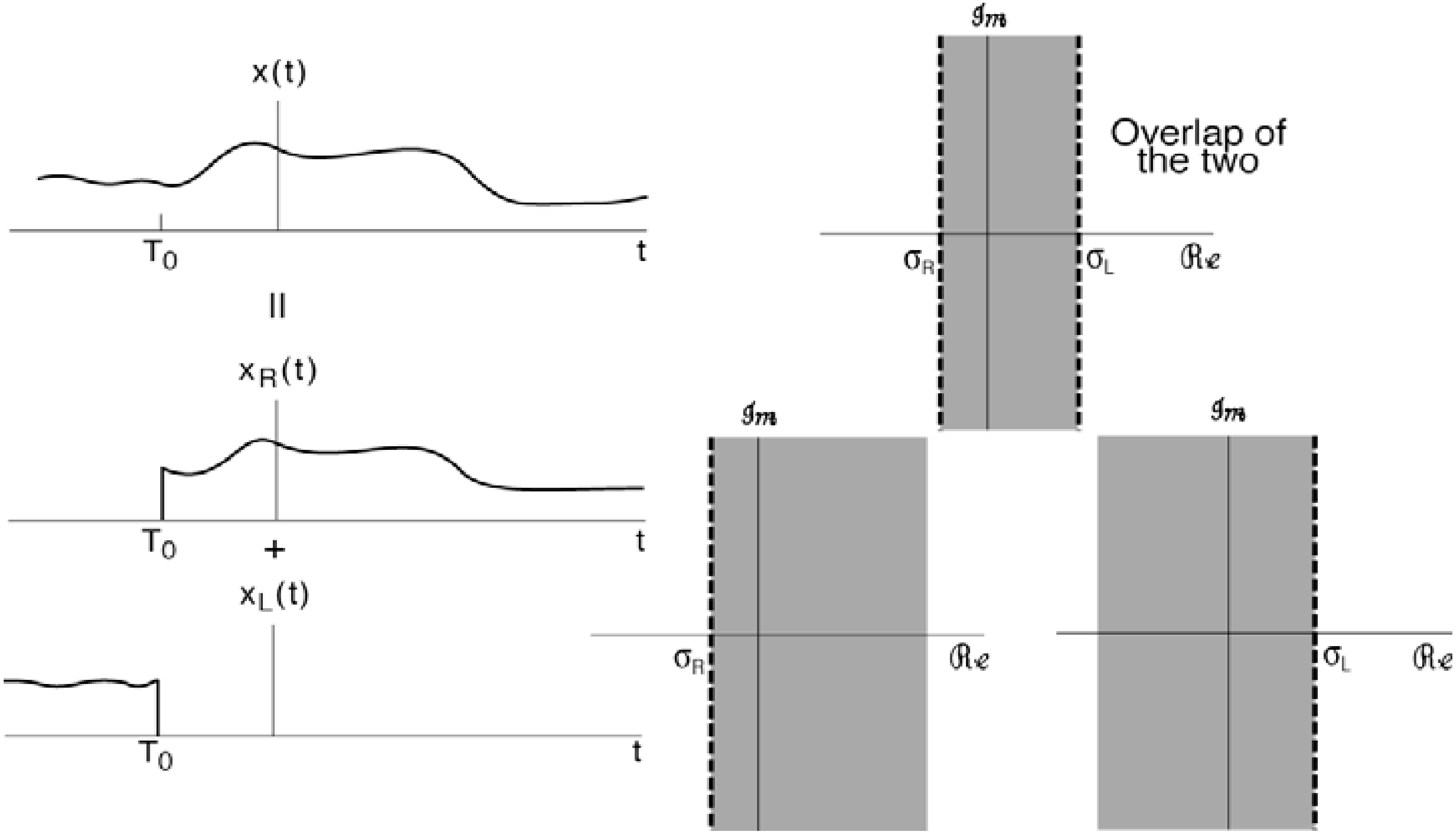
$$X(s) = 1 - \frac{4}{3}\frac{1}{s+1} + \frac{1}{3}\frac{1}{s-2} = \frac{(s-1)^2}{(s+1)(s-2)}$$

ROC: Re{s} > 2



Two-sided Signals

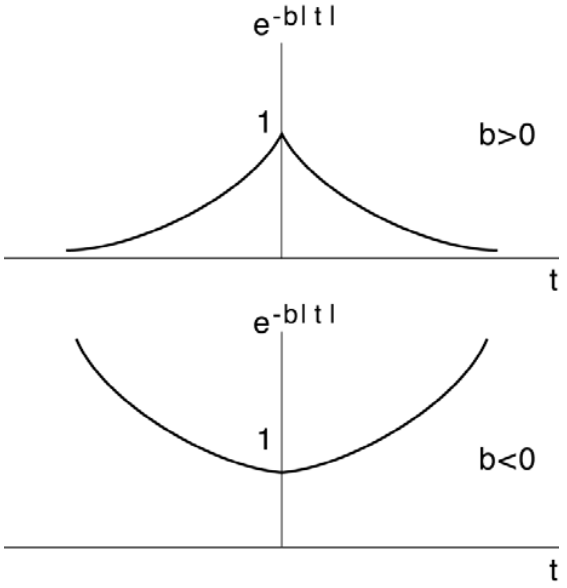
If x(t) is two-sided, then the ROC consists of the intersection of a left-sided and right-sided version of x(t), which is a strip in the *s*-plane:



https://www.isip.piconepress.com/courses/temple/ece_3512/lectures/

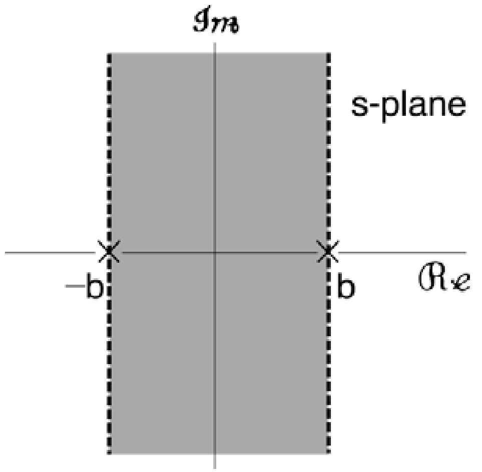
Example 9.7

$$x(t) = e^{-b|t|} = e^{bt}u(-t) + e^{-bt}u(t)$$



$$X(s) = -\frac{1}{s-b} + \frac{1}{s+b} \quad \begin{matrix} 1^{st}\ term: \mathrm{Re}\{s\} < b \\ 2^{nd}\ term: \mathrm{Re}\{s\} > -b \end{matrix}$$

$$= \frac{-2b}{s^2 - b^2} \quad (b > 0) \text{ with ROC below}$$

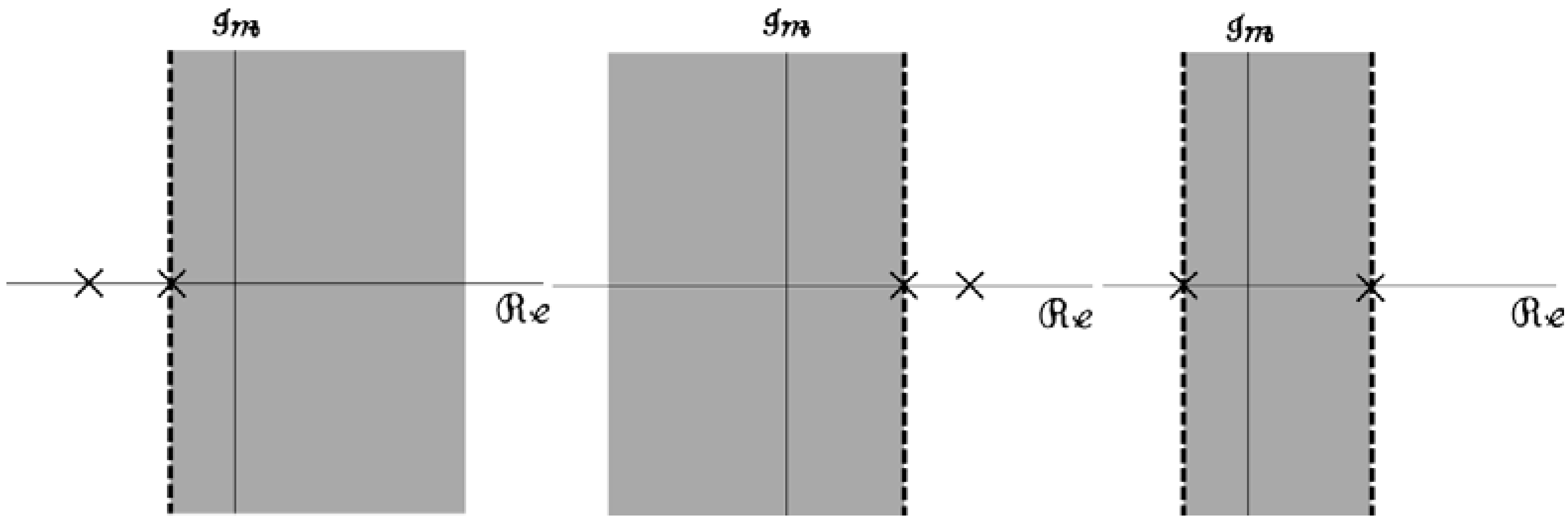


- If b < 0, the Laplace transform does not exist.
- Hence, the ROC plays an integral role in the Laplace transform.

ROC does not include poles

- Since the ROC cannot include poles, the ROC is bounded by the poles for a rational transform.

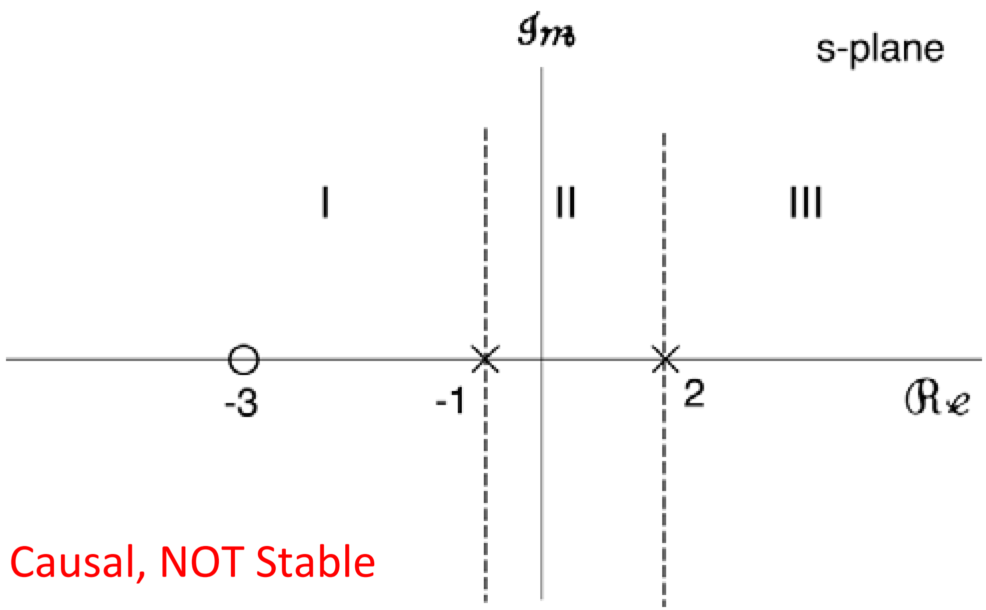
- If $x(t)$ is right-sided, the ROC begins to the right of the rightmost pole. If $x(t)$ is left-sided, the ROC begins to the left of the leftmost pole. If $x(t)$ is double-sided, the ROC will be the intersection of these two regions.



- If the ROC includes the $j\omega$-axis, then the Fourier transform of $x(t)$ exists. Hence, the Fourier transform can be considered to be the evaluation of the Laplace transform along the $j\omega$-axis.

Inverse Laplace transform

- Consider the Laplace transform: $X(s) = \dfrac{(s+3)}{(s+1)(s-2)}$
- Can we uniquely determine the original signal, $x(t)$?
- There are three possible ROCs:



- ROC III: only if $x(t)$ is right-sided. Causal, NOT Stable
- ROC I: only if $x(t)$ is left-sided. NOT causal, NOT stable
- ROC II: only if $x(t)$ has a Fourier transform. STABLE

Find x(t) for different ROCs

$$X(s) = \frac{(s+3)}{(s+1)(s-2)}$$

$$\frac{s+3}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} \Rightarrow s+3 = A(s-2) + B(s+1)$$

$$\Rightarrow s+3 = (A+B)s - 2A + B$$

$$\Rightarrow A+B = 1; -2A+B = 3$$

$$\Rightarrow A = -2/3, B = 5/3$$

$$X(s) = \frac{(s+3)}{(s+1)(s-2)} = -\frac{2}{3}\frac{1}{s+1} + \frac{5}{3}\frac{1}{s-2}$$

ROC III → Causal, not stable, x(t) right sided $x(t) = -\frac{2}{3}e^{-t}u(t) + \frac{5}{3}e^{2t}u(t)$

ROC I → Not causal, not stable, x(t) left sided $x(t) = +\frac{2}{3}e^{-t}u(-t) - \frac{5}{3}e^{2t}u(-t)$

ROC II → Not causal, stable, x(t) two sided $x(t) = -\frac{2}{3}e^{-t}u(t) - \frac{5}{3}e^{2t}u(-t)$

Example 9.9

Find inverse Laplace transform of $X(s)=\frac{1}{(s+1)(s+2)}$; $\text{Re}\{s\}>-1$

$$X(s)=\frac{1}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2}\Rightarrow A=1, B=-1\Rightarrow X(s)=\frac{1}{s+1}-\frac{1}{s+2}$$

$$x(t)=\left[e^{-t}-e^{-2t}\right]u(t)$$

Find inverse Laplace transform of $X(s)=\frac{1}{(s+1)(s+2)}$; $\text{Re}\{s\}<-2$

$$X(s)=\frac{1}{(s+1)(s+2)}=\frac{A}{s+1}+\frac{B}{s+2}\Rightarrow A=1, B=-1\Rightarrow X(s)=\frac{1}{s+1}-\frac{1}{s+2}$$

$$x(t)=\left[-e^{-t}+e^{-2t}\right]u(-t)$$

Linearity and Shifting of Laplace Transform

Linear

IF $x_1(t) \overset{L}{\leftrightarrow} X_1(s)$ ROC=R_1

AND $x_2(t) \overset{L}{\leftrightarrow} X_2(s)$ ROC=R_2

THEN $ax_1(t) + bx_2(t) \overset{L}{\leftrightarrow} aX_1(s) + bX_2(s)$ ROC= $R_1 \cap R_2$

Time shifting

IF $x(t) \overset{L}{\leftrightarrow} X(s)$ ROC=R

THEN $x(t - t_0) \overset{L}{\leftrightarrow} e^{-st_0} X(s)$ ROC=R

Example: Linearity and Shifting

Consider the following signal, which is a linear sum of two time-shifted sinusoids.

$$x(t) = 2x_1(t-2.5) - 0.5x_1(t-4)$$
$$x_1(t) = \sin(\omega_0 t)u(t)$$

Laplace transform of $x_1(t)$: $X_1(s) = \frac{\omega_0}{s^2 + \omega_0{}^2}; \quad \mathrm{Re}\{s\} > 0$

Using linearity and time-shifting properties of Laplace transform, we get:

$$X(s) = \left(2e^{-2.5s} - 0.5e^{-4s}\right)\frac{\omega_0}{s^2 + \omega_0^2} \qquad \mathrm{Re}\{s\} > 0$$

Convolution

- The Laplace transform also has the multiplication property, i.e.

$$x(t) \overset{L}{\leftrightarrow} X(s) \qquad \text{ROC}=R_1$$

$$h(t) \overset{L}{\leftrightarrow} H(s) \qquad \text{ROC}=R_2$$

$$x(t) * h(t) \overset{L}{\leftrightarrow} X(s)H(s) \qquad \text{ROC containing } R_1 \cap R_2$$

Convolution in time-domain becomes multiplication in Laplace domain.

- Note that pole-zero cancellation may occur between $H(s)$ and $X(s)$ which extends the ROC

$$X(s) = \frac{s+1}{s+2} \qquad \Re\{s\} > -2$$

$$H(s) = \frac{s+2}{s+1} \qquad \Re\{s\} > -1$$

$$X(s)H(s) = 1 \qquad -\infty < \Re\{s\} < \infty$$

Differentiation in Time-Domain

$$x(t) \overset{L}{\leftrightarrow} X(s) \qquad \text{ROC}=R$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}ds$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} sX(s)e^{st}ds$$

$$\frac{dx(t)}{dt} \overset{L}{\leftrightarrow} sX(s) \qquad \text{ROC containing } R$$

Problem 9.21

(a) Determine the Laplace transform and ROC and pole-zero plot:

$$x(t) = e^{-2t}u(t) + e^{-3t}u(t)$$

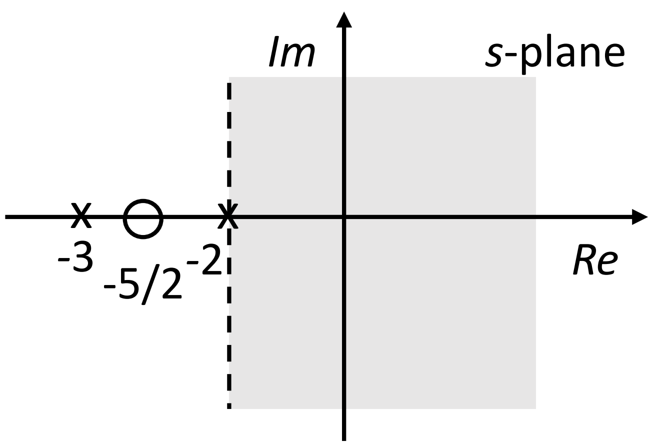
Zero: s = -5/2

$$X(s) = \frac{1}{s+2} + \frac{1}{s+3} = \frac{2s+5}{(s+2)(s+3)}$$

Poles: s = -2, -3

ROC: Re{s} > -2

Because of u(t), ROC is on the right-side of s-plane



CAUSAL: because X(s) is rational and ROC is on the right-side of the right-most pole

STABLE: because ROC contains imaginary axis (jw)

Problem 9.21

(b) Determine the Laplace transform and ROC and pole-zero plot:

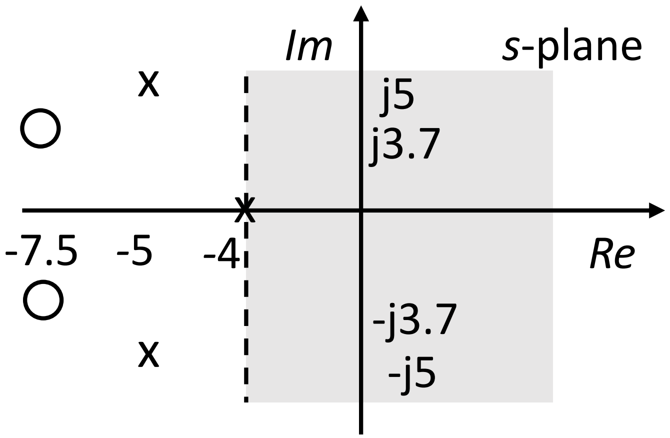
$$x(t) = e^{-4t}u(t) + e^{-5t}(\sin 5t)u(t)$$

$$X(s) = \frac{1}{s+4} + \frac{5}{(s+5)^2 + 5^2} = \frac{1}{(s+4)} + \frac{5}{s^2 + 10s + 50} = \frac{s^2 + 15s + 70}{(s+4)\left(s^2 + 10s + 50\right)}$$

Poles: solving s + 4 = 0 → s = -4; and $s^2+10s+50 = 0$ → $s = -5 \pm j\,5$

Zeros: solving $s^2+15s+70 = 0$ → $s = -7.5 \pm j\,3.7$

ROC: Re{s} > -4



CAUSAL
STABLE

Problem 9.21

(c) Determine the Laplace transform and ROC and pole-zero plot:

$$x(t) = e^{2t}u(-t) + e^{3t}u(-t)$$

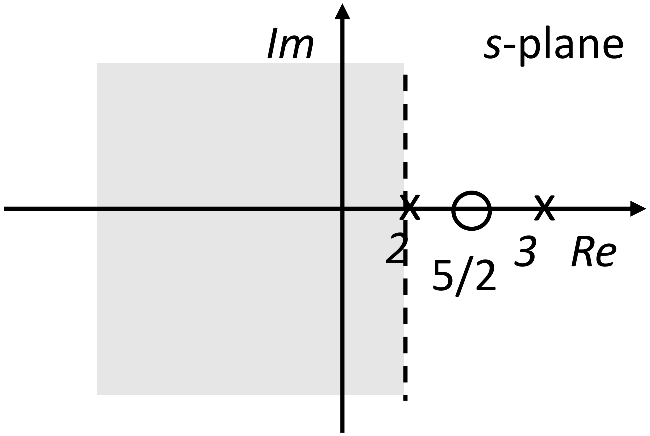
Zero: s = 5/2

$$X(s) = -\frac{1}{s-2} - \frac{1}{s-3} = \frac{5-2s}{(s-2)(s-3)}$$

Poles: s = 2, 3

ROC: Re{s} < 2

Because of u(-t), ROC is on the left-side of s-plane



NOT CAUSAL;
STABLE

Problem 9.22

Find inverse Laplace transform of $X(s) = \frac{1}{s^2+9}$

$$X(s) = \frac{1}{s^2+9} = \frac{1}{s^2+3^2} = \frac{1}{3}\frac{3}{s^2+3^2}$$

For ROC: Re{s} > 0,	$x(t) = \frac{1}{3}\sin(3t)u(t)$
For ROC: Re{s} < 0,	$x(t) = -\frac{1}{3}\sin(3t)u(-t)$

Find inverse Laplace transform of $X(s) = \frac{s}{s^2+9}$

$$X(s) = \frac{s}{s^2+9} = \frac{s}{s^2+3^2} \Rightarrow$$

For ROC: Re{s} > 0,	$x(t) = \cos(3t)u(t)$
For ROC: Re{s} < 0,	$x(t) = -\cos(3t)u(-t)$

Problem 9.22

Find inverse Laplace transform of $X(s) = \frac{s+2}{s^2+7s+12}$; $ROC: -4 < \mathrm{Re}\{s\} < -3$

$$X(s) = \frac{s+2}{s^2+7s+12} = \frac{s+2}{(s+4)(s+3)} = \frac{A}{s+4} + \frac{B}{s+3} = \frac{2}{s+4} - \frac{1}{s+3}$$

$$x(t) = 2e^{-4t}u(t) - (-)e^{-3t}u(-t) = 2e^{-4t}u(t) + e^{-3t}u(-t)$$

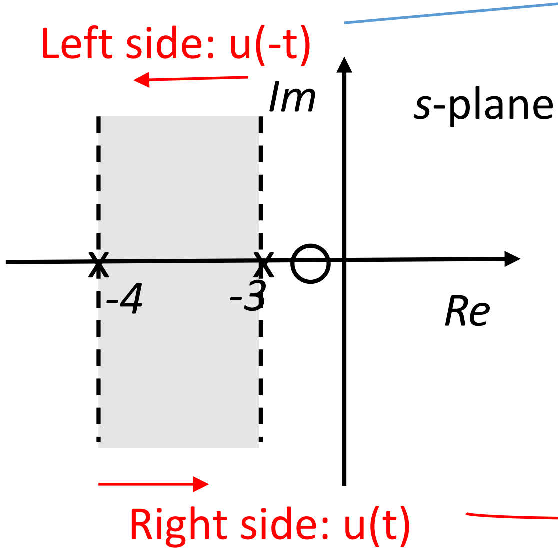
s+2 = A(s+3) + B(s+4)

s+2 = (A+B)s + 3A+4B

A+B = 1

3A+4B = 2

∴ A = 2, B = -1



Problem 9.26

Determine $Y(s)$, when

$$y(t) = x_1(t-2) * x_2(-t+3)$$
$$x_1(t) = e^{-2t}u(t)$$
$$x_2(t) = e^{-3t}u(t)$$

$$e^{-at}u(t) \overset{L}{\Longrightarrow} \frac{1}{s+a}; Re\{s\} > -a$$

$$x(t-t_o) \overset{\mathcal{L}}{\Longrightarrow} e^{-st_o}X(s)$$

$$x(-t) \overset{\mathcal{L}}{\Longrightarrow} X(-s)$$

Solution:

$$X_1(s) = \frac{1}{s+2}; \quad ROC > -2$$
$$X_2(s) = \frac{1}{s+3}; \quad ROC > -3$$

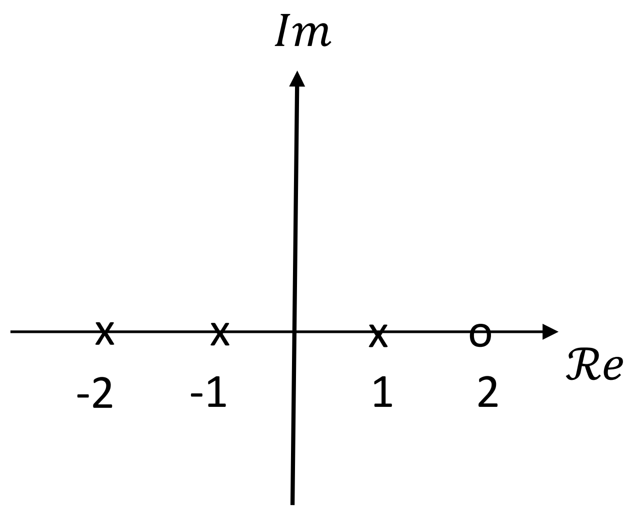
$$x_1(t-2) \overset{\mathcal{L}}{\Longrightarrow} e^{-2s}X_1(s) = e^{-2s}\left[\frac{1}{s+2}\right] = \frac{e^{-2s}}{s+2}; \quad ROC > -2$$

$$x_2(-t+3) = x_2(-(t-3)) \overset{\mathcal{L}}{\Longrightarrow} e^{-3s}X_2(-s) = e^{-3s}\left[\frac{1}{-s+3}\right] = \frac{e^{-3s}}{-s+3}; \quad ROC > -3$$

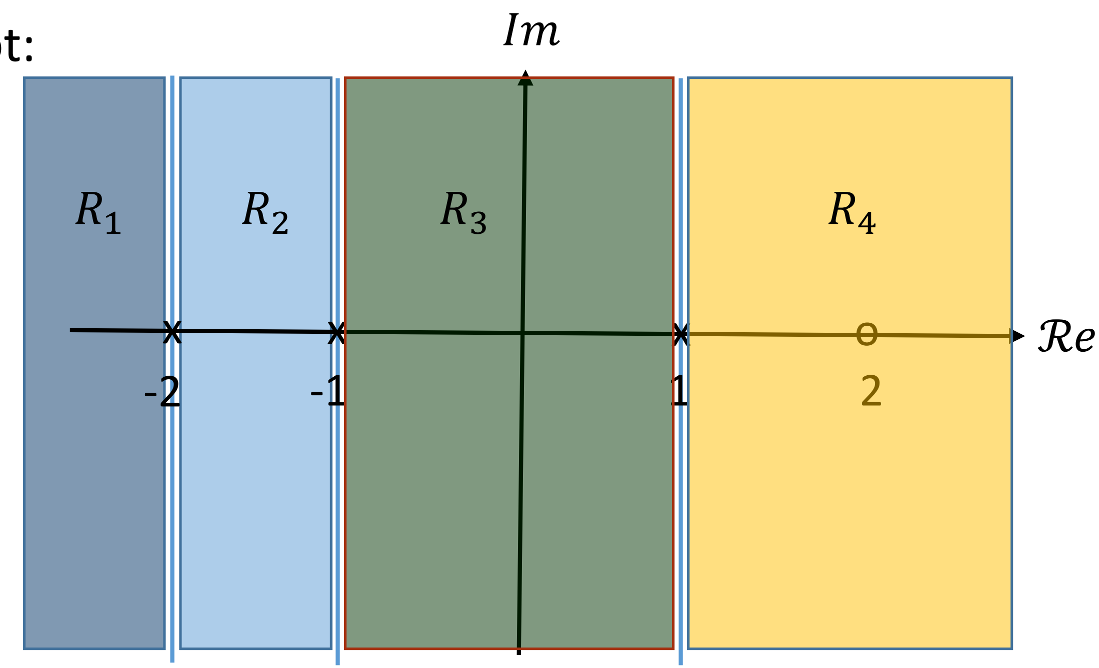
$$Y(s) = \left[\frac{e^{-2s}}{s+2}\right]\left[\frac{e^{-3s}}{-s+3}\right] = \frac{e^{-5s}}{(s+2)(-s+3)}$$

Problem 9.28

An LTI system H(s) has pole-zero plot:



a) Indicate all possible ROCs
b) Specify whether the system: Stable and/or Causal from part a)



R_1: NOT Causal and NOT Stable ROC: Re{s} < -2

R_2: NOT Causal and NOT Stable ROC: -2 < Re{s} < -1

R_3: NOT Causal and Stable ROC: -1 < Re{s} < 1

R_4: Causal and NOT Stable ROC: Re{s} > 1

Problem 9.31

LTI system has differential equation:

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2\,y(t) = x(t)$$

a) Determine H(s) as a ratio of two polynomials in *s*. and Sketch the pole-zero plot.

$$\frac{d}{dt}x(t) \overset{\mathcal{L}}{\Longrightarrow} sX(s)$$

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2\,y(t) = x(t)$$

$$s^2\,Y(s) - s\,Y(s) - 2\,Y(s) = X(s)$$

$$Y(s)\,[s^2 - s - 2] = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2} = \frac{1}{(s-2)(s+1)}$$

$$\frac{1}{(s-2)(s+1)} = \frac{A}{(s-2)} + \frac{B}{(s+1)}$$

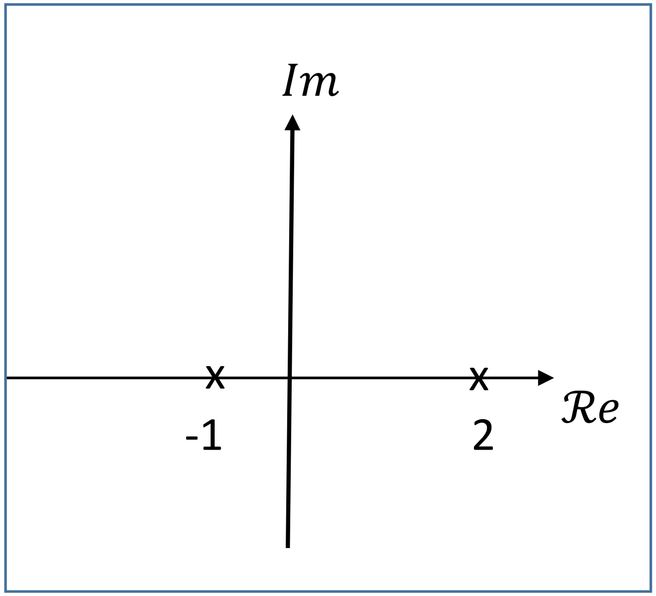
$$1 = (s-2)(s+1)\left[\frac{A}{(s-2)} + \frac{B}{(s+1)}\right]$$

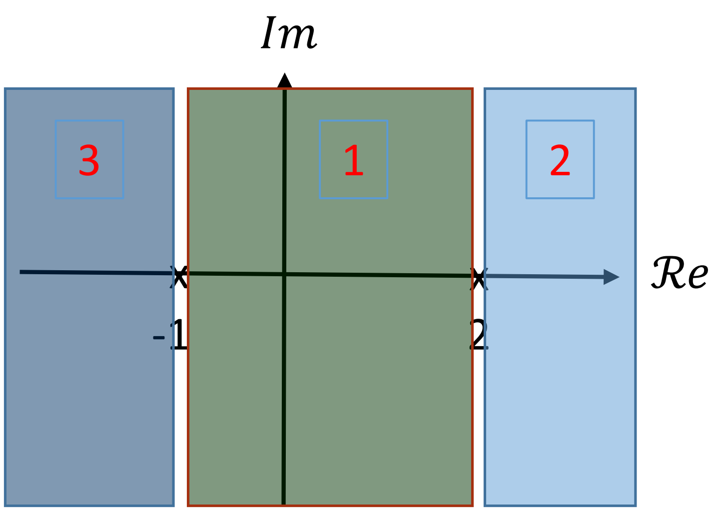
$$1 = A\,(s+1) + B\,(s-2)$$

A = 1/3, *B* = -1/3

$$H(s) = \frac{\frac{1}{3}}{(s-2)} - \frac{\frac{1}{3}}{(s+1)}$$

Poles: s = 2, -1





Problem 9.31

b) Determine $h(t)$ for the following cases:

1. The system is Stable
2. The system is Causal
3. The system is neither Stable nor Causal

$$H(s) = \frac{\frac{1}{3}}{(s-2)} - \frac{\frac{1}{3}}{(s+1)}$$

- **Stable:**

$$-1 < ROC < 2$$

$$h_1(t) = -\frac{1}{3}\, e^{2t}\, u(-t) - \frac{1}{3}\, e^{-t}\, u(t)$$

- **Causal:**

$$ROC > 2$$

$$h_2(t) = \frac{1}{3}\, e^{2t}\, u(t) - \frac{1}{3}\, e^{-t}\, u(t)$$

- **NOT Stable and NOT Causal:**

$$ROC < -1$$

$$h_3(t) = -\frac{1}{3}\, e^{2t}\, u(-t) + \frac{1}{3}\, e^{-t}\, u(-t)$$