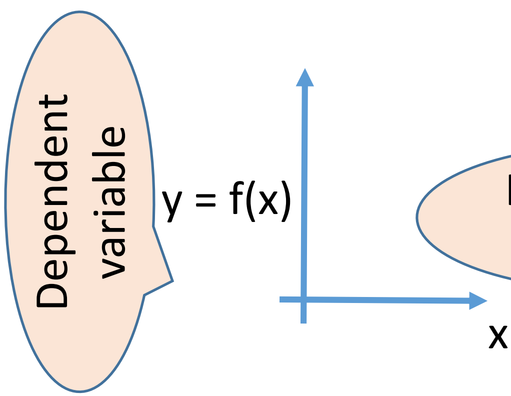
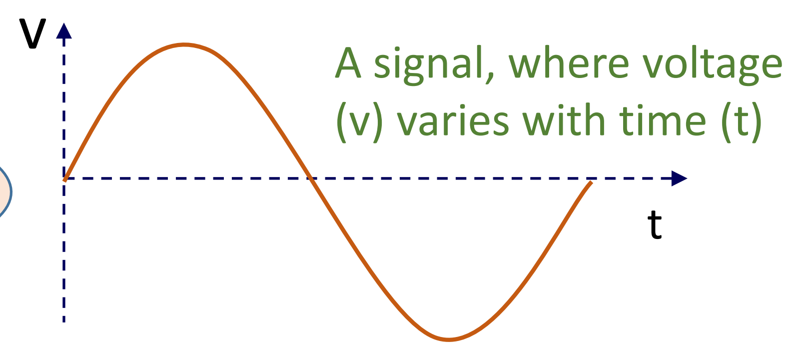
Introduction to Signals and Systems

What is a signal?

- ❑Signals may describe a wide variety of physical phenomena.
- ❑The information in a signal is contained in a pattern of variations of some form.
- ❑A signal is represented mathematically as a function of one or more independent variables.

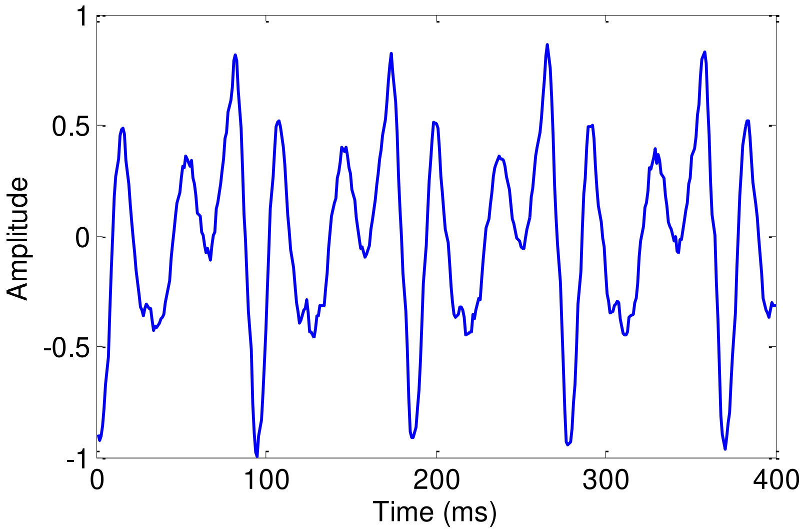




Examples of Signals - 1

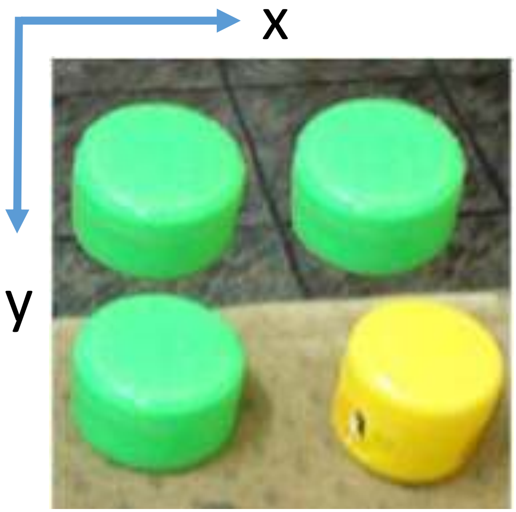
One dimensional signal, because there is only one independent variable, such as time.

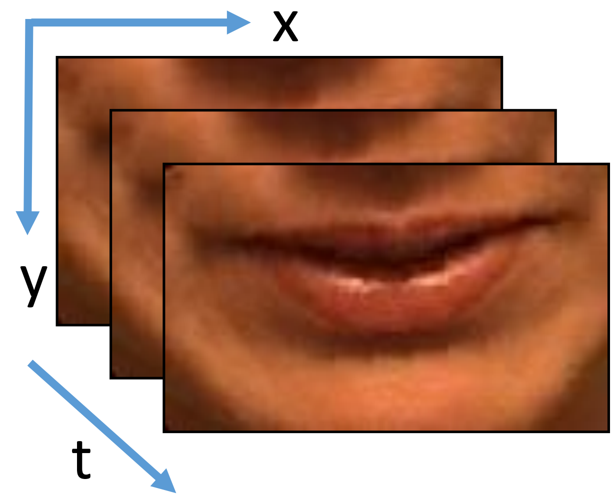
Speech Signal



ECG (Electrocardiogram) Signal

Examples of Signals - 2





Intensity of the image at location (x, y) can be expressed as *I*(x, y). As there are two independent variables (x and y), the image is a two dimensional signal.

A video has three independent variables (x, y, and t (time)), therefore, it is a three dimensional signal. A video is a sequence of frames (images).

Two Basic Types of Signals

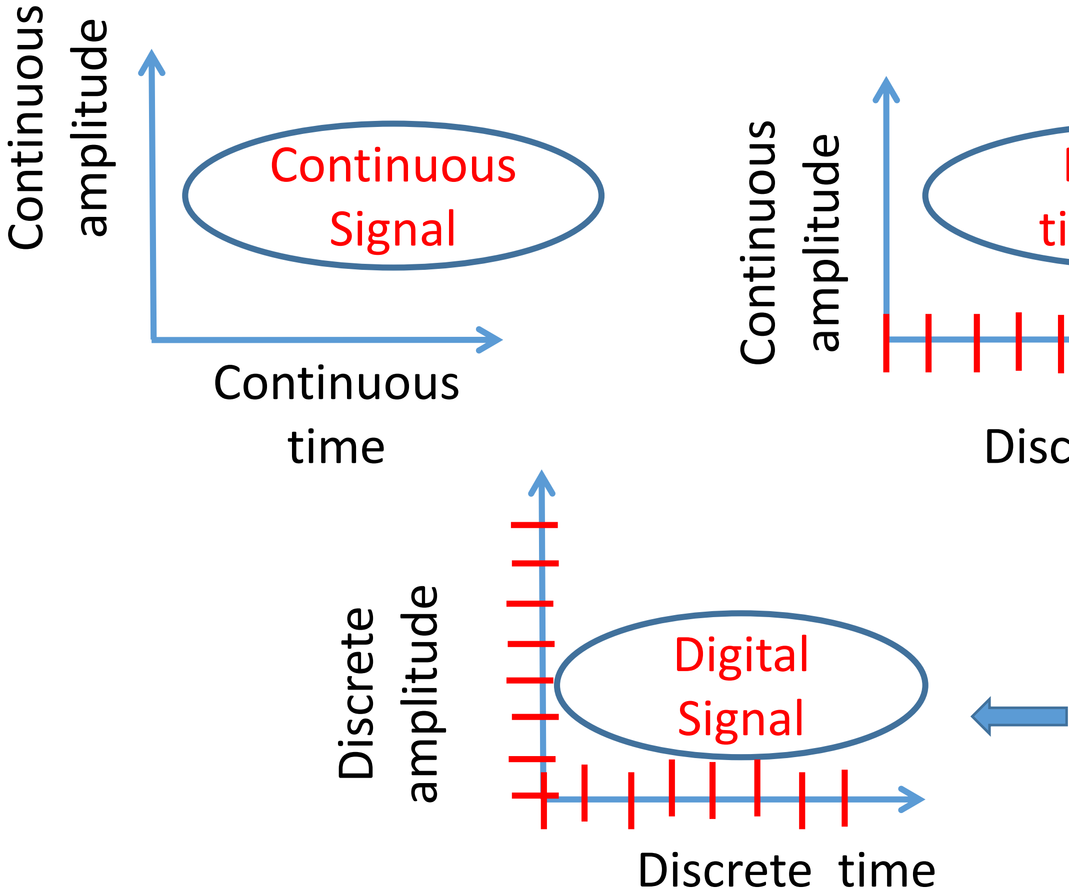
Continuous Signal

A continuous-time (CT) signal is one that is present at all instants in time or space, such as oscillating voltage signal.

Discrete-time Signal

A discrete-time (DT) signal is only present at discrete points in time or space. For example closing stock market average is a signal that changes only at discrete points in time (at the close of each day).

Continuous, discrete-time, & Digital Signals

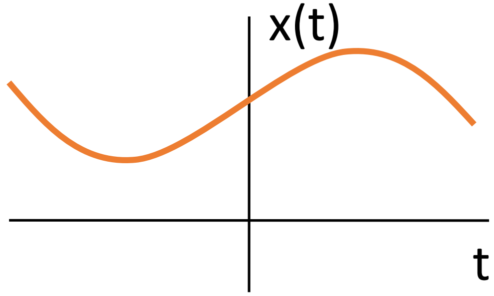


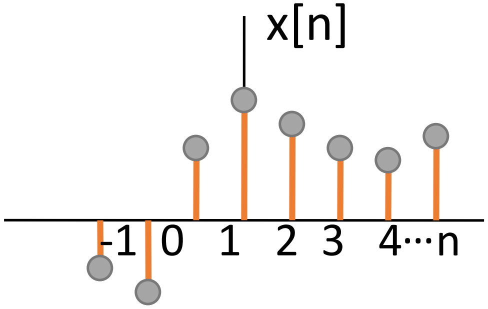
You will learn it in CEN 352.

Notation of Continuous and discrete-time Signals

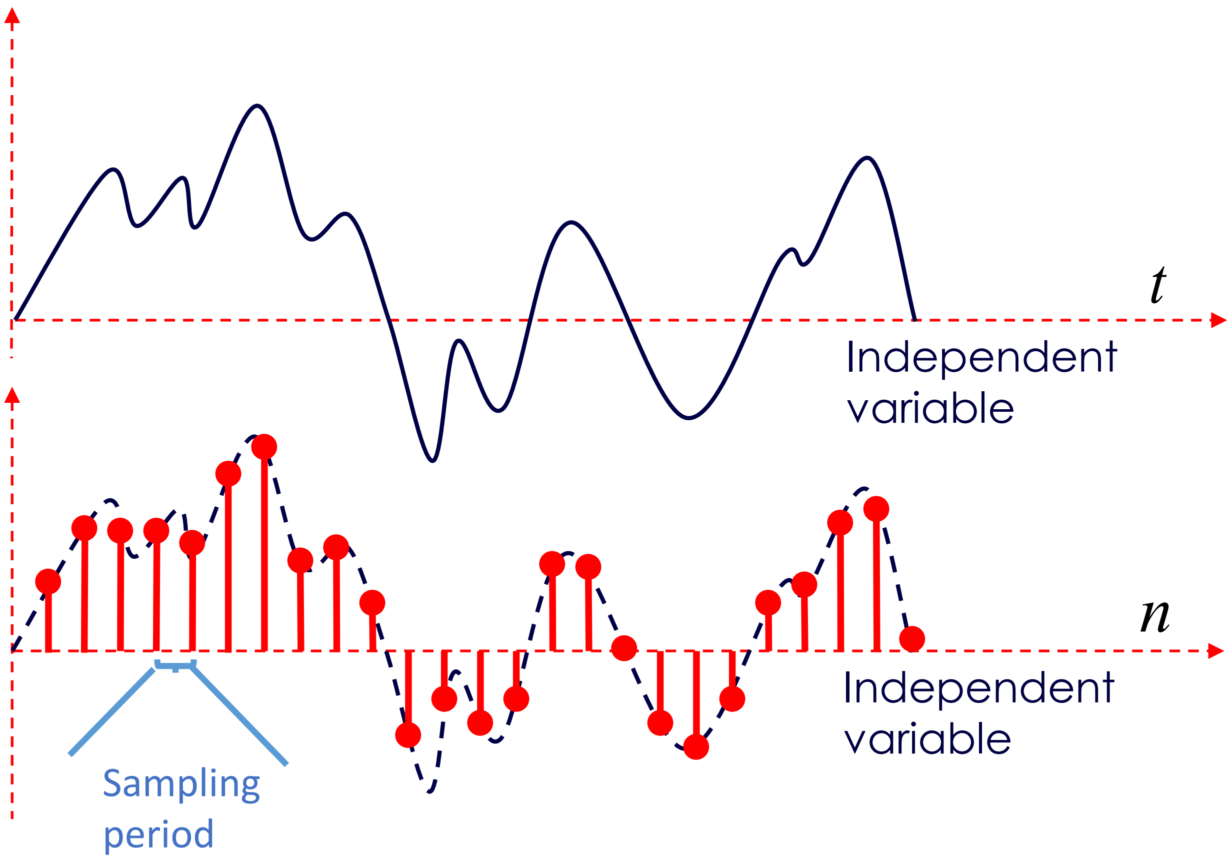
To distinguish between continuous-time and discrete-time signals, we will use

- The symbol ‘t’ to denote the continuous-time independent variable and
- ‘n’ to denote the discrete-time independent variable.
- We will enclose the independent variable in parentheses ‘(.)’ and for discrete-time signals, we will use brackets ‘[.]’





Continuous and discrete-time Signals



We can convert a continuous signal into a discrete-time signal by 'sampling'.

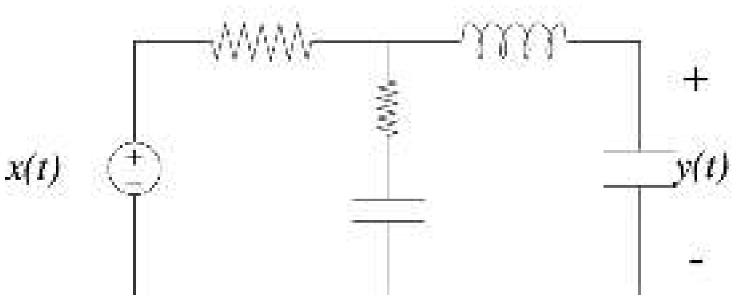
'*n*' is always an integer.

Systems

- A system is an abstraction of anything that takes an input signal, operates on it, and produces an output signal.
  - A system generally establishes a relationship between its input and its output.
  - Examples could be car, camera, etc.
- Systems that operate on continuous-time signal are known as continuous-time (CT) systems.
- Systems that operate on discrete-time signals are known as discrete-time (DT) systems.

Examples of Systems

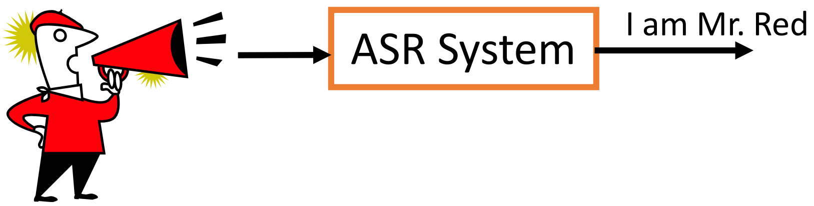
An RLC circuit



Courtesy of Prof. Alan S. Willsky

- What is the input signal?
  - *x(t) (the D.C. source)*
- What is the output signal?
  - *y(t) (the signal across capacitor)*
- What is the system?
  - *The whole RLC network*

Automatic speech recognition (ASR) system



Imaging system

Drill - 1

1. Most of the signals in this physical world is …………… (CT signals / DT signals). Choose the right one.
2. Mention four systems other than those mentioned in the slides.
3. Mention three signals other than those mentioned in the slides.
4. How can we convert a CT signal into a DT signal?
5. Can a system have multiple inputs and multiple outputs?
6. What do you mean by time-domain signal and spatial-domain signal?

MATLAB

- Matlab ® is a software tool for computation in science and engineering.
- Developed, published and trademarked by The MathWorks, Inc.
- Originally developed as a “Matrix Laboratory” but now used in applications in almost all areas of science and engineering.
- It has a rich collection of tool boxes covering basic mathematics, graphics, differential equations, electric/electronic circuits, partial differential equations, simulation problems, control systems, signal processing, image processing, statistics, symbolic computations, etc.
- http://www.mathworks.com/help/pdf_doc/matlab/getstart.pdf
- http://www.mathworks.com/academia/student_center/tutorials/launchpad.html

1.1.2 Signal Power and Energy

Continuous-time (CT) signal

- The total energy over the time internal $t_1 \le t \le t_2$ in a continuous-time signal $x(t)$ is defined as

$$\int_{t_1}^{t_2} |x(t)|^2 \, dt$$

where $|x|$ denotes the magnitude of the (possibly complex) number x.

- The time averaged power is given by $$\frac{1}{(t_2 - t_1)} \int_{t_1}^{t_2} |x(t)|^2 dt$$

Over an infinite time interval, i.e., for $-\infty < t < +\infty$

- Total energy: $$E_\infty \triangleq \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

- Total averaged power: $$P_\infty \triangleq \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

1.1.2 Signal Power and Energy

Discrete-time (DT) signal

- The total energy in a discrete-time signal $x[n]$ over the time interval $n_1 \leq n \leq n_2$ is defined as

$$\sum_{n=n_1}^{n_2} |x[n]|^2$$

- The average power over the interval in this case is given by

$$\frac{1}{n_2 - n_1 + 1} \sum_{n=n_1}^{n_2} |x[n]|^2$$

Over an infinite time interval, i.e., for $-\infty < t < +\infty$

- Total energy: $$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

- Total averaged power: $$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2$$

Three Important Cases

Case 1: Signals with finite total energy, i.e., $E_\infty < \infty$:

Such a signal must have zero average power. For example, in continuous case, if $E_\infty < \infty$, then

$$P_\infty = \lim_{T \to \infty} \frac{E_\infty}{2T} = 0$$

An example of a finite-energy signal is a signal that takes on the value of 1 for $0 \le t \le 1$ and 0 otherwise. In this case, $E_\infty = 1$ and $P_\infty = 0$.

Case 2: Signals with finite average power, i.e., $P_\infty < \infty$:

For example, consider the constant signal where $x[n] = 4$. This signal has infinite energy, as

$$E_\infty = \lim_{N \to \infty} \sum_{n=-N}^{+N} |x[n]|^2 = \lim_{N \to \infty} \sum_{n=-N}^{+N} 4^2 = \cdots + 16 + 16 + 16 \ldots$$

Three Important Cases - continued

However, the total average power is finite,

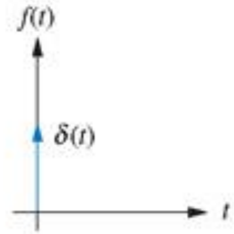
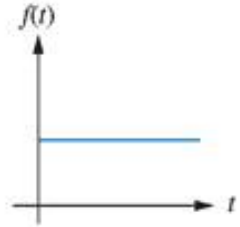
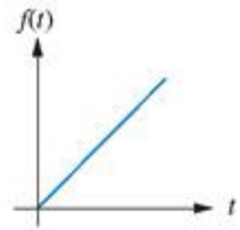
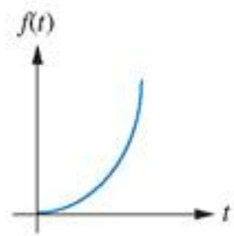
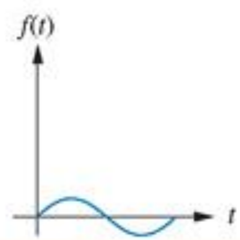
$$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} 4^2$$

$$= \lim_{N\to\infty} \frac{16}{2N+1} \sum_{n=-N}^{+N} 1 \ = \lim_{N\to\infty} \frac{16(2N+1)}{2N+1} \ = \lim_{N\to\infty} 16 = 16$$

Case 3: Signals with neither E_∞ nor P_∞ finite:

A simple example of such a case could be $x(t) = t$. In this case both E_∞ and P_∞ are infinite

Some Frequently Used Signals

Input	Function	Description	Sketch
Impulse	$\delta(t)$	$\delta(t) = \infty$ for $0- < t < 0+$ $= 0$ elsewhere $\int_{0-}^{0+} \delta(t)\,dt = 1$	
Step	$u(t)$	$u(t) = 1$ for $t > 0$ $= 0$ for $t < 0$	
Ramp	$tu(t)$	$tu(t) = t$ for $t \geq 0$ $= 0$ elsewhere	
Parabola	$\frac{1}{2}t^2u(t)$	$\frac{1}{2}t^2u(t) = \frac{1}{2}t^2$ for $t \geq 0$ $= 0$ elsewhere	
Sinusoid	$\sin \omega t$		

Example: Power and Energy

Problem 1: Find P_∞ and E_∞ for the signal, $x_1(t) = e^{-2t}u(t)$

Solution:

$$E_\infty = \int_{-\infty}^{\infty} |x_1(t)|^2\,dt = \int_{-\infty}^{\infty} |e^{-2t}u(t)|^2\,dt = \int_{0}^{\infty} |e^{-2t}|^2\,dt$$

$$= \int_{0}^{\infty} |e^{-4t}|\,dt = -\frac{1}{4}\left(-\frac{1}{e^{4(0)}} + \frac{1}{e^{4(\infty)}}\right) = -\frac{1}{4}(-1+0) = \frac{1}{4}$$

P_∞ is zero, because $E_\infty < \infty$

1.2 Transformations of the Independent Variable

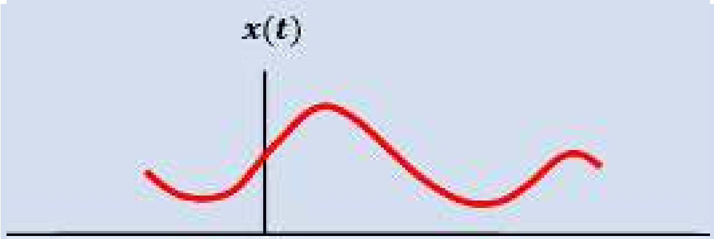
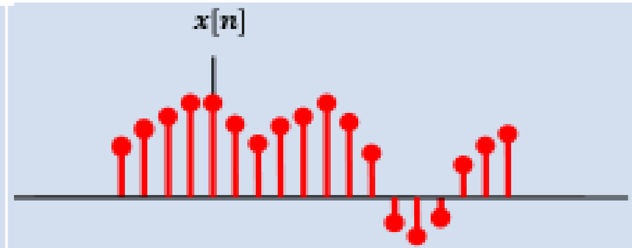
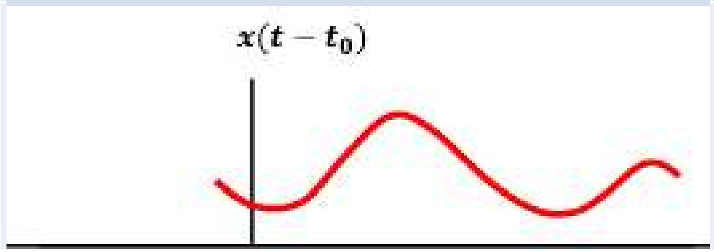
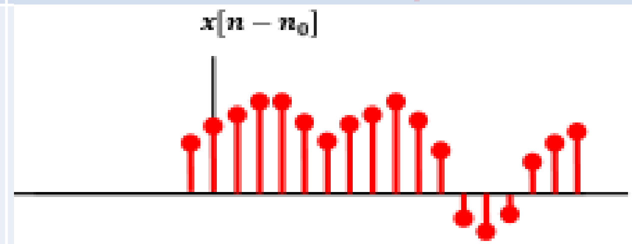
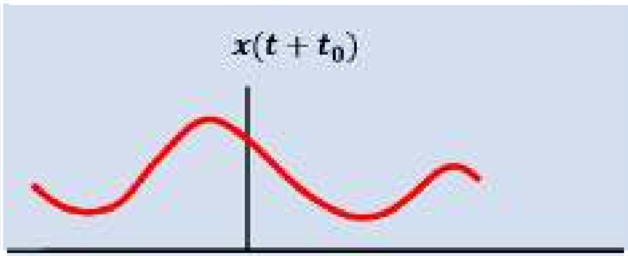
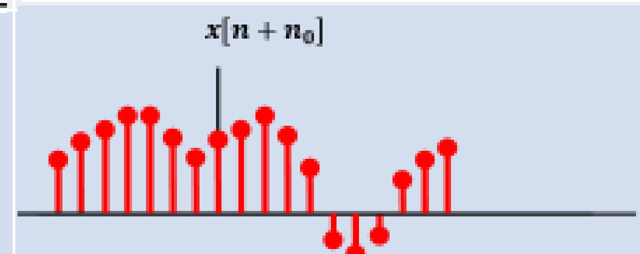
The ***transformation of a signal*** is one of the central concepts in the field of signals and systems.

We will focus on a very limited but important class of signal transformations that involves the *modifications of the independent variable, i.e., the time axis*.

(A) Time Shift

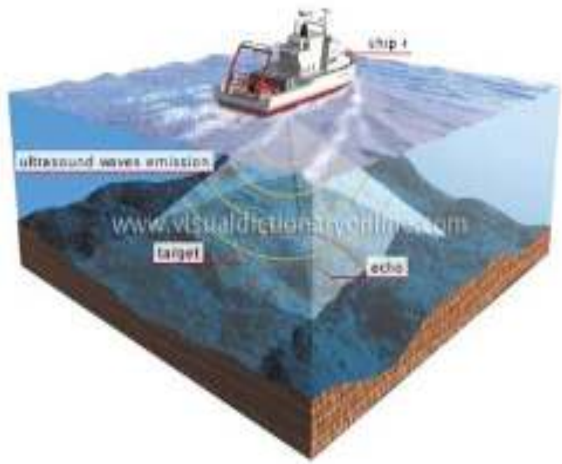
The original and the shifted signals are identical in shape, but are displaced or shifted along the time-axis with respect to each other. Signals could be termed as delayed or advanced in this case.

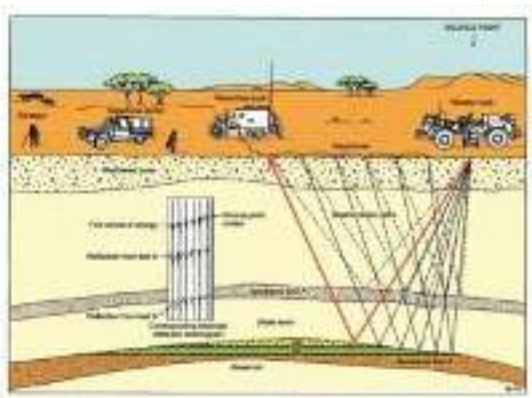
Time Shift

Signals	Continuous time	Discrete time
Original		
Delayed		
Advance		

Time Shift - continued

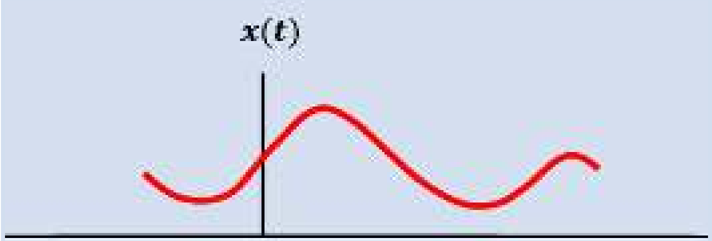
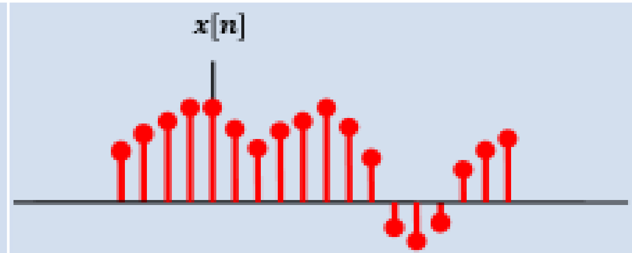
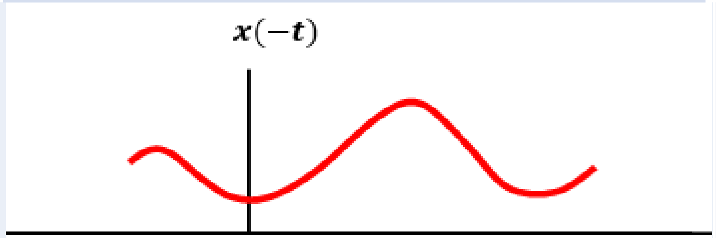
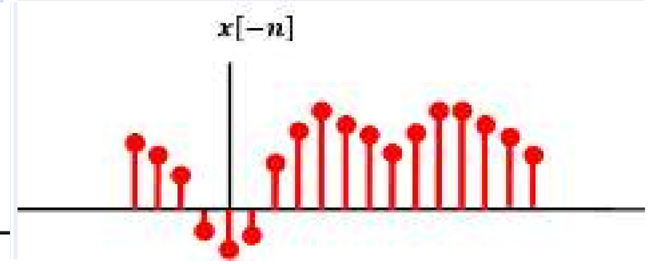
Such signals arise in applications such as radar, sonar and seismic signal processing. Several receivers placed at different locations receive the time shifted signals due to the transmission time they take while passing through a medium (air, water or rock etc.).





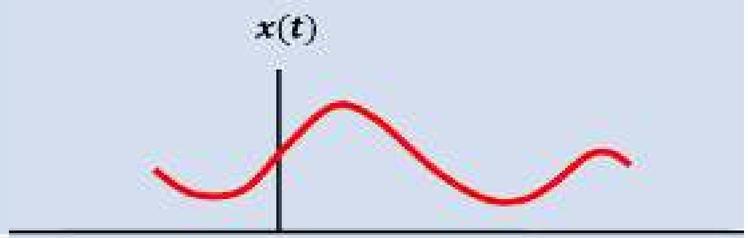
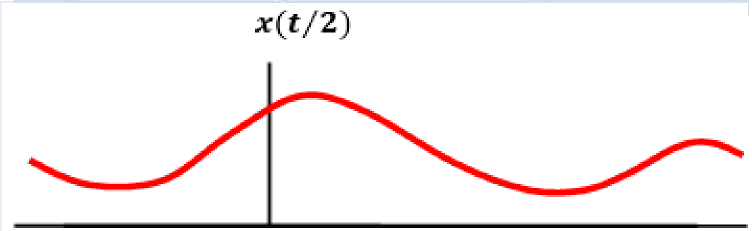
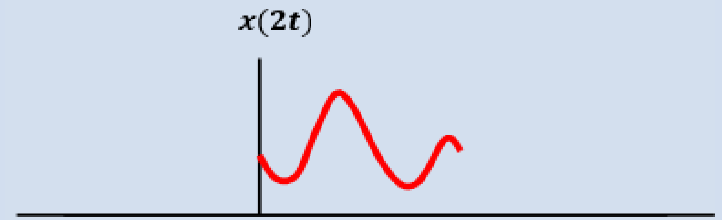
Time Reversal (Reflection)

In this case, the original signal is reflected about the time = 0. For example, if the original signal is some audio recording, then the time reversed signal would be the audio recording played backward.

Signals	Continuous time	Discrete time
Original		
Time reversed		

Time Scaling

In this case, if the original signal is $x(t)$, the time variable is multiplied with a constant to get a time-scaled signal, e.g., $x(2t)$, $x(5t)$, or $x(t/2)$. If we think of the signal $x(t)$ as audio recording, then $x(2t)$ is the audio recording played at twice the speed and $x(t/2)$ is the recording played at half of the speed.

Signals	Continuous time
Original	
Stretching	
Compressing	

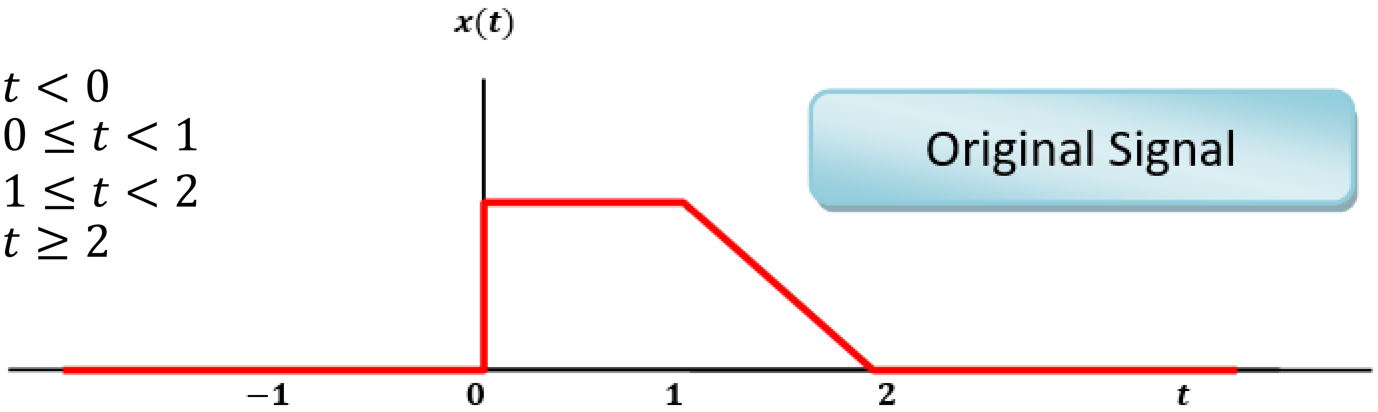
General Case of the Transformation of the Independent Variable

A general case for the transformation of independent variable is the one in which for the original signal $x(t)$ is changed to the form $x(\alpha t + \beta)$, where α and β are given numbers. It has the following effects on the original signal:

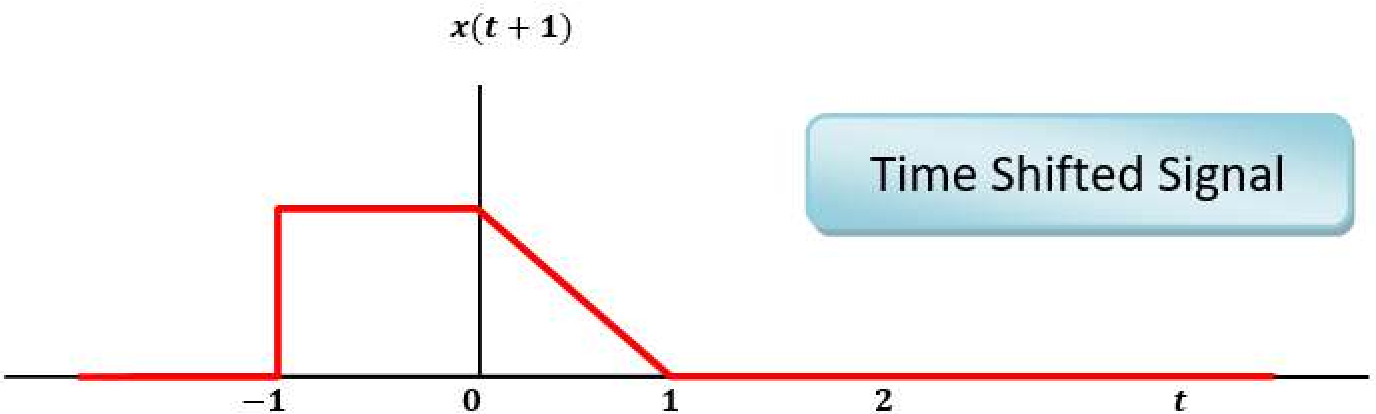
- The general shape of the signal is preserved.
- The signal is linearly stretched if $|\alpha| < 1$.
- The signal is linearly compressed if $|\alpha| > 1$.
- The signal is delayed (shifted in time) if $\beta < 0$.
- The signal is advanced (shifted in time) if $\beta > 0$.
- The signal is reversed in time (reflected) if $\alpha < 0$.

Example: Time Shift: (1)

$$x(t) = \begin{cases} 0 & if\ t < 0 \\ 1 & if\ 0 \le t < 1 \\ 2 - t & if\ 1 \le t < 2 \\ 0 & if\ t \ge 2 \end{cases}$$



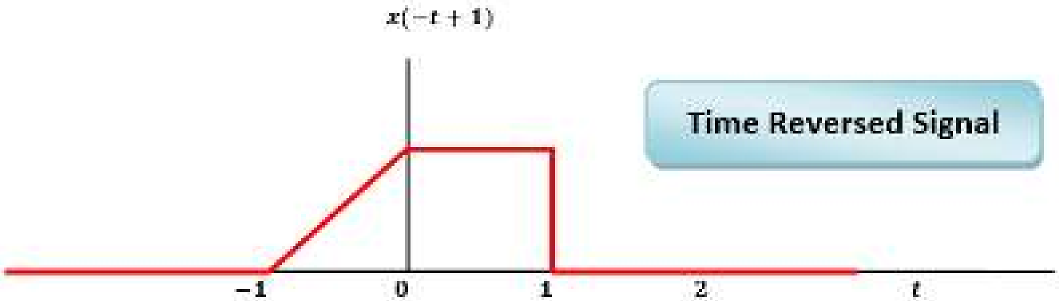
The signal $x(t + 1)$ can be obtained by shifting the given signal to the left by one unit



Example: Time Shift: (2)

The signal $x(-t+1)$ can be obtained using the mathematical definition or figure of the original signal $x(t)$. If we use the mathematical definition, then making the following table could be useful.

t	$-t+1$	$x(-t+1)$
-2	3.0	0
-1.5	2.5	0
-1	2.0	0
-0.5	1.5	0.5
0	1.0	1
0.5	0.5	1
1	0.0	1
1.5	-0.5	0
2	-1.0	0
2.5	-1.5	0
3	-2.0	0



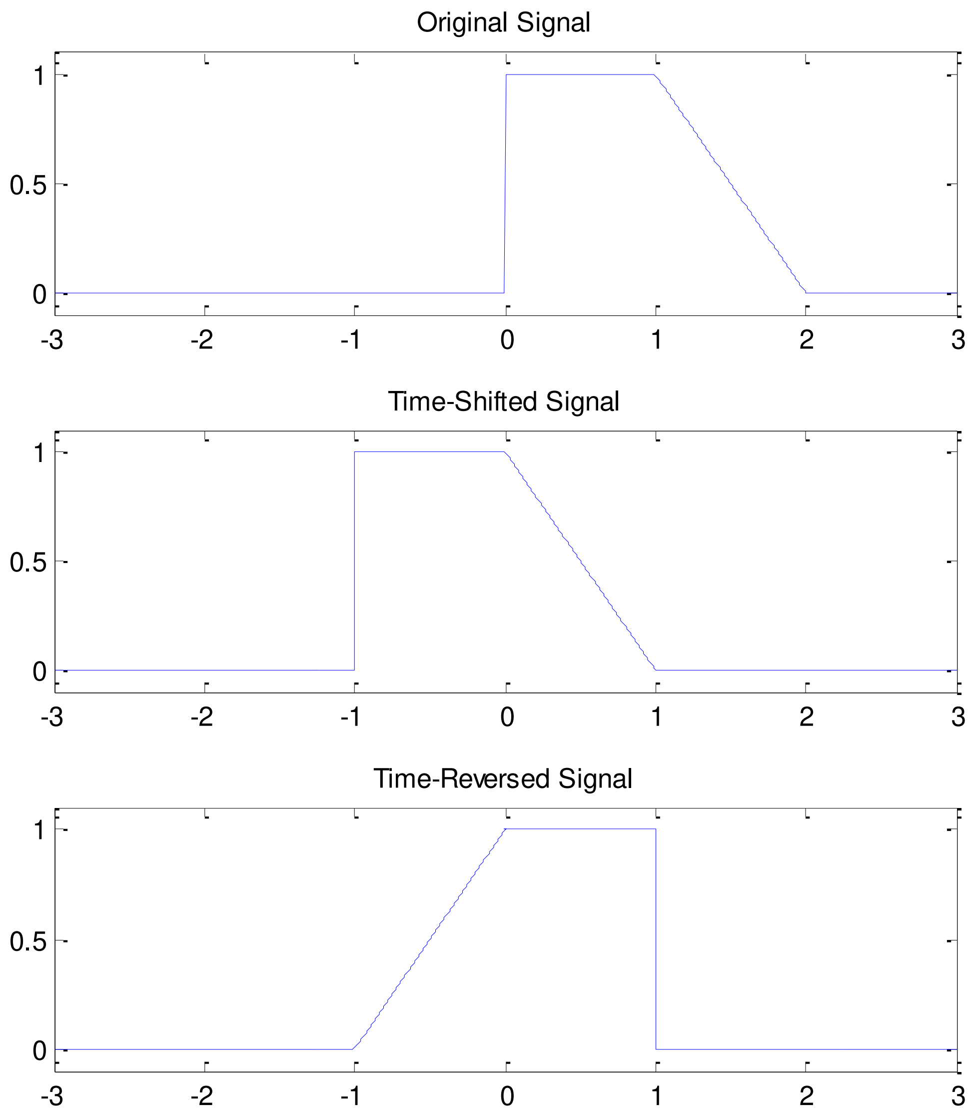
First plot x(t+1), then reflect.

MATLAB Drill - 1

In MATLAB®, the original signal can be written as an inline function. This function can then be used to plot the original signal, the shifted signal and the time-reversed signal using the following MATLAB® code.

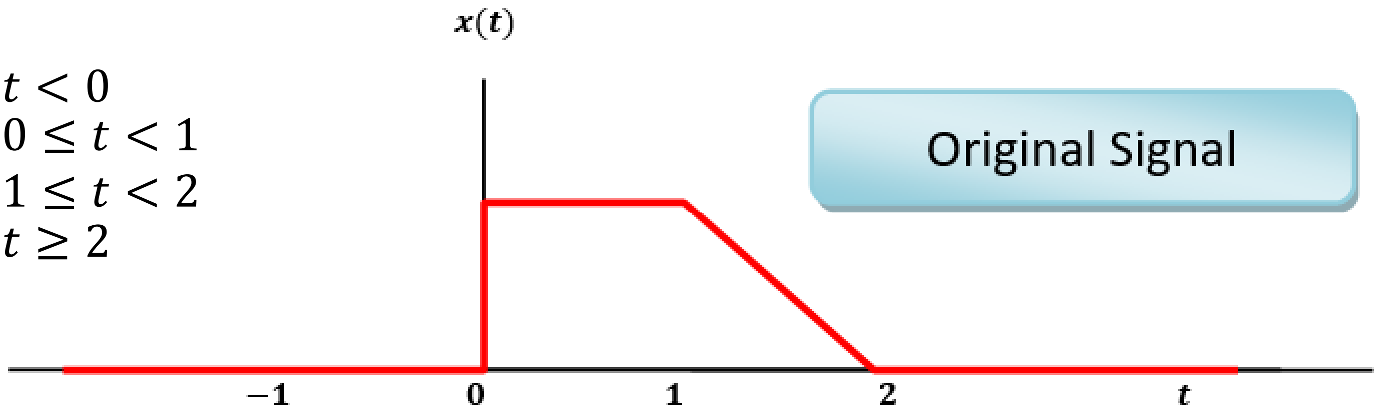
```
>>g = inline(' ((t>=0)&(t<1)) + (2-t).*((t>=1) & (t<2))','t');
>>t = -3:0.001:3;
>>subplot(3,1,1), plot(t, g(t)), axis([-3 3 -0.1 1.1]), title('Original Signal')
>>subplot(3,1,2), plot(t, g(t+1)), axis([-3 3 -0.1 1.1]), title('Time-Shifted Signal')
>>subplot(3,1,3),plot(t, g(-t+1)),axis([-3 3 -0.1 1.1]), title('Time-Reversed Signal')
```

MATLAB Drill – 1: continued



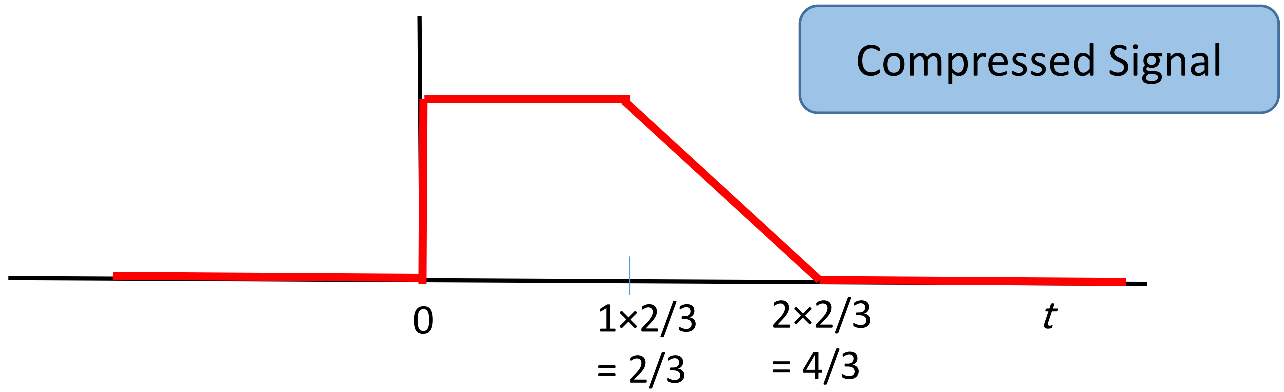
Example: Time Compression: (1)

$$x(t) = \begin{cases} 0 & if\ t < 0 \\ 1 & if\ 0 \le t < 1 \\ 2 - t & if\ 1 \le t < 2 \\ 0 & if\ t \ge 2 \end{cases}$$



Find $x\left(\frac{3}{2}t\right)$

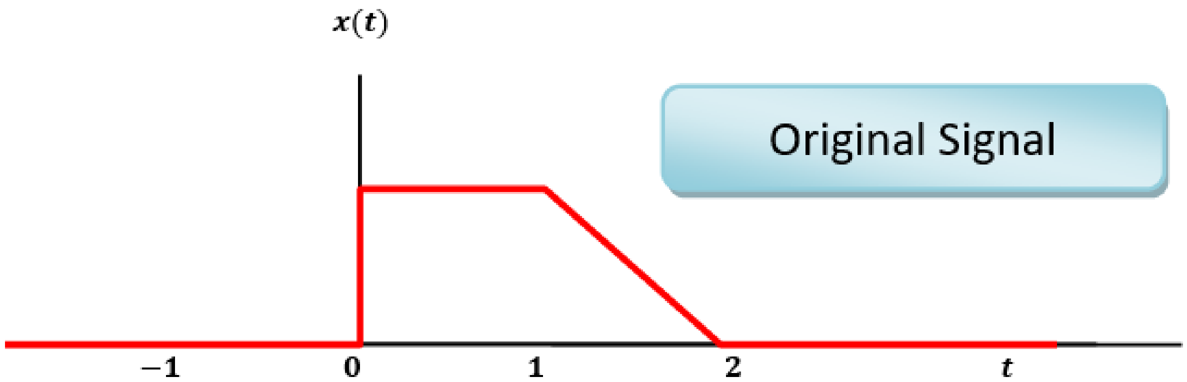
$x(\alpha t + \beta)$; $|\alpha| > 1$, so linear compression by a factor of 1 / (3/2) = 2/3

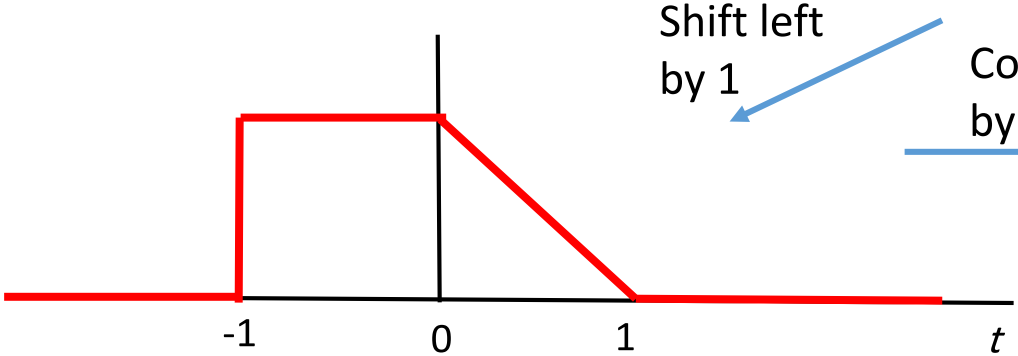


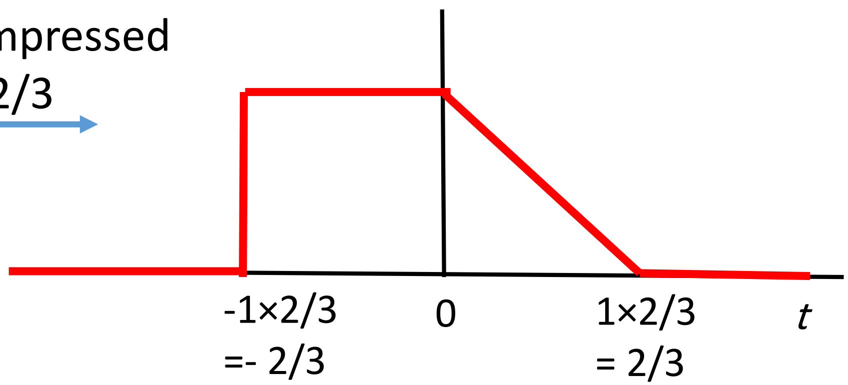
Example: Time Compression: (2)

Find $x\left(\frac{3}{2}t+1\right)$

Compressed by a factor of 2/3, and shift left by 1







Final Signal

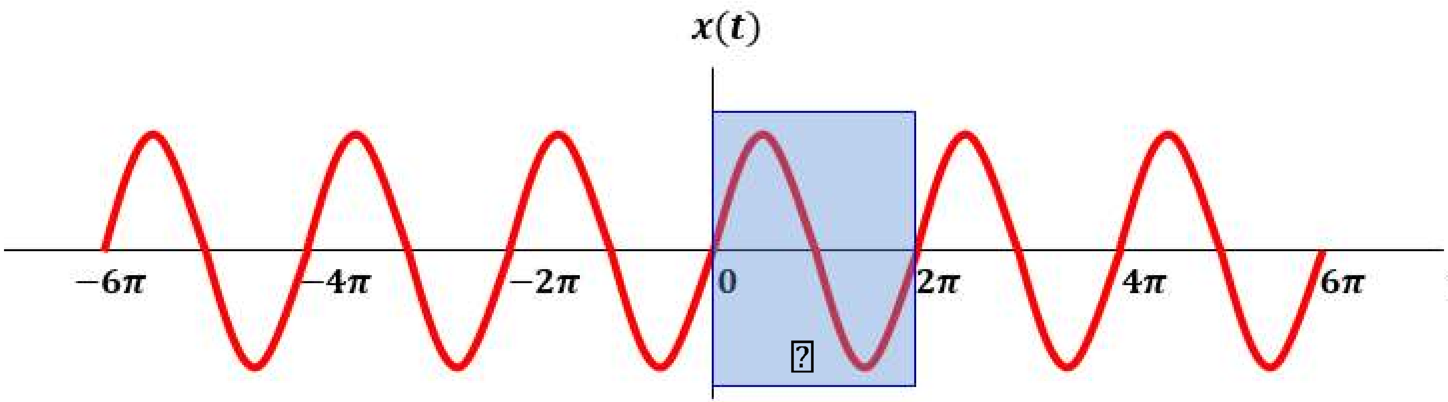
1.2.2 Periodic Signals

A periodic continuous-time signal $x(t)$ is defined as

$$x(t) = x(t + T)$$

where T is a positive number called the period.

A typical example is that of a sinusoidal signal $x(t) = \sin(t)$ for $-\infty < t < +\infty$.



For the above signal, the period is $T = 2$. It can be noticed that for any time t:

$$\boldsymbol{sin(t + 2\pi) = sin(t)} \qquad \boldsymbol{sin(t + m2\pi) = sin(t)}$$

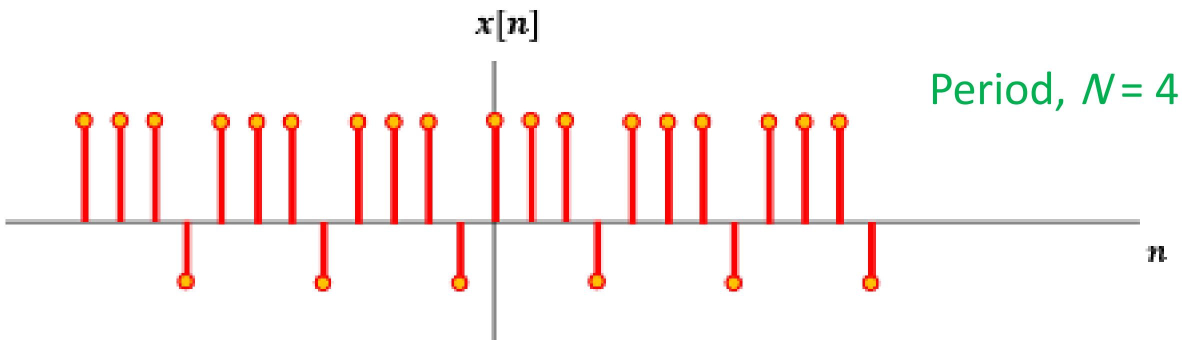
where m is a positive number.

Periodic Signals - continued

The ***fundamental period*** T_0 of $x(t)$ is the smallest positive value of T for which the equation $x(t) = x(t + T)$ holds.

A discrete-time signal $x[n]$ is periodic with period N, where N is a positive integer, if it is unchanged by a time-shift of N, i.e., if

$$x[n] = x[n + N] \quad \text{for all values of } n.$$



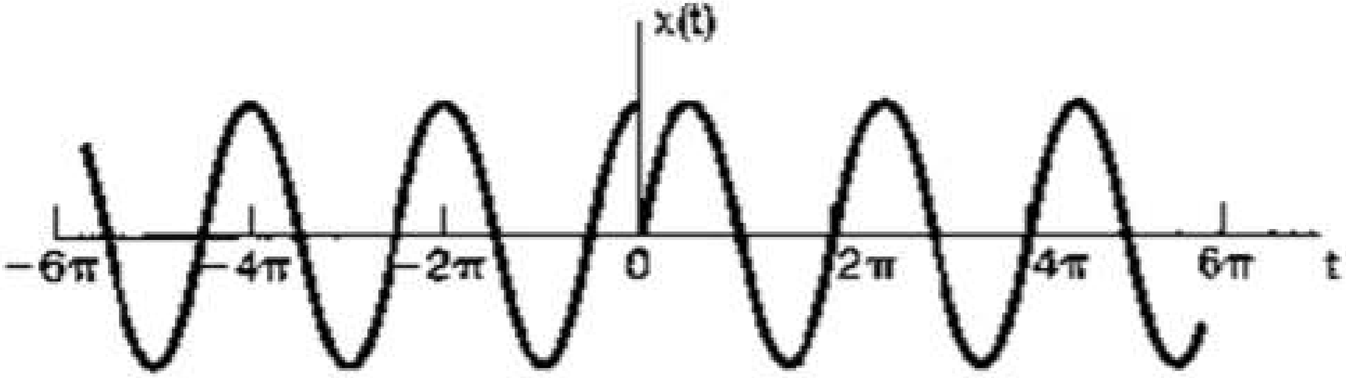
The ***fundamental period*** N_0 of $x[n]$ is the smallest positive value of N for which the equation $x[n] = x[n + N_0]$ holds.

Periodic Signals - Example

$$x(t) = \begin{cases} \cos(t) & \text{if } t < 0 \\ \sin(t) & \text{if } t \geq 0 \end{cases}$$

Since, cos(2π+t) = cos(t) and sin(2π+t) = sin(t), considering $t < 0$ and $t \geq 0$ separately, the signal repeats itself in every interval of 2π.

But, if we look at the following figure of x(t), we find there is a discontinuity at t = 0, which does not occur at any other time. Therefore, x(t) is not periodic.

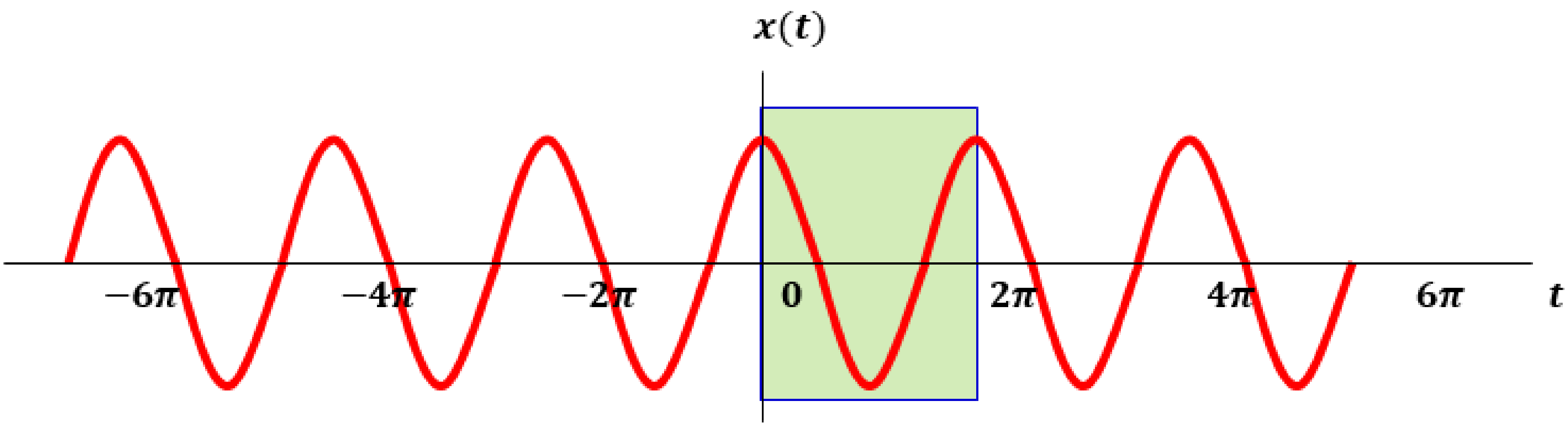


1.2.3 Even and Odd Signals

Even Signals

A signal $x(t)$ or $x[n]$ is defined as an even signal if it is identical to its time-reversed counterpart, i.e., with its reflection about the origin.

Even continuous-time Signal	$x(-t) = x(t)$
Even Discrete-time Signal	$x[-n] = x[n]$

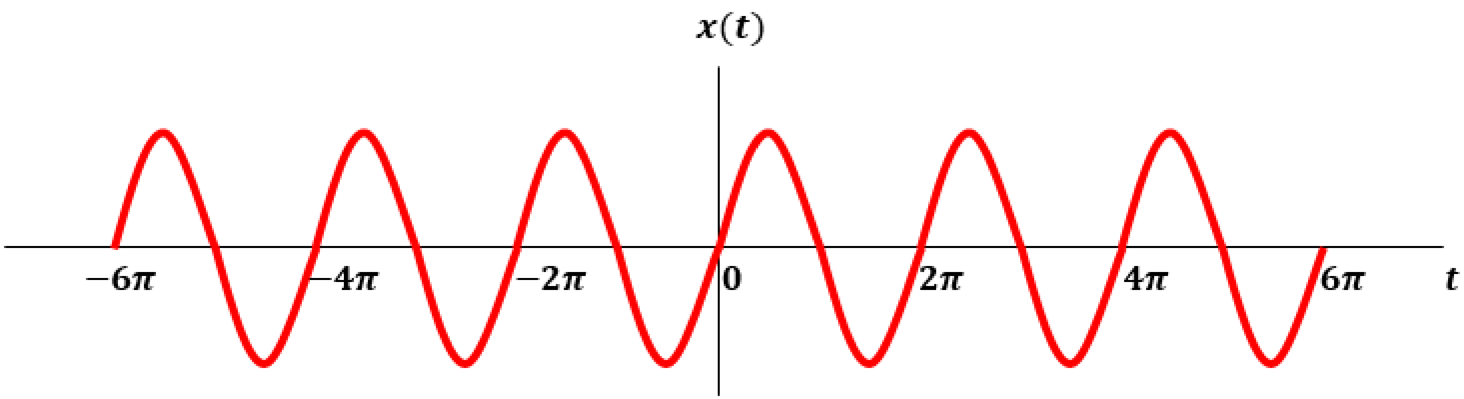


Odd Signals

A signal $x(t)$ or $x[n]$ is defined as an odd signal if,

Odd continuous-time Signal	$x(-t) = -x(t)$
Odd Discrete-time Signal	$x[-n] = -x[n]$

As a special case, the odd signal must be zero at $t = 0$ or $n = 0$.

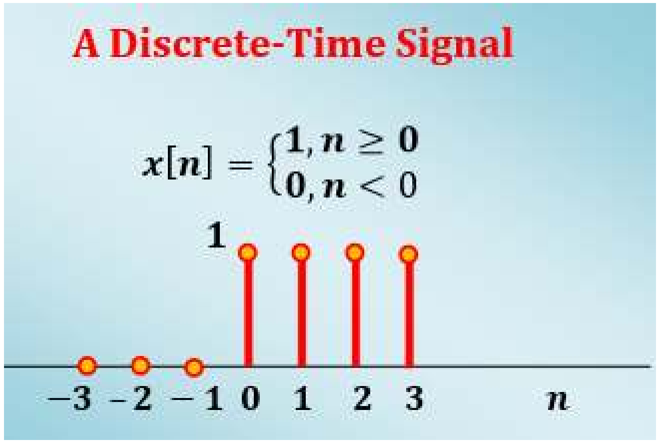


Decomposing a Signal into Even and Odd Parts

An important fact is that any signal (continuous-time or discrete-time) can be broken into a sum of two signals: even and odd.

Signal	Component	Mathematical Form
Continuous-time Signal $x(t)$	Even Part	$\mathcal{E}_v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$
	Odd Part	$\mathcal{O}_d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$
Discrete-time Signal $x[n]$	Even Part	$\mathcal{E}_v\{x[n]\} = \frac{1}{2}[x[n] + x[-n]]$
	Odd Part	$\mathcal{O}_d\{x[n]\} = \frac{1}{2}[x[n] - x[-n]]$

Decomposing a Signal into Even and Odd Parts Example



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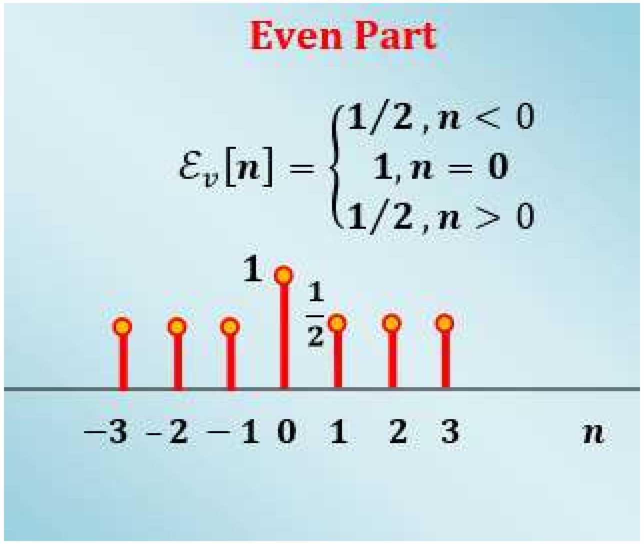

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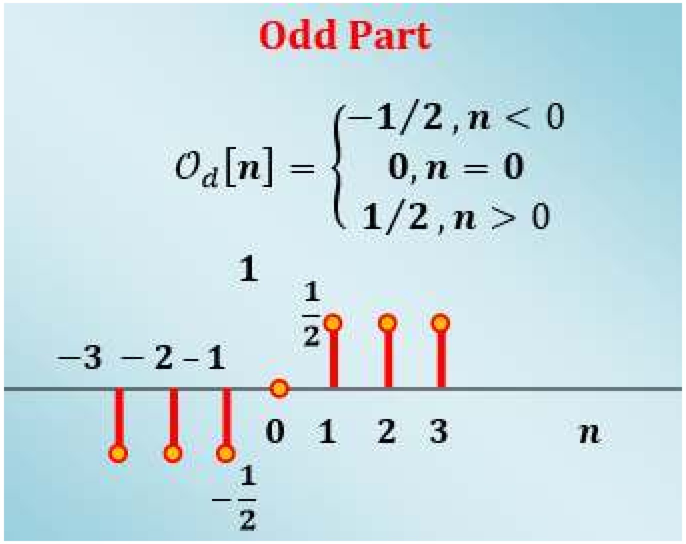


1.3 Exponential and Sinusoidal Signals

Continuous-Time Complex Exponential and Sinusoidal Signals

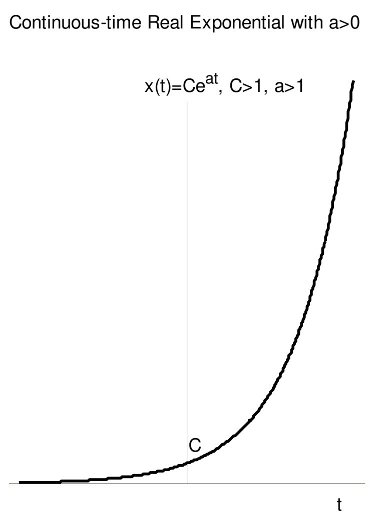
A continuous-time complex signal $x(t)$ can be written as

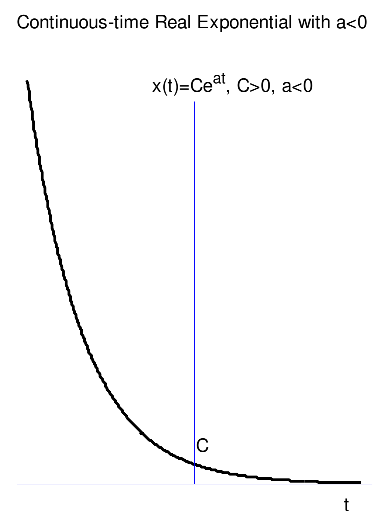
$$x(t) = Ce^{at}$$

where C and a are, in general, complex numbers.

Real Exponential Signals

In this case both C and a are real numbers, and $x(t)$ is called a real exponential.





Periodic Complex Exponential and Sinusoidal Signals

Now we consider the case of complex exponentials where a is purely imaginary. More, specifically, we consider:

$$x(t) = e^{j\omega_0 t}$$

An important property of this signal is that it is periodic.

$$x(t) = x(t+T) \Rightarrow e^{j\omega_0 t} = e^{j\omega_0 (t+T)} = e^{j\omega_0 t} e^{j\omega_0 T} \Rightarrow e^{j\omega_0 T} = 1$$

This equation can be true,

1. If, $\omega_0 = 0$, then $x(t) = 1$, which is periodic for any value of T.

2. If, $\omega_0 \neq 0$, then the fundamental period T_0 of $x(t)$, i.e. the smallest value of T for which the above equation holds, is

$$T_0 = \frac{2\pi}{|\omega_0|}$$

Periodic Signals

Replacing the value of T with this T_0, and using Euler's formula, that is,

$$e^{j\omega_0 T} = \cos(\omega_0 T) + j\sin(\omega_0 T)$$

We get

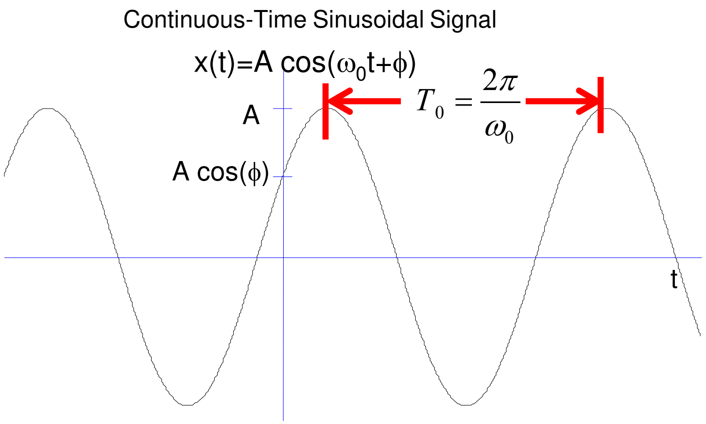
$$e^{j\omega_0 T} = \cos(2\pi) + j\,\sin(2\pi) = 1 + j0 = 1$$

Therefore, the signal $x(t)$ is a periodic signal.

Similarly, the signal $x(t) = e^{-j\omega_0 t}$ has the same fundamental period.

Sinusoidal Signal:

$$x(t) = A\cos(\omega_0 t + \phi)$$



Sinusoid Signals

$$A\cos(\omega_0 t + \phi) = A\left(\frac{e^{j(\omega_0 t+\phi)} + e^{-j(\omega_0 t+\phi)}}{2}\right) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

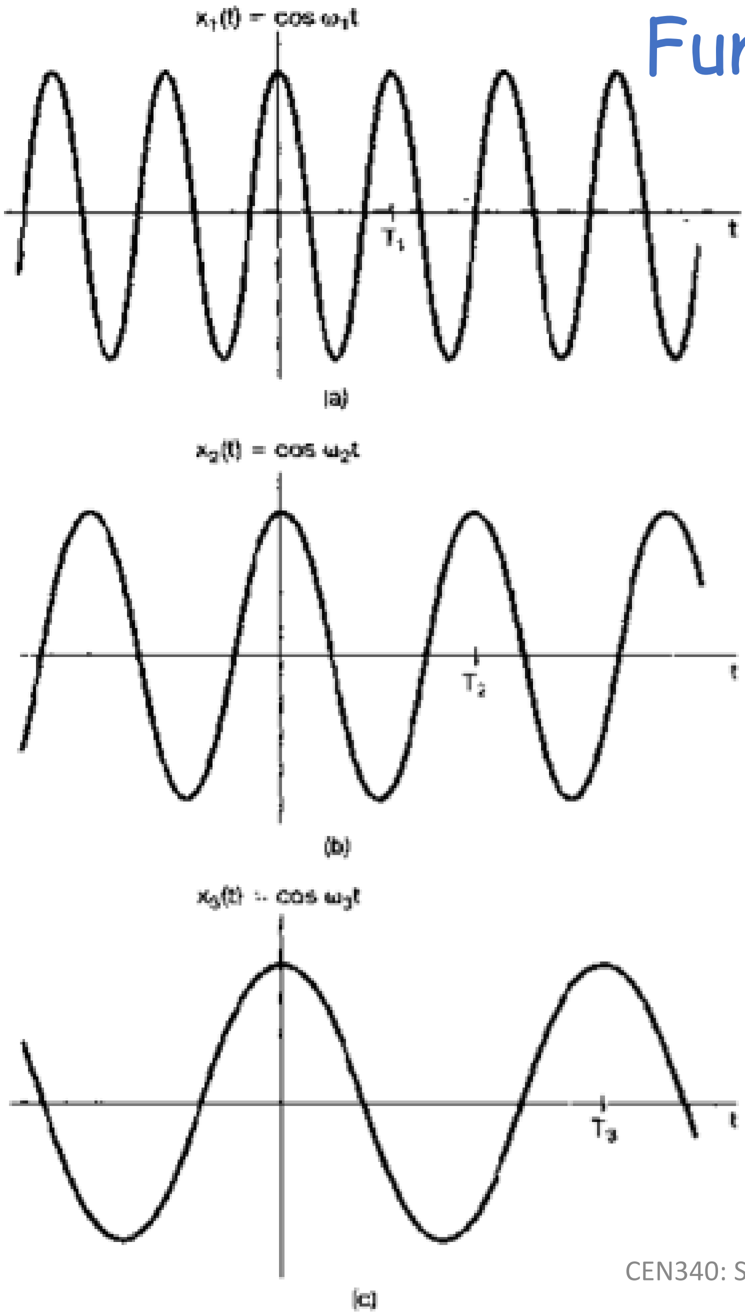
$$A\cos(\omega_0 t + \phi) = A\,\Re e\left\{e^{j(\omega_0 t+\phi)}\right\}$$

$$A\sin(\omega_0 t + \phi) = A\,\Im m\left\{e^{j(\omega_0 t+\phi)}\right\}$$

The fundamental period T_0 of a continuous-time sinusoidal or a periodic complex exponential signal, is inversely proportional to the $|\omega_0|$, which is called the *fundamental frequency*.

$$T_0 = \frac{2\pi}{|\omega_0|}$$

Fundamental Period and Frequency



If we decrease the value of the magnitude of ω_0, we slow down the rate of oscillations and hence increase the period T_0. Alternatively, if we increase the value of the magnitude of ω_0, we increase the rate of oscillations and hence decrease the period T_0.

$$\omega_1 > \omega_2 > \omega_3$$
$$T_1 < T_2 < T_3$$

Energy & Power of Sinusoid / Complex Exp Signals

Over the one fundamental period T_0 of a continuous-time sinusoidal or a periodic complex exponential signal, the signal energy and power can be determined as:

$$E_{period} = \int_0^{T_0} \left|e^{j\omega_0 t}\right|^2 dt = \int_0^{T_0} 1dt = T_0$$

$$P_{period} = \frac{1}{T_0}\int_0^{T_0} \left|e^{j\omega_0 t}\right|^2 dt = \frac{1}{T_0}\int_0^{T_0} 1dt = \frac{T_0}{T_0} = 1$$

As there are an infinite number of periods as t ranges from $-\infty$ to $+\infty$, the total energy integrated over all time is infinite. The total average power is however remains 1, as by definition,

$$P_\infty = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} \left|e^{j\omega_0 t}\right|^2 dt = \lim_{T\to\infty}\frac{1}{2T}2T = 1$$

Harmonics of a Periodic Complex Exponential

We have noted that, $e^{j\omega T_0} = 1$

which implies that ωT_0 is a multiple of 2π, i.e.,

$$\omega T_0 = 2\pi k \quad \text{where } k = 0,\ \pm 1,\ \pm 2,\ \cdots$$

This shows that ω must be an integer multiple of ω_0, i.e., the fundamental frequency. We can therefore, write

$$\phi_k(t) = e^{jk\omega_0 t} \quad \text{where } k = 0,\ \pm 1,\ \pm 2,\ \cdots$$

This is called the k-harmonic of the complex exponential signal.

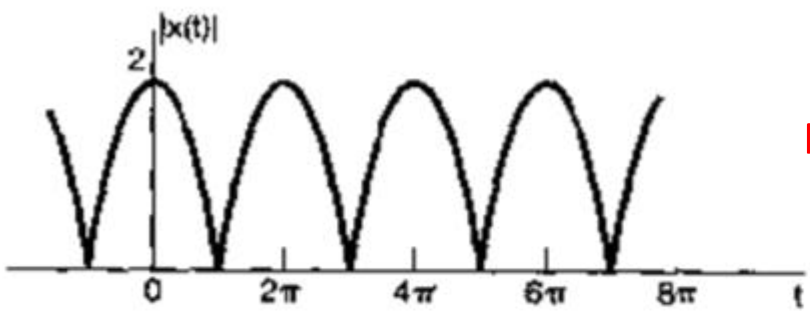
Expressing Two Complex Exponentials into a Product of One Complex Exp. & One Sinusoidal

$$x(t) = e^{j2t} + e^{j3t}$$

$$x(t) = e^{j2.5t}\left(e^{-j0.5t} + e^{j0.5t}\right) = 2e^{j2.5t}\cos(0.5t)$$

The magnitude of x(t) is:

$$|x(t)| = 2|\cos(0.5t)|$$



Full-wave rectified sinusoid.

General Complex Exponential Signals

The general complex exponential signals are of the form

$$x(t) = Ce^{at}$$

Where both C and a are complex numbers. Let us represent them as

$$C = |C|e^{j\theta} \qquad a = r + j\omega_0$$

Polar form   Cartesian form

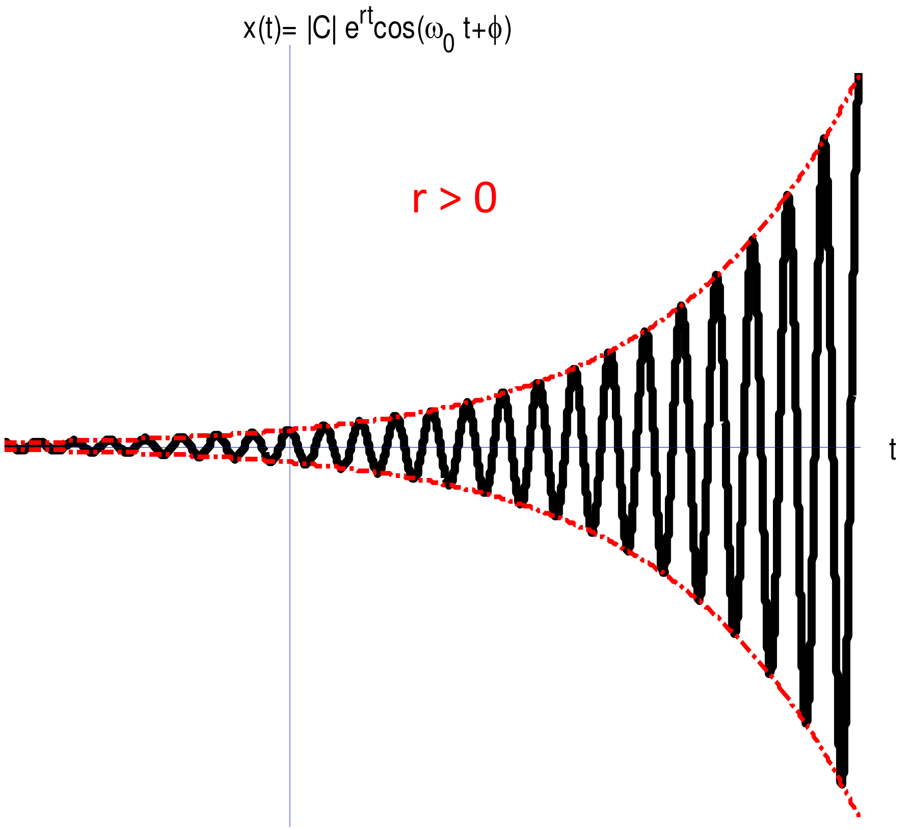
$$x(t) = Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t} = |C|e^{rt}e^{j(\omega_0 t+\theta)}$$

$$x(t) = Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta)$$

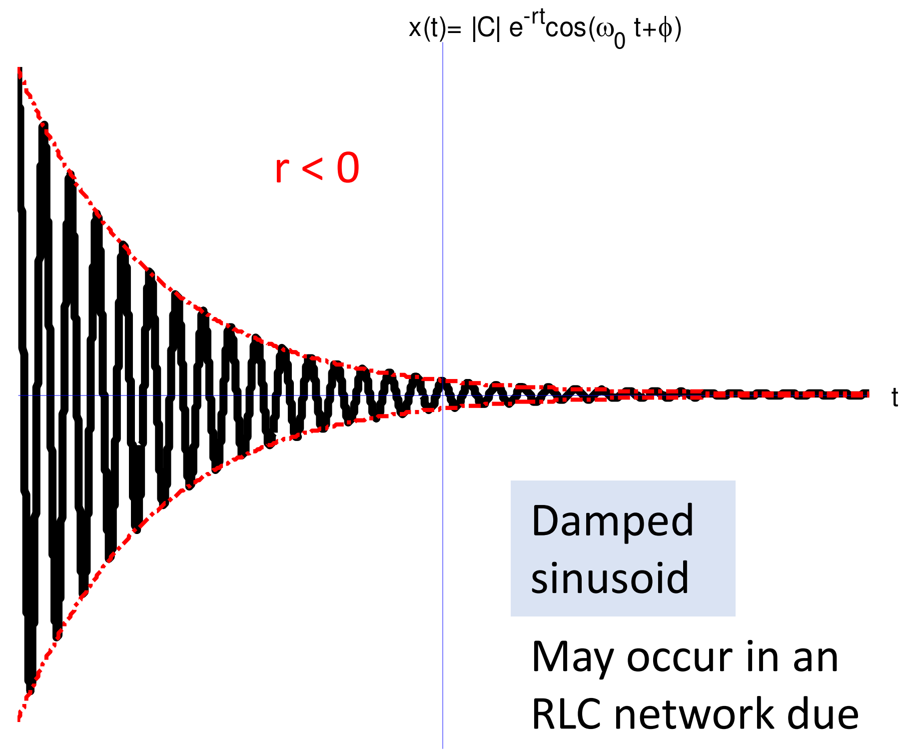
1. For $r = 0$, the real and imaginary parts of a complex exponential are sinusoidal.
2. For $r > 0$, they correspond to sinusoidal signals multiplied with growing exponential.
3. For $r < 0$, they correspond to sinusoidal signals multiplied with decreasing exponentials.

Example: General Complex Exponential Signals

Sinusoid with growing exponential



Sinusoid with decaying exponential



Damped sinusoid

May occur in an RLC network due to resistors

1.3.2 Discrete-Time Complex Exponential and Sinusoidal Signals

A discrete-time complex exponential signal or sequence $x[n]$ can be written as

$$x[n] = C\alpha^n$$

where C and α are, in general, complex numbers. This could also be written as

$$x[n] = Ce^{\beta n}$$

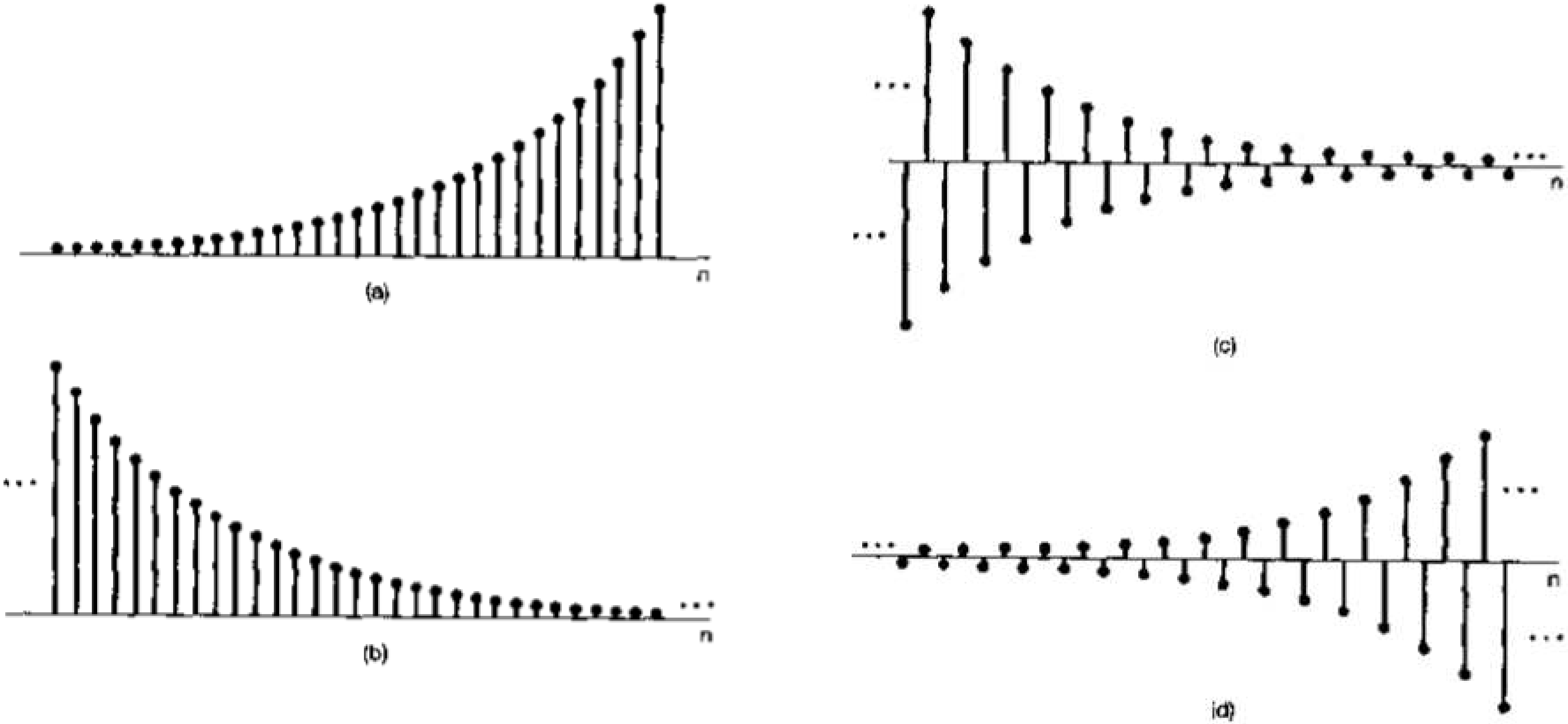
$$\text{where} \quad \alpha = e^{\beta}$$

Real Exponential Signals

In this case both C and α are real numbers, and $x[n]$ is called a real exponential.

USAGE: Real-valued discrete-time exponentials are often used to describe population growth as a function of generation, and total return on investment as a function of day, month, a quarter.

Example: Real Exponential Signals



$x[n] = C\alpha^n$

(a) $\alpha > 1$; (b) $0 < \alpha < 1$; (c) $-1 < \alpha < 0$; (d) $\alpha < -1$

What will happen if (i) $\alpha = 1$, and (ii) $\alpha = -1$?

Discrete-Time Sinusoid Signals

$$x[n] = e^{j\omega_0 n} = \cos(\omega_0 n) + j\sin(\omega_0 n)$$

Therefore, a discrete-time sinusoid signal can be written as:

$$A\cos(\omega_0 n + \phi) = A\left(\frac{e^{j(\omega_0 n+\phi)} + e^{-j(\omega_0 n+\phi)}}{2}\right) = \frac{A}{2}e^{j\phi}e^{j\omega_0 n} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 n}$$

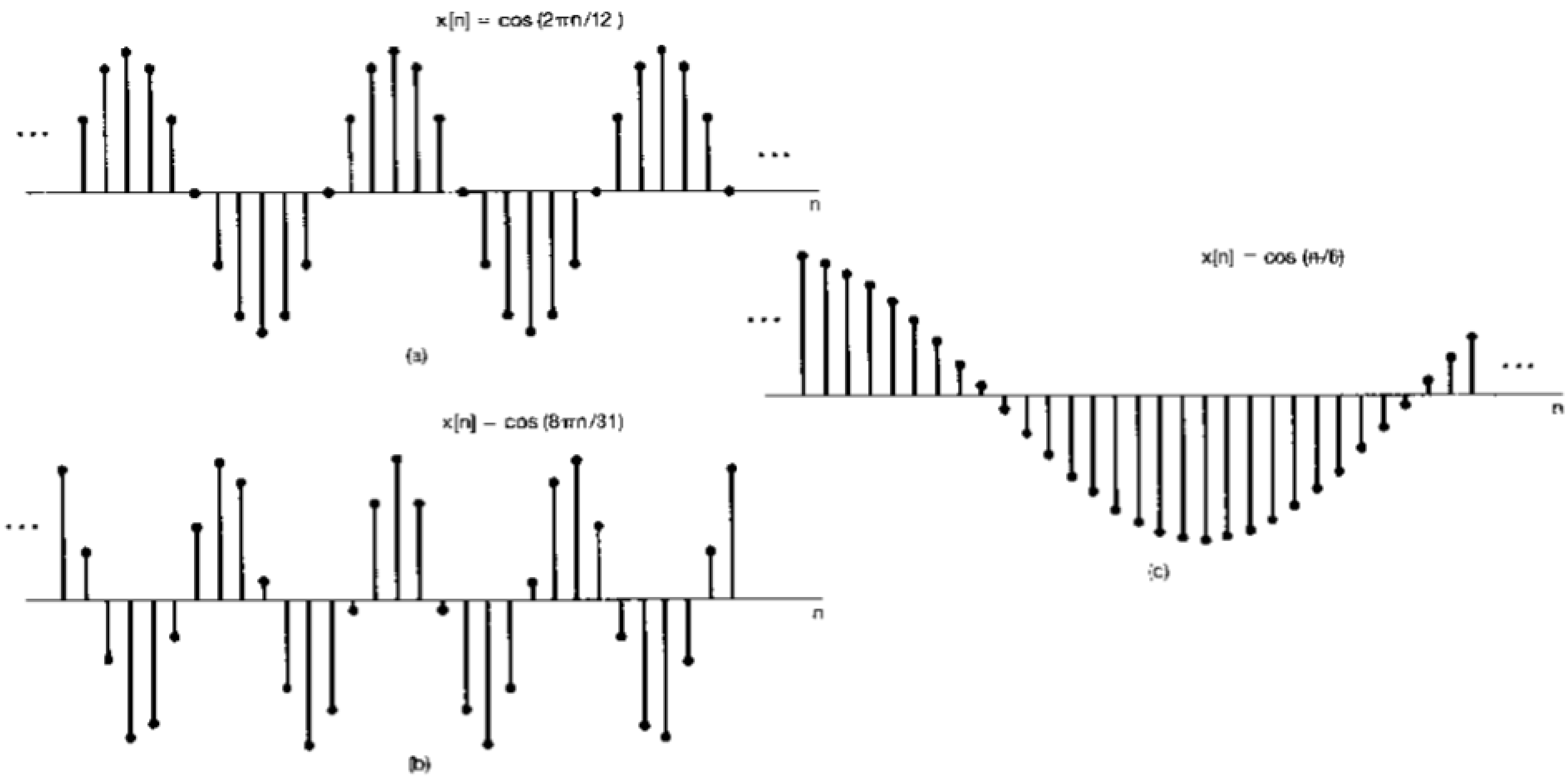
Using real and imaginary parts, we find:

$$A\cos(\omega_0 n + \phi) = A\,\Re e\left\{e^{j(\omega_0 n+\phi)}\right\}$$

$$A\sin(\omega_0 n + \phi) = A\,\Im m\left\{e^{j(\omega_0 n+\phi)}\right\}$$

Both the shaded signals have infinite total energy, but finite average power. For example, for every sample, $|e^{j\omega_0 n}|^2 = 1$, so it contributes to the total energy, making it infinite; however, per point time, the average power is 1.

Example: Discrete-Time Sinusoid Signals



Discrete-Time Complex Exponential Signals

The general discrete-time complex exponential signals are of the form

$$x[n] = C\alpha^n$$

where both C and α are complex numbers. Let us represent them as

$$\left.\begin{aligned} C &= |C|e^{j\theta} \\ \alpha &= |\alpha|e^{j\omega_0} \end{aligned}\right\} \quad \text{Polar form}$$

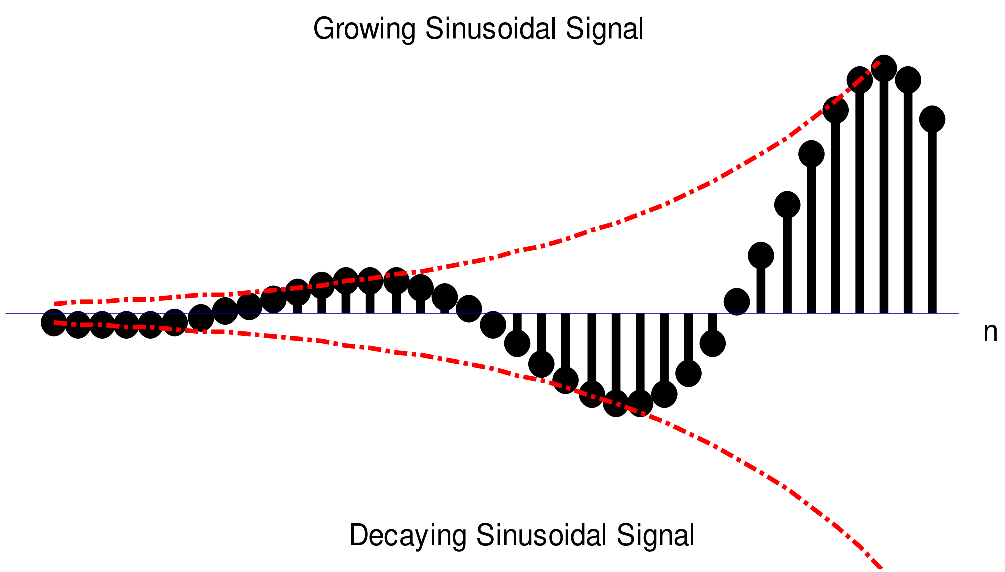
$$x[n] = C\alpha^n = |C|e^{j\theta}|\alpha|^n e^{j\omega_0 n} = |C||\alpha|^n e^{j(\omega_0 n+\theta)}$$

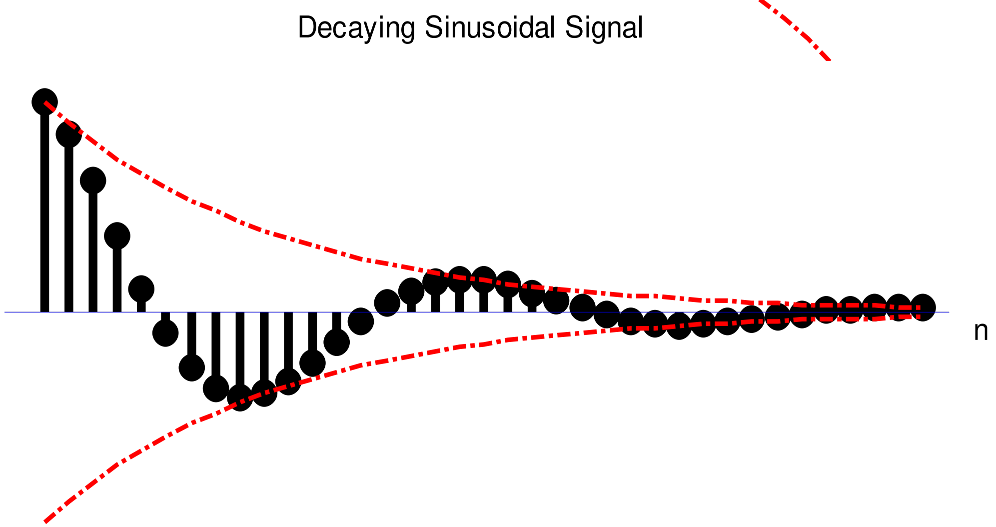
Using Euler's formula, it can be written as

$$x[n] = C\alpha^n = |C||\alpha|^n \cos(\omega_0 n + \theta) + j|C||\alpha|^n \sin(\omega_0 n + \theta)$$

1. For $|\alpha| = 1$, the real and imaginary parts of a complex exponential are sinusoidal.
2. For $|\alpha| > 1$, they correspond to sinusoidal signals / sequences multiplied with growing exponential.
3. For $|\alpha| < 1$, they correspond to sinusoidal signals / sequences multiplied with decreasing exponentials.

Discrete-Time Complex Exponential Signals





Discrete-Time Complex Exponential Signals

There are many similarities between continuous-time and discrete-time signals. But also there are many important differences. One of them is related with the discrete-time exponential signal $e^{j\omega_0 n}$

The following properties were found with regard to the continuous-time exponential signal $e^{j\omega_0 t}$:

1. The larger the magnitude of ω_0, the higher is the rate of oscillations in the signal;
2. $e^{j\omega_0 t}$ is periodic for any value of ω_0.

To see the difference for the first property, consider the discrete-time complex exponential:

$$e^{j(\omega_0+2\pi)n} = e^{j2\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}$$

This shows that the exponential at $\omega_0 + 2\pi$ is the same as that at frequency ω_0

Discrete-Time Complex Exponential Signals

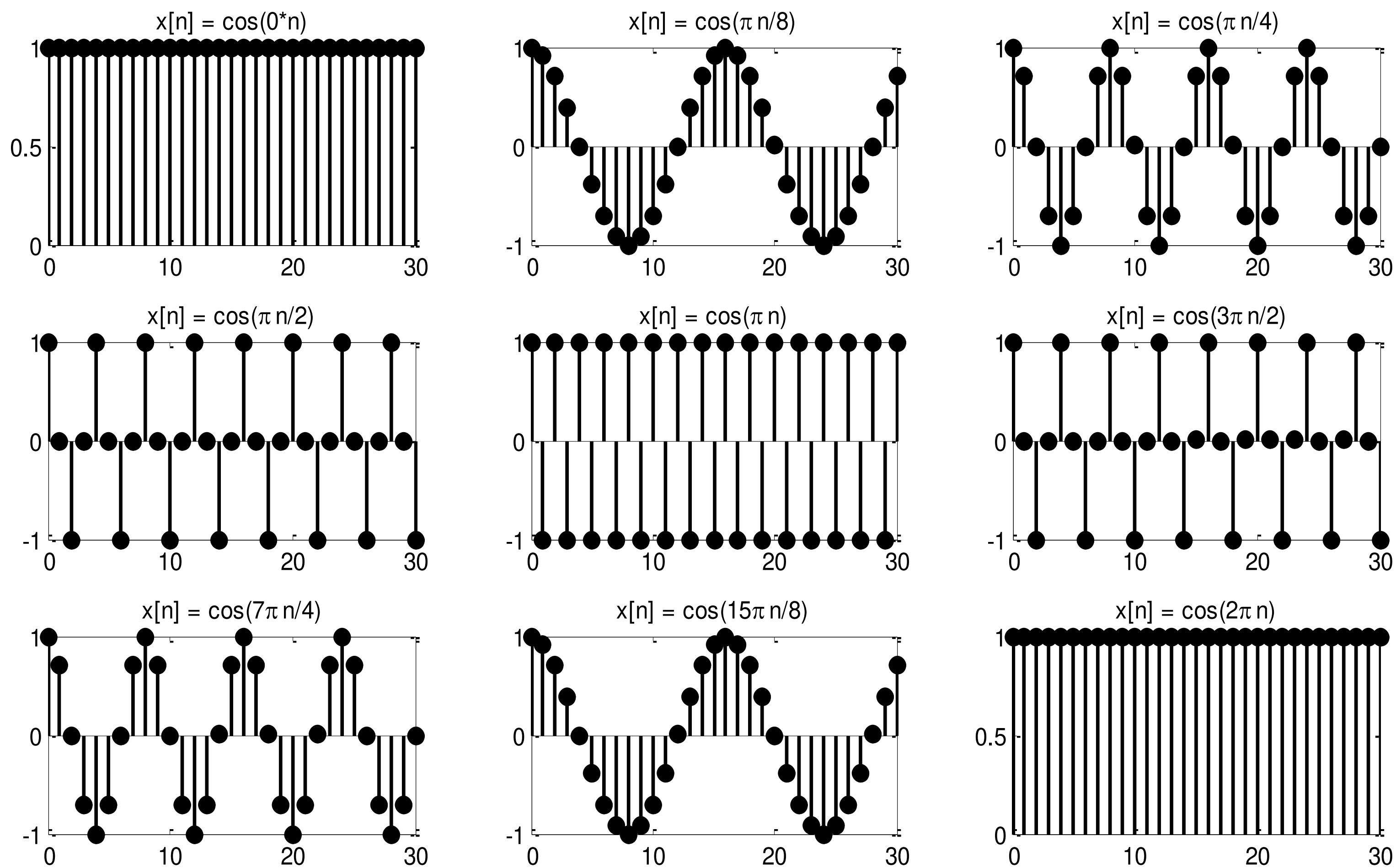
In case of continuous-time exponential, the signals $e^{j\omega_0 t}$ are all distinct for distinct values of ω_0.

- In discrete-time, these signals are not distinct. In fact, the signal with frequency ω_0 is identical to signals with frequencies $\omega_0 \pm 2\pi$, $\omega_0 \pm 4\pi$ and so on. Therefore, in considering discrete-time complex exponentials, we need only consider a frequency interval of size 2π. The most commonly used 2π intervals are $0 \leq \omega_0 \leq 2\pi$ or the interval $-\pi \leq \omega_0 \leq \pi$.
- As ω_0 is gradually increased, the rate of oscillations in the discrete-time signal does not keep on increasing. If ω_0 is increased from 0 to 2π, the rate of oscillations first increase and then decreases.
- Note in particular that for $\omega_0 = \pi$ or for any odd multiple of π,

$$e^{j\pi n} = \left(e^{j\pi}\right)^n = (-1)^n$$

so that the signal oscillates rapidly, changing sign at each point in time.

Discrete-Time Complex Exponential Signals



Start decreasing from here

Periodicity of Discrete-Time Complex Exponential Signals

$$e^{j\omega_0(n+N)} = e^{j\omega_0 n} \rightarrow e^{j\omega_0 N} = 1$$

It is true if $\omega_0 N$ is a multiple of 2π

$$\omega_0 N = 2\pi m \rightarrow \frac{\omega_0}{2\pi} = \frac{m}{N}$$

It means that the discrete-time signal $e^{j\omega_0 n}$ is periodic only when $\frac{\omega_0}{2\pi}$ is a rational number.

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct values of ω_0.	Identical signals for values of ω_0 separated by multiples of 2π.
Periodic for any choice of ω_0.	Periodic only if $\omega_0 = 2\pi m/N$ for some integer $N > 0$ and m.
Fundamental frequency ω_0.	Fundamental frequency ω_0/m.
Fundamental period **$\omega_0 = 0$: undefined** **$\omega_0 \neq 0$: $\frac{2\pi}{\omega_0}$**	Fundamental period $\omega_0 = 0$: undefined $\omega_0 \neq 0$: $m\left(\frac{2\pi}{\omega_0}\right)$

Periodicity; Workout - (1)

Find fundamental period of the signal: $x[n] = e^{j(2\pi/3)n} + e^{j(3\pi/4)n}$

The first term

$$\omega_0 = 2\pi / 3$$

$$N = m\left(\frac{2\pi}{\omega_0}\right) = m\left(\frac{2\pi}{2\pi/3}\right)$$

$$\frac{N}{m} = \frac{3}{1} \Rightarrow N = 3$$

The second term

$$\omega_0 = 3\pi / 4$$

$$N = m\left(\frac{2\pi}{\omega_0}\right) = m\left(\frac{2\pi}{3\pi/4}\right)$$

$$\frac{N}{m} = \frac{8}{3} \Rightarrow N = 8$$

LCM(3, 8) = 24

Therefore, the fundamental period = 24

Cartesian to Polar & Vice Versa; Workout - (2)

1.1

$$\frac{1}{2}e^{j\pi}$$
$$= (1/2)\left(\cos\pi + j\sin\pi\right)$$
$$= (1/2)(-1 + j\,0)$$
$$= -(1/2) + j\,(0)$$

$$\frac{1}{2}e^{-j\pi}$$
$$= (1/2)\left(\cos(-\pi) + j\sin(-\pi)\right)$$
$$= (1/2)(-1 + j\,0)$$
$$= -(1/2) + j\,(0)$$

1.2

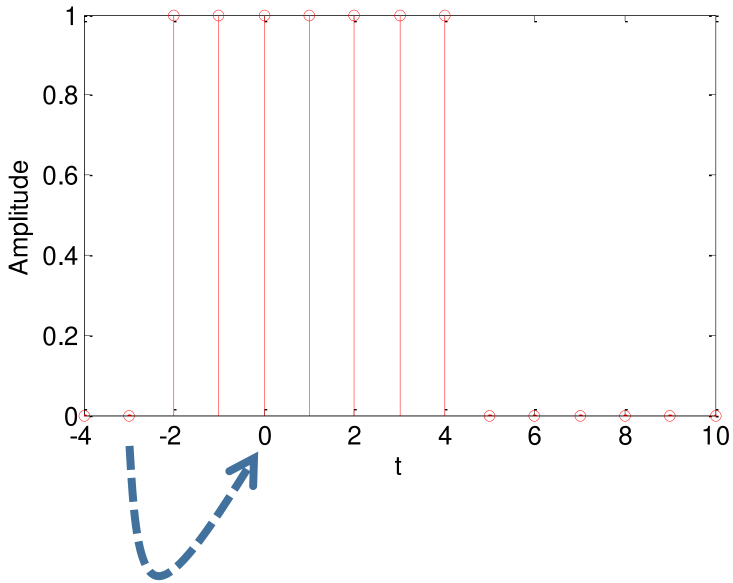
$$-3j$$
$$Ce^{-j\theta} = C\cos\theta - Cj\sin\theta = 0 - 3j\,(1)$$
$$\therefore C = 3$$
$$\sin\theta = 1 \Rightarrow \theta = \pi/2$$
$$-3j = 3e^{-j\pi/2}$$

1.4 (a)

Workout - (3)

Let, x[n] be a signal with x[n] = 0 for n < -2 and n > 4. For a signal x[n-3], determine the value of *n* for which it is guaranteed to be zero.

x[n-3] means shifting the signal towards right by 3 samples.



$n < -2 \rightarrow n+3 < -2+3\ (=1)$

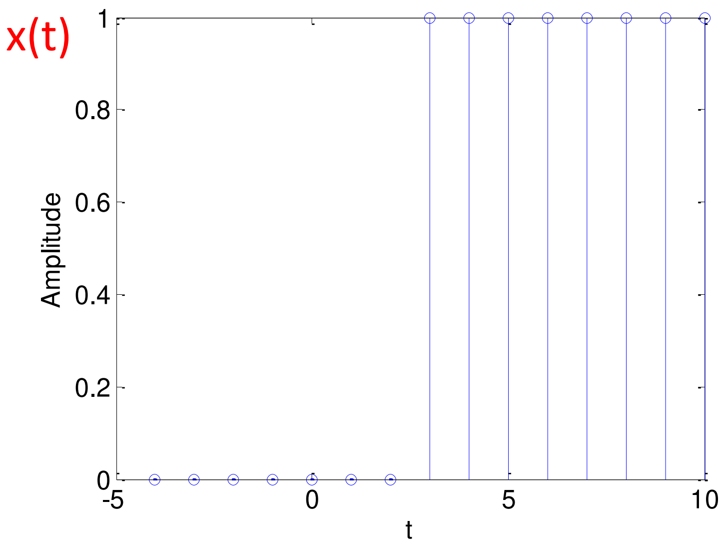
$n > 4 \rightarrow n+3 > 4+3\ (=7)$

The shifted signal will be zero for n < 1 and n > 7.

1.5 (a)

Workout - (4)

Let x(t) be a signal with x(t) = 0 for t < 3. For the signal x(1 – t), determine the value of t for which it is guaranteed to be zero.



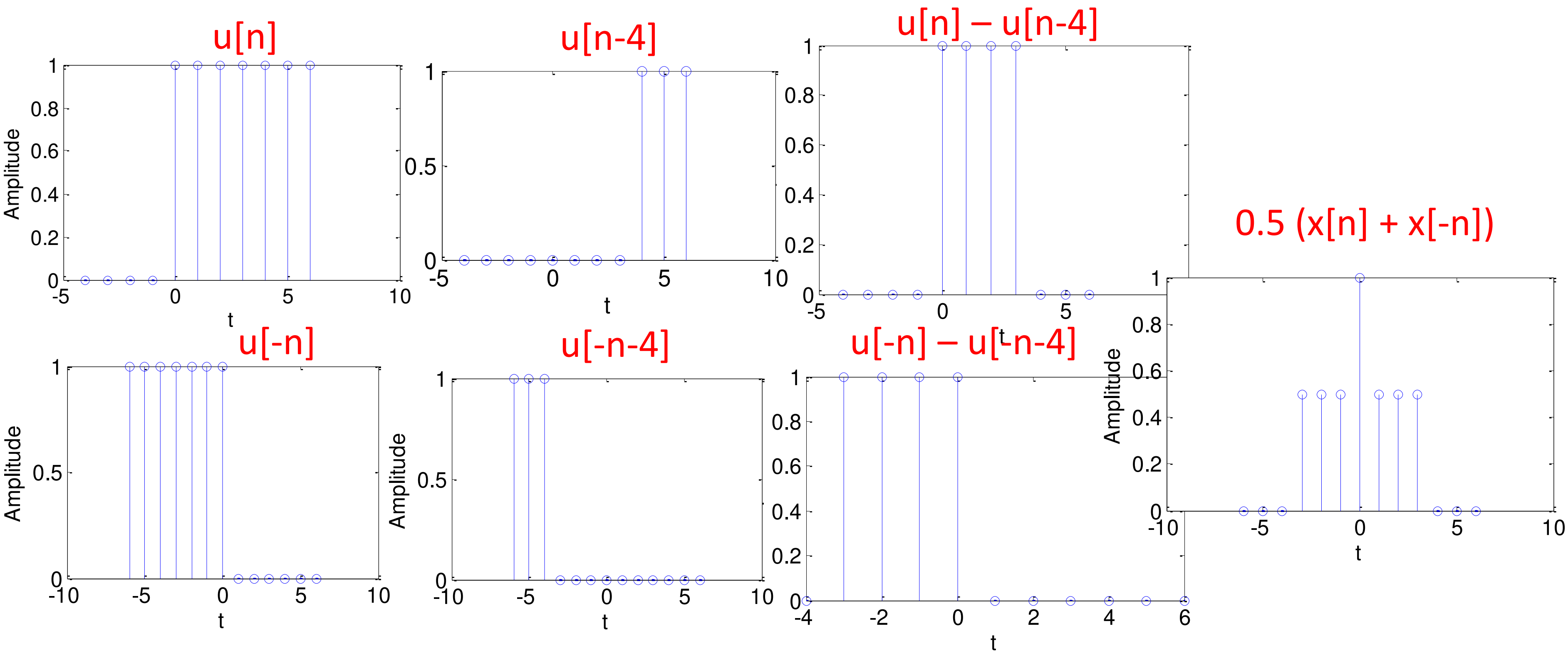
For t > -2, the signal is zero.

1.7 (a)

Workout - (5)

For a signal x[n] = u[n] – u[n-4], determine the values of the independent variable at which the even part of the signal is guaranteed to be zero.

EVEN{ x[n]} = 0.5 (x[n] + x[-n]) = 0.5 (u[n] – u[n-4] + u[-n] – u[-n-4])

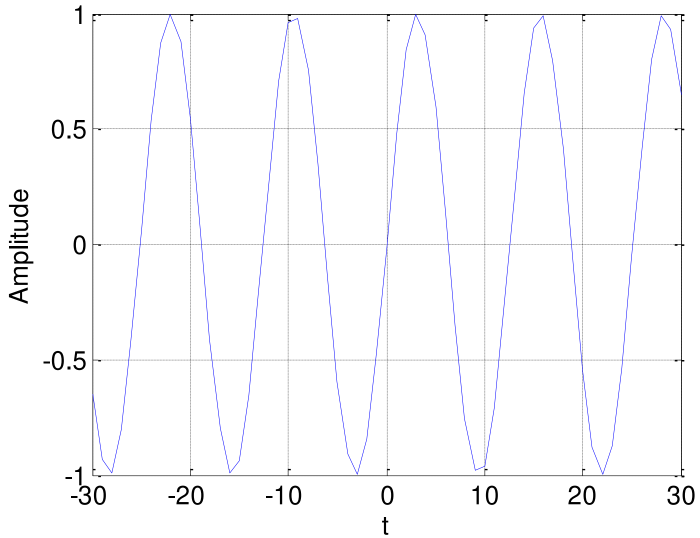


Zero for n > 3 and n <-3

1.7 (b)

Workout – (6)

For a signal x(t) = sin(0.5t), determine the values of the independent variable at which the even part of the signal is guaranteed to be zero.



It is always an odd signal, so the even part is zero for all values of t.

1.8 (a)

Workout - (7)

Express the real part of the signal, x(t) = -2, in the form $Ae^{-at}\cos(\omega t+\phi)$, where A, a, ω, and ϕ are real numbers with A > 0, and $-\pi < \phi < \pi$.

$$x(t) = A\,e^{-at}\cos(\omega t + \phi) = -2 = 2 \times 1 \times (-1) = 2\,e^{-0t}\cos(0t + \pi)$$

A = 2, a = 0, ω = 0, and $\phi = \pi$

The above problem when the signal is $x(t) = \sqrt{2}e^{j\pi/4}\cos(3t + 2\pi)$

$$x(t) = \sqrt{2}e^{j\pi/4}\cos(3t + 2\pi) = \sqrt{2}\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)\cos(3t + 2\pi)$$

$$\text{Real part} = \sqrt{2}\cos\frac{\pi}{4}\cos(3t + 2\pi) = \sqrt{2}\times\frac{1}{\sqrt{2}}\times\cos 3t = \cos 3t$$

$$= 1\times e^{0t}\times\cos(3t + 0)$$

A = 1, a = 0, ω = 3, and ϕ = 0

1.9 (a)

Workout - (8)

If the signal x(t) is periodic, find the fundamental period.

$$x(t) = je^{j10t}$$
$$= j(\cos 10t + j\sin 10t) = j\cos 10t - \sin 10t$$
$$= j\sin\left(10t + \pi/2\right) + \cos\left(10t + \pi/2\right)$$
$$= e^{\left(10t + \pi/2\right)}$$

Fundamental period: $$T_0 = \frac{2\pi}{|\omega_0|} = \frac{2\pi}{10} = \frac{\pi}{5}$$

1.10

Workout - (9)

If the signal x(t) is periodic, find the fundamental period.

$$x(t) = 2\cos(10t+1) - \sin(4t-1)$$

$$T_0 = \frac{2\pi}{10} = \frac{\pi}{5} \qquad T_0 = \frac{2\pi}{4} = \frac{\pi}{2}$$

Fundamental period:

LCM ($\pi/5$, $\pi/2$) = LCM (π, π) / HCF (5, 2) = $\pi/1$ = π

$$LCM\left(\frac{a}{b},\frac{c}{d}\right) = LCM(a,c)/HCF(b,d)$$

1.11

Workout - (10)

Determine the fundamental period of the following signal x[n].

$$x[n] = 1 + e^{j4\pi n/7} - e^{j2\pi n/5}$$

$$N_0 = m\left(\frac{2\pi}{\omega_0}\right)$$

$$N_0(\text{first part}) = 1$$

$$N_0(\text{second part}) = m\left(\frac{2\pi}{4\pi/7}\right) = m(7/2) = 7$$

$$N_0(\text{third part}) = m\left(\frac{2\pi}{2\pi/5}\right) = m(5/2) = 5$$

$$N_0 = LCM(1,7,5) = \boxed{35}$$

Acknowledgement

The slides are prepared based on the following textbook:

- Alan V. Oppenheim, Alan S. Willsky, with S. Hamid Nawab, *Signals & Systems*, 2^{nd} Edition, Prentice-Hall, Inc., 1997.

Special thanks to

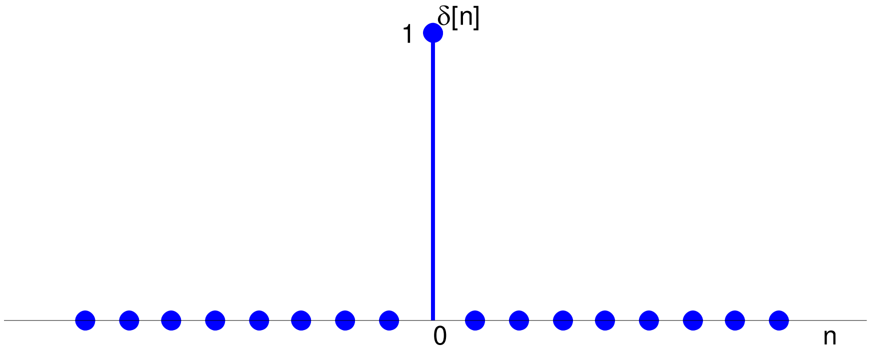
- *Prof. Anwar M. Mirza*, former faculty member, College of Computer and Information Sciences, King Saud University
- *Dr. Abdul Wadood Abdul Waheed*, faculty member, College of Computer and Information Sciences, King Saud University

1.4 Unit Step & Unit Impulse Functions

1.4.1 The Discrete-Time Unit Impulse and Unit-Step Sequences

Unit Impulse Function:

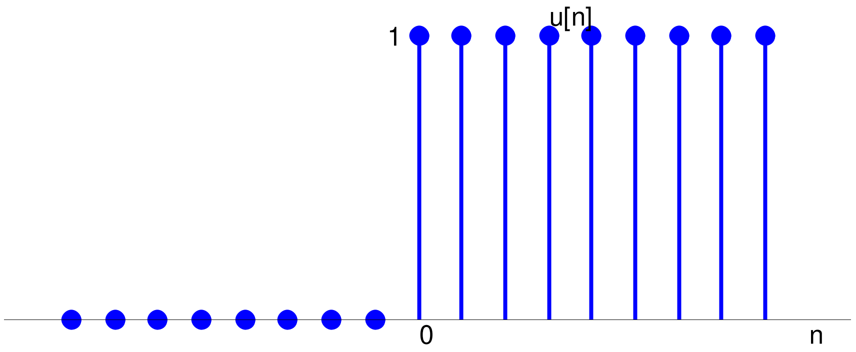
$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



Unit Step & Unit Impulse

Unit Step Function:

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



There is a close relationship between the discrete-time unit impulse and unit step signals. The discrete-time unit impulse can be written as the first-difference of the discrete-time unit step

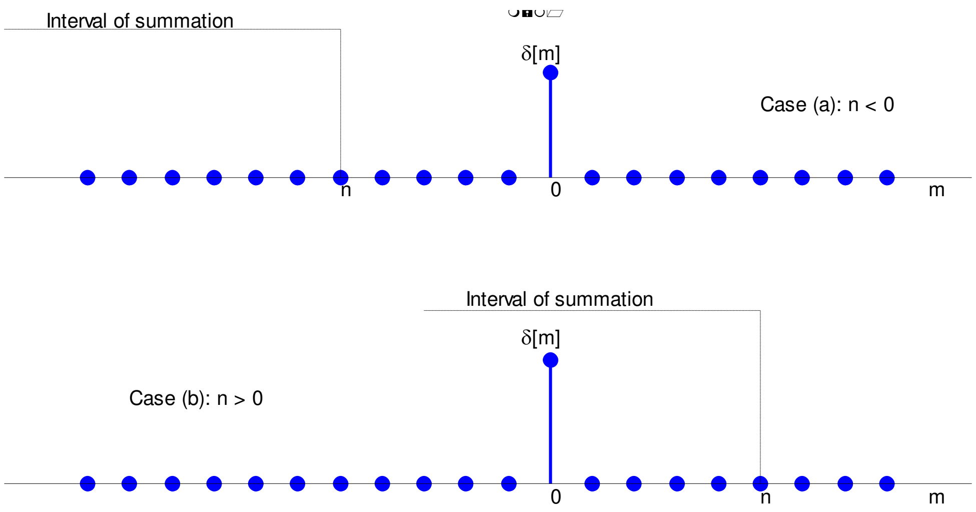
$$\delta[n] = u[n] - u[n-1]$$

Conversely, the discrete-time unit step is the running sum of the unit sample

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

Unit Step & Unit Impulse – contd.

0 for $n < 0$ and 1 for $n \geq 0$



The unit impulse sequence can be used to sample the value of a signal at $n = 0$. In particular, since $\delta[n]$ is non-zero (and equal to 1) only for $n = 0$, therefore

$$x[n]\delta[n] = x[0]\delta[n]$$

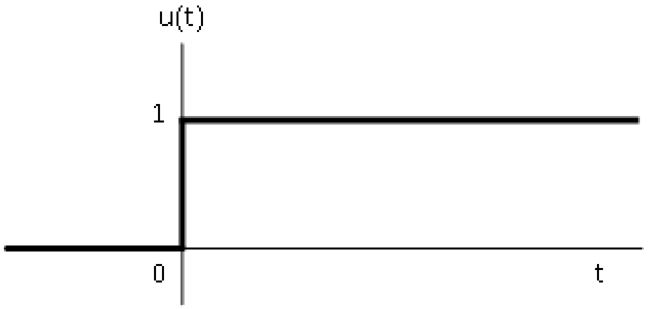
More generally, if we consider a unit impulse $\delta[n - n_0]$ at $n = n_0$, then

$$x[n]\delta[n - n_0] = x[n_0]\delta[n - n_0]$$

1.4.2 The Continuous-Time Unit Impulse and Unit-Step Sequences

The continuous-time unit step function, denoted by $u(t)$ is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



The unit step can be written as the running integral of the unit impulse,

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

The unit impulse in the continuous-time can be written as the first derivative of the unit step in continuous time

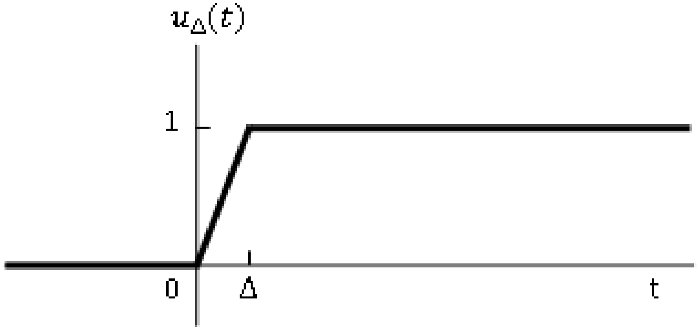
$$\delta(t) = \frac{du(t)}{dt}$$

The Continuous-Time Unit Impulse and Unit-Step Sequences

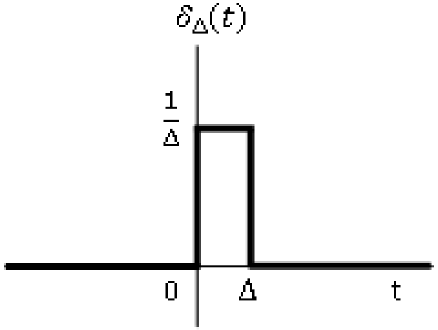
We notice that u(t) is discontinuous at t=0 (and consequently cannot be differentiated at t=0), therefore, there is some formal difficulty with this equation in the previous slide.

Therefore, we interpret equation by considering an approximation to the unit step $u_\Delta(t)$ in which the function rises from 0 to 1 in a short time interval of length Δ. The step function $u(t)$ can be considered as an idealization of $u_\Delta(t)$ for Δ so short that its duration doesn't matter for any practical purpose. More formally, $u(t)$ is the limit of $u_\Delta(t)$ as $\Delta \rightarrow 0$.

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}$$



Continuous-time approximation to the unit step function, $u_\Delta(t)$

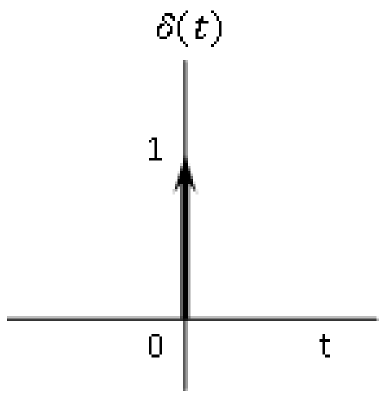


Derivative of $u_\Delta(t)$

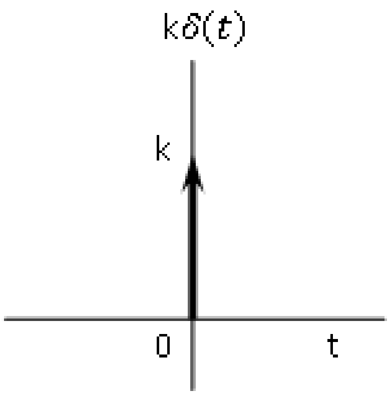
Continuous-Time Unit Impulse

It could be noticed that $\delta_\Delta(t)$ is short pulse of duration Δ and with unit area for any value of Δ. If we gradually decrease the value of Δ, the pulse will become narrower and the height will increase (to maintain the area to unity). Therefore, in the limiting case, we can write

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$



Continuous-time unit impulse



Continuous-time scaled impulse

$$\int_{\tau=-\infty}^{\tau=t} k\delta(\tau)d\tau = ku(t)$$

1.12

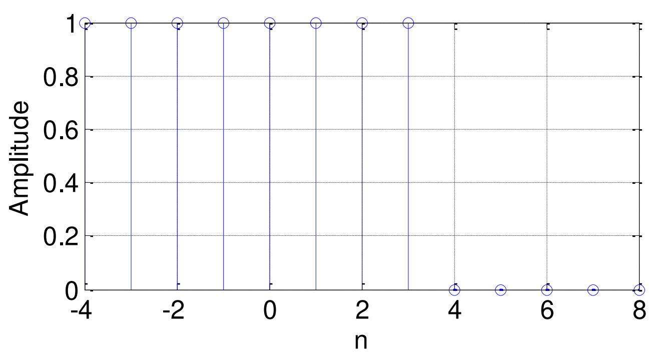
Workout - (11)

For $x[n]=1-\sum_{k=3}^{\infty}\delta[n-1-k]$ Determine the values of M and n_0 so that $x[n]=u[M_n-n_0]$

$$x[n]=1-\left(\delta[n-1-3]+\delta[n-1-4]+\delta[n-1-5]+\ldots+\delta[n-1-\infty]\right)$$
$$=1-\left(\delta[n-4]+\delta[n-5]+\delta[n-6]+\ldots+\delta[n-\infty]\right)$$

$$x[-4]=1-\left(\delta[-8]+\delta[-9]+\delta[-10]+\ldots\right)=1-(0+0+0+\ldots)=1$$
$$x[0]=1-\left(\delta[-4]+\delta[-5]+\delta[-6]+\ldots\right)=1-(0+0+0+\ldots)=1$$
$$x[4]=1-\left(\delta[0]+\delta[-1]+\delta[-2]+\ldots\right)=1-(1+0+0+\ldots)=0$$
$$x[5]=1-\left(\delta[1]+\delta[0]+\delta[-1]+\ldots\right)=1-(0+1+0+\ldots)=0$$

...



u[n] is shifted by +3, and then reflected.

$$x[n]=u[M_n-n_0]=u[-n+3]$$

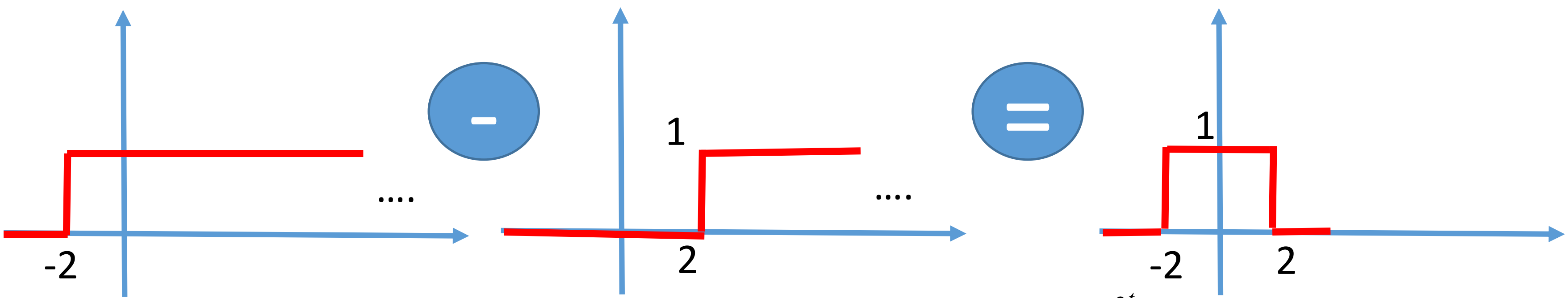
Therefore, M_n = -1 and n_0 = -3

1.13

Workout - (12)

Given $x(t) = \delta(t+2) - \delta(t-2)$ Calculate E_∞ for $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau = \int_{-\infty}^{t} \big(\delta(\tau+2) - \delta(\tau-2)\big)\, d\tau$$



$$\int_{-\infty}^{t} \big(\delta(\tau+2)\big)\, d\tau \quad - \quad \int_{-\infty}^{t} \big(\delta(\tau-2)\big)\, d\tau \quad = \quad \int_{-\infty}^{t} \big(\delta(\tau+2) - \delta(\tau-2)\big)\, d\tau$$

$$E_\infty = \int_{-2}^{2} |1|^2\, dt = 2 - (-2) = 4$$

1.14

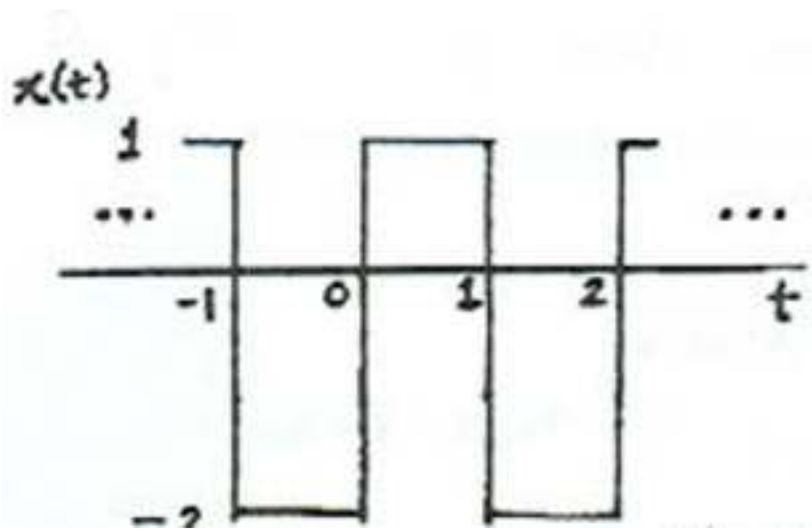
Workout - (13)

Consider a periodic signal with period T = 2: $x(t)=\begin{cases}1, & 0\le t\le 1\\ -2, & 1<t<2\end{cases}$

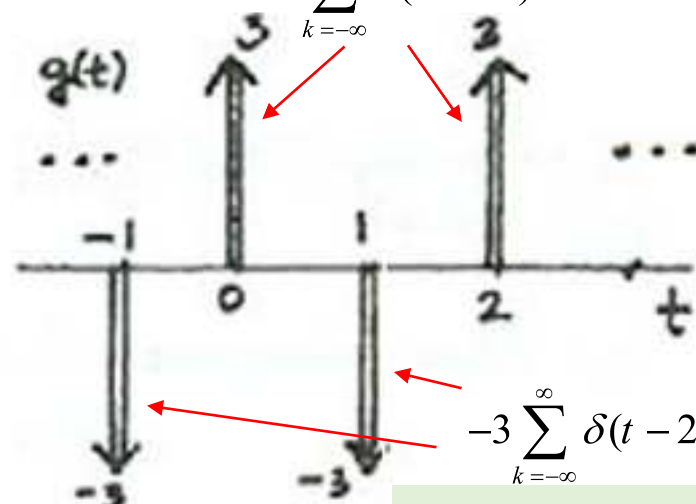
The derivative of this signal is related to the “impulse train” with period T = 2: $g(t)=\sum_{k=-\infty}^{\infty}\delta(t-2k)$

It can be shown that $$\frac{dx(t)}{dt}=A_1 g(t-t_1)+A_2 g(t-t_2)$$

Determine the values of A_1, t_1, A_2 and t_2.



$$3\sum_{k=-\infty}^{\infty}\delta(t-2k)$$



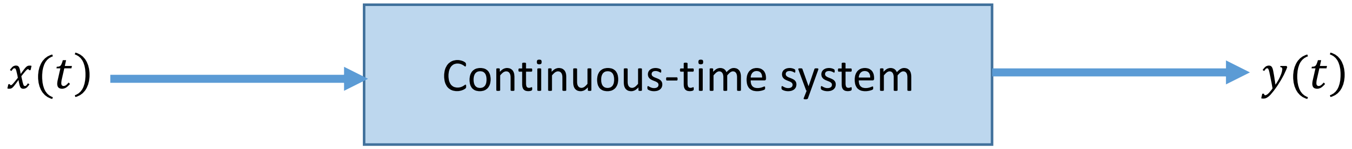
$$-3\sum_{k=-\infty}^{\infty}\delta(t-2k-1)$$

$A_1=3$, $t_1=0$, $A_2=-3$, $t_2=1$

1.5 Continuous-Time Discrete-Time Systems

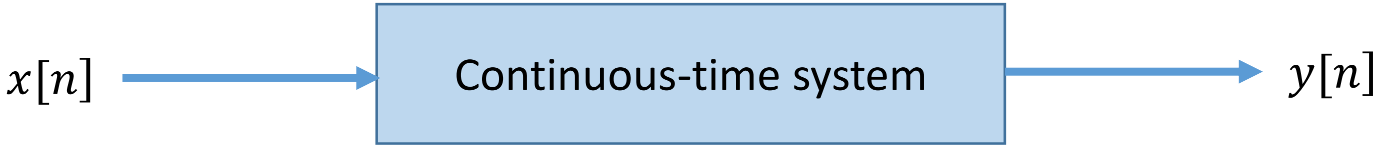
A **continuous-time system** is a system in which continuous-time input signals are applied and result in continuous-time output signals. The input-output relation of such systems can be represented by the notation:

$$x(t) \rightarrow y(t)$$

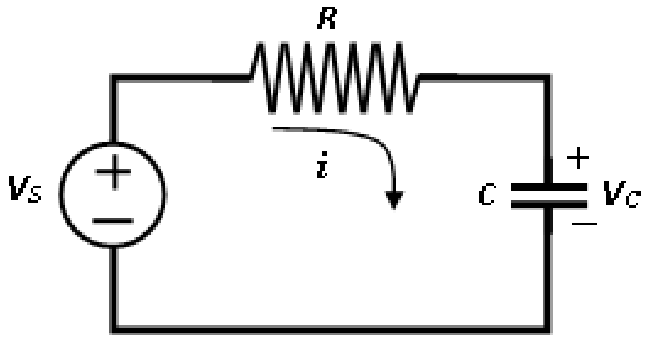


A **discrete-time system** is a system in which discrete-time input signals are applied and result in discrete-time output signals. The input-output relation of such systems can be represented by the notation:

$$x[n] \rightarrow y[n]$$



Simple Example of Systems



A simple ***RC*** circuit with source $\boldsymbol{V_s}$ and capacitor voltage $\boldsymbol{V_c}$

$$i(t) = \frac{v_s(t) - v_c(t)}{R}$$

$$i(t) = C\frac{dv_c(t)}{dt}$$

The differential equation giving a relationship between the input $v_s(t)$ and the output $v_c(t)$,

$$\frac{dv_c(t)}{dt} + \frac{1}{RC}v_c(t) = \frac{1}{RC}v_s(t)$$

Simple Example of Systems - contd.

An automobile responding to an applied force ***f*** from the engine and to a retarding force ***ρv*** proportional to the automobile's velocity ***v***.

Here we regard the force $f(t)$ as the input and velocity $v(t)$ as the output. If we let m denote the mass of the automobile and $m\rho v$ the resistance due to friction, then equating acceleration i.e. the time derivative of velocity, with net force divided by mass, we get

$$\frac{dv(t)}{dt} = \frac{1}{m}[f(t) - \rho v(t)] \quad \Rightarrow \quad \frac{dv(t)}{dt} + \frac{\rho}{m}v(t) = \frac{1}{m}f(t)$$

Example 1: Systems

As a simple example of a discrete-time system, consider a simple model for the balance in a bank account from month to month. Specifically, let $y[n]$ denote the balance at the end of the n-th month, and suppose that $y[n]$ evolves from month to month according to the equation

$$y[n] = y[n-1] + 0.01y[n-1] + x[n] = 1.01y[n-1] + x[n]$$

Or,

$$y[n] - 1.01y[n-1] = x[n]$$

Where $x[n]$ represents the net deposit (i.e., deposits minus withdrawals) during the month and the term $1.01y[n-1]$ is the 1% profit each month.

Example 2: Systems

As a second example, consider the digital simulation of the differential equation in which we resolve the time into discrete intervals of length Δ and approximate the derivative $dv(t)/dt$ at $t = n\Delta$ by the first backward difference, i.e.,

$$\frac{v(n\Delta) - v((n-1)\Delta)}{\Delta}$$

In this case, if we let

$$v[n] = v(n\Delta) \quad \text{and} \quad f[n] = f(n\Delta)$$

We obtain the following discrete-time model relating the sampled signals $f[n]$ and $v[n]$:

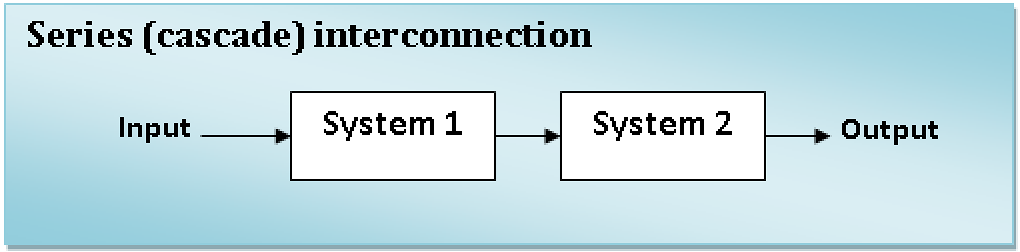
$$v[n] - \frac{m}{(m+\rho\Delta)} v[n-1] = \frac{\Delta}{(m+\rho\Delta)} f[n]$$

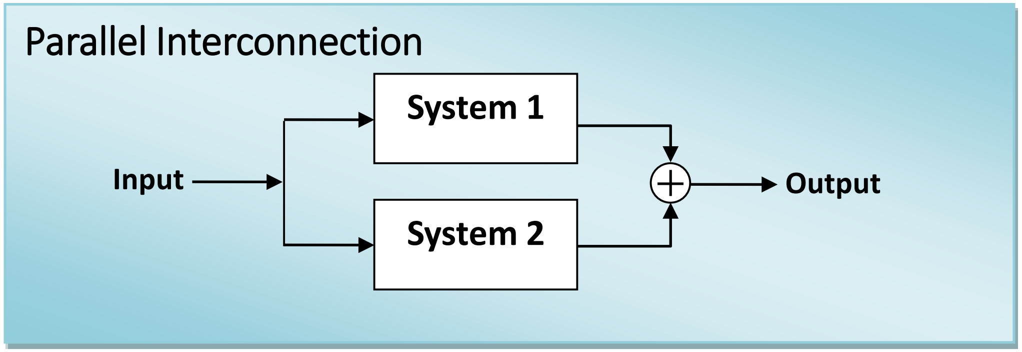
Interconnection of Systems

Many real systems are built as interconnections of several subsystems.

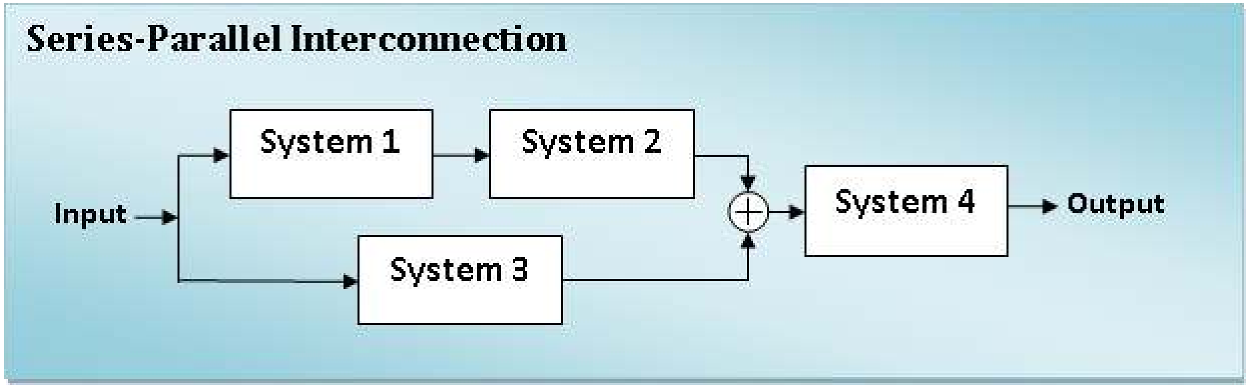
For example, a modern digital telephone system involves the interconnections of a microphone receiver, audio to digital converter, a transmitter, a receiver, a digital to audio convertor and one or more speakers (apart from several other sub-systems).

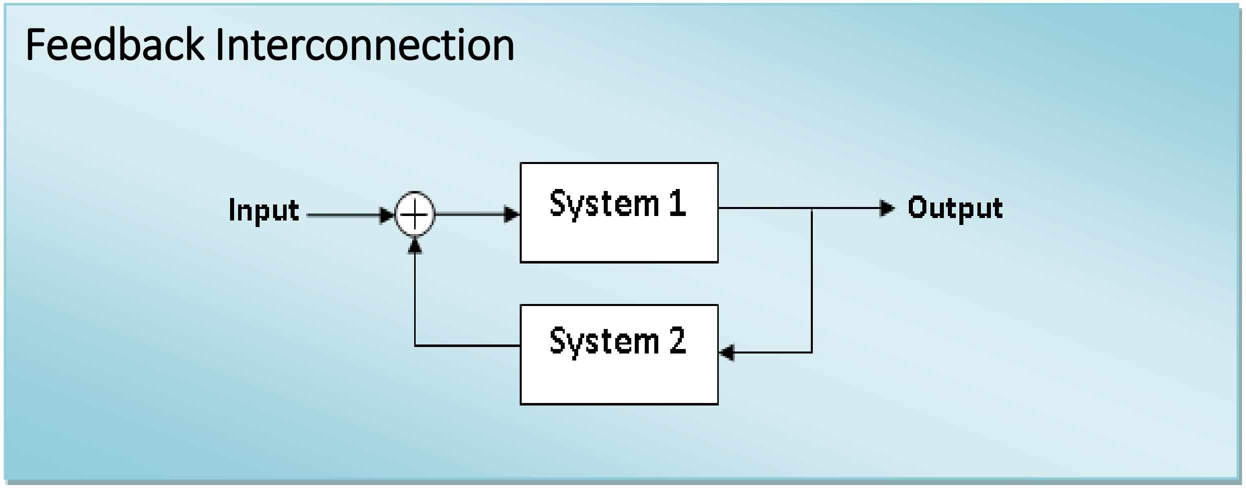
There are several basic system interconnections that are encountered more frequently:





Interconnection of Systems - contd.





1.6 Basic System Properties

1.6.1 Systems with and without memory

A system is said to be *memoryless* if its output for each value of the independent variable at a given time is dependent on the input at only that same time.

For example, the system specified by the relationship

$$y[n] = (2x[n] - x^2[n])^2$$

is memoryless, as the value of $y[n]$ at any particular time n_0 depends on the value of $x[n]$ only at that time, i.e. $x[n_0]$.

As a particular case, a resistor can be considered as a memoryless system: with the input $x(t)$ taken as the current and with voltage taken as the output $y(t)$, the input-output relationship for a resistor is,

$$y(t) = Rx(t)$$

where R is the resistance.

Systems with and without memory

Another particular, simple memoryless system is the ***identity system***, whose output is identical with the input. That is, the input-output relationship for the continuous-time identity system is

$$y(t) = x(t)$$

and the corresponding relationship in discrete-time is

$$y[n] = x[n]$$

An example of a discrete-time system with memory is an ***accumulator*** or ***summer***

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

and a second example is a delay

$$y[n] = x[n-1]$$

Systems with memory

A capacitor is an example of a continuous-time system with memory; since if the input is taken to be the current and the output is the voltage, then

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$

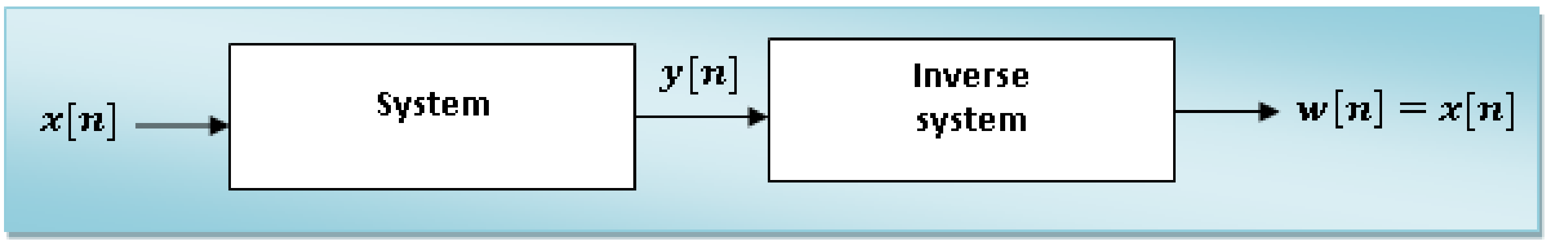
where C is the capacitance

Examples:

- Delay
- Accumulator
- Storage of energy
- Memory dependent on the future values of the input and the output

Invertibility and Inverse Systems

A system is said to be invertible if distinct inputs lead to distinct outputs.



Concept of an inverse system for a general invertible system

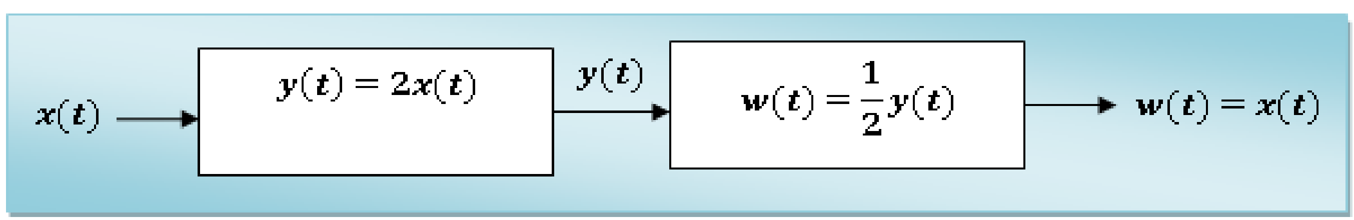
If a system is invertible, than an inverse system exists that, when cascaded with the original system, yields an output $w[n]$ equal to the input $x[n]$ to the first system.

System

$$y(t) = 2x(t)$$

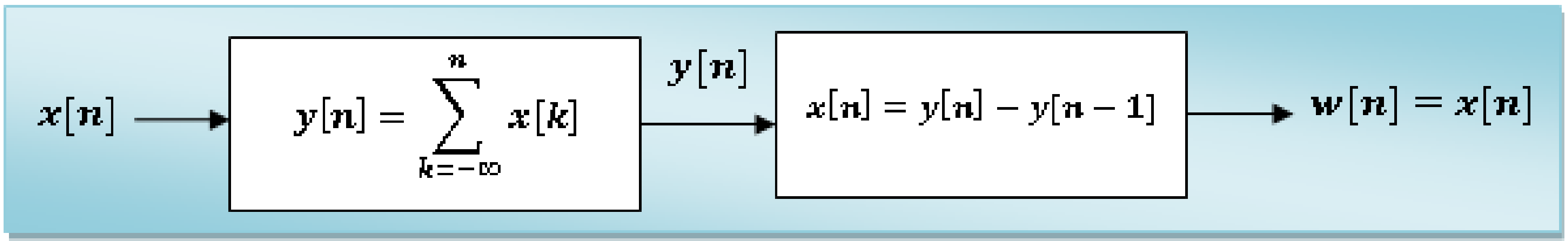
Inverse System

$$x(t) = \frac{1}{2}y(t)$$



Example: Invertible Systems

Accumulator



Example: Non-invertible Systems

- A system that produces a zero output sequence for any input sequence.

$$y(t) = 0$$

- A system where the output is the square of the input.

$$y(t) = x^2(t)$$

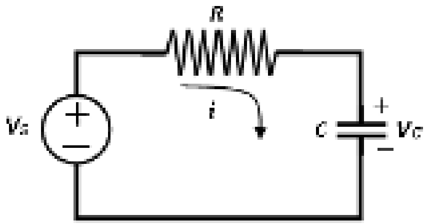
Because, t = ±n will produce the same output.

The concept of invertibility is important in many applications. One particular example is that of systems used for encoding in a variety of communication systems.

Causality

A system is causal if the output at any time depends on values of the input at only the present and past times.

Such systems are also referred to as ***non-anticipative***, as the system output does not anticipate future values of the input.



- The RC circuit (see figure) is causal, since the capacitor voltage depends only on the present and past values of the source voltage.
- The motion of an automobile is causal, as it does not anticipate future actions of the driver.
- The systems described by these equations are also causal.
- All memoryless systems are causal, since the output responds only to the current value of the input.

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

$$y[n] = x[n-1]$$

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)d\tau$$

Causality - contd.

The following systems are not causal:

$$y[n] = x[n] - x[n+1]$$

$$y(t) = x(t+1)$$

Causality is not an essential consideration in applications where the independent variable is not time, such as in image processing.

- In processing data that have been collected previously, as often is the case with speech, geophysical or meteorological signals etc., we are by no means constrained to causal processing.
- An example of noncausal averaging system is

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{+M} x[n-k]$$

Causality - contd.

Consider the following system:

$$y[n] = x[-n]$$

Checking for the negative time, e.g. $n = -4$, we see that $y[-4] = x[4]$, so that the output at this time depends on a future value of the input. Hence the system is not causal.

Consider the following system:

$$y(t) = x(t)\cos(t+1)$$

In this system, the output at any time equals the input at that same time multiplied with a number that fluctuate with time. Specifically, we can re-write

$$y(t) = x(t)\mathrm{g}(\mathrm{t})$$

where $\mathrm{g}(\mathrm{t})$ is a time-varying function, namely $\mathrm{g}(\mathrm{t}) = \cos(\mathrm{t}+1)$. Thus, only the current value of the input influences the current value of the output, and we conclude that this system is causal (and, also memoryless).

Stability

A system is said to be *stable* if a small input leads to a response that does not diverge.

There are several examples of stable systems. “Stability of physical systems generally results from the presence of mechanisms that dissipate energy”.

For example, in the ***RC*** circuit shown before, the resistor dissipates energy and this circuit is a stable system.

More specifically,

If the input to a *stable system* is bounded (i.e., if its magnitude does not grow without bounds), then the output must also be bounded, and therefore cannot diverge.

Stability - contd.

$$y[n] = \frac{1}{2M+1} \sum_{k=-M}^{+M} x[n-k]$$

If the input $x[n]$ to the system is bounded (say by a number $\boldsymbol{B}$), for all values of n, then according to Equation above, the output $y[n]$ of the system is also bounded by $\boldsymbol{B}$. This is because the output $y[n]$ is the average of a finite set of values of the input. Therefore, the output $y[n]$ is bounded and the system is stable.

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

This systems sums all of the past values of the input rather than just a finite set of values, and the system is unstable, since the sun can grow even if the input $x[n]$ is bounded.

Example: Stability

Suppose we suspect that a particular system is unstable, then a useful strategy is to look for a specific bounded input that leads to an unbounded output for that system.

$$S_1: \; y(t) = tx(t)$$

$$S_2: \; y(t) = e^{x(t)}$$

Now, for system S_1, a constant input $x(t) = 1$ yields $y(t) = t$, which is unbounded: since no matter what finite constant input we pick, $|y(t)|$ will exceed that constant for some t. Therefore, the system S_1 is unstable.

For system S_2, let us the input $x(t)$ be bounded by a positive number B, i.e.

$$|x(t) < B| \longrightarrow \quad -B < x(t) < B \qquad \text{for all } t.$$

Using the definition of S_2, we can write $\quad e^{-B} < |y(t)| < e^{B}$

The system S_2 is therefore, stable.

Time Invariance

A system is said to be *time invariant* if a time shift in the input signal leads to an identical time shift in the output signal.

if $y[n]$ is the output of a discrete-time, time-invariant system when $x[n]$ is the input, then $y[n-n_0]$ is the output when $x[n-n_0]$ is applied as an input.

In continuous-time when $y(t)$ is the output corresponding to the input $x(t)$, a time-invariant system will have $y(t-t_0)$ as the output when $x(t-t_0)$ is the input.

Consider now the discrete-time system defined by $y[n] = n\,x[n]$

Suppose, we consider the input signal $x_1[n] = \delta[n]$, which yields an output $y_1[n] = 0$ (since $n\delta[n] = 0$).

However, the input $x_2[n] = \delta[n-1]$ yields the output $y_2[n] = n\delta[n-1] = \delta[n-1]$. Thus, while $x_2[n]$ is a shifted version of $x_1[n]$, $y_2[n]$ is not a shifted version of $y_1[n]$.

Time Invariance - contd.

Consider the continuous-time system defined by $y(t) = \sin[x(t)]$

To check this system is time invariant, we must determine whether the time-invariance property holds for any input and any time shift t_0. Thus, let $x(t)$ be an arbitrary input to this system, and let

$$y_1(t) = \sin[x_1(t)]$$

to be the corresponding output. Then, consider a second input obtained by shifting $x_1(t)$ in time

$$x_2(t) = x_1(t + t_0)$$

The corresponding output to this new input

$$y_2(t) = \sin[x_2(t)] = \sin[x_1(t + t_0)]$$

$$y_1(t + t_0) = \sin[x_1(t + t_0)]$$

We see that $y_2(t) = y_1(t + t_0)$, and therefore, the system is time invariant.

Linearity

A linear system, in continuous-time or discrete-time, is a system that possesses the important property of superposition: If an input consists of the weighted sum of several signals, then the output is the superposition – that is, the weighted sum – of the responses of the system to each of those signals.

Let $y_1(t)$ be the response of a continuous-time system to an input $x_1(t)$, and let $y_2(t)$ be the response of a continuous-time system to an input $x_2(t)$. Then the system is linear if,

1. The response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$.
2. The response to $ax_1(t)$ is $ay_1(t)$, where a is any complex constant.

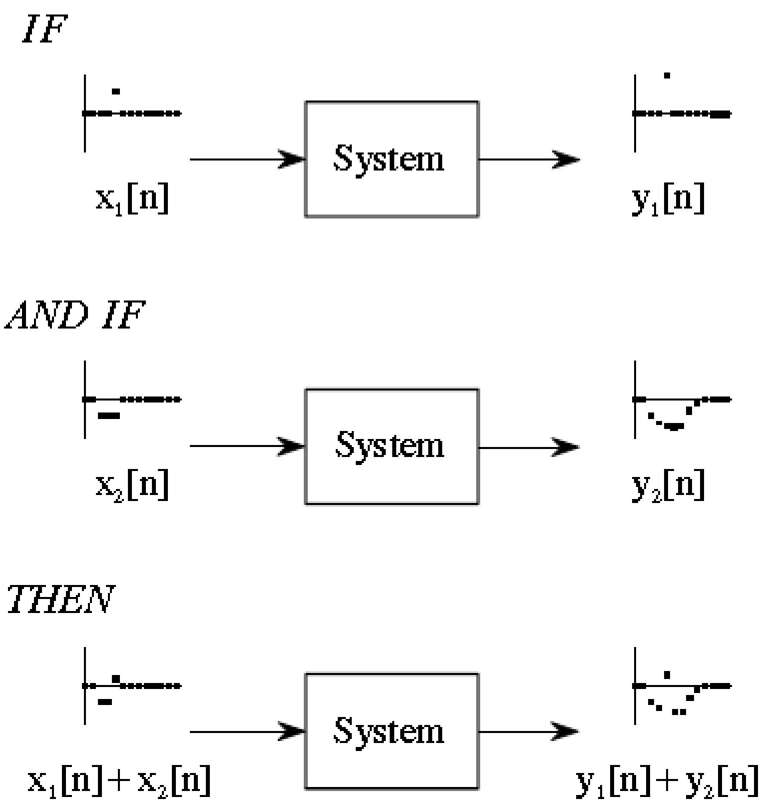
The first of these two properties is called the ***additivity*** property and the second is known as the ***scaling*** or ***homogeneity*** property.

The two properties defining a linear system can be combined into a single statement:

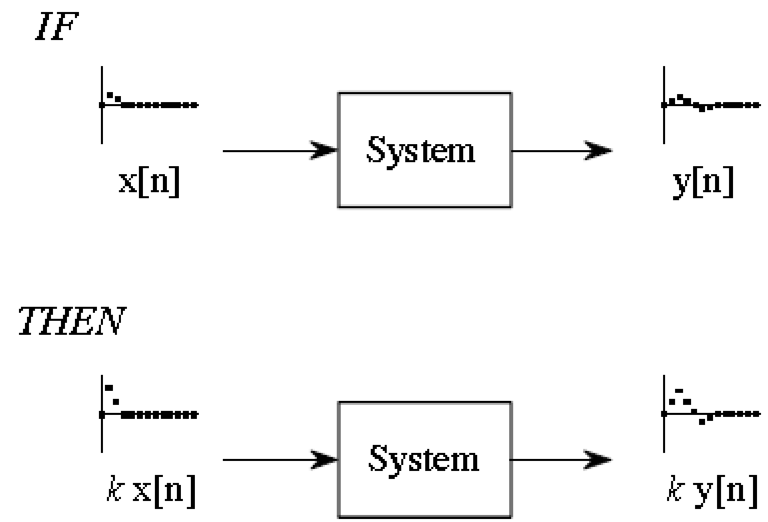
$$x_1(t) + bx_2(t) = ay_1(t) + by_2(t)$$

Linearity - contd.

Additivity



Homogeneity



Example: Linearity

Consider a system S whose input $x(t)$ and output $y(t)$ are related by

$$y(t) = tx(t)$$

To determine whether or not S is linear, we consider two arbitrary inputs $x_1(t)$ and $x_2(t)$,

$$x_1(t) \rightarrow y_1(t) = tx_1(t)$$

$$x_2(t) \rightarrow y_2(t) = tx_2(t)$$

Let $x_3(t)$ be a linear combination of $x_1(t)$ and $x_2(t)$. That is,

$$x_3(t) = ax_1(t) + bx_2(t)$$

where a and b are arbitrary scalars. If $x_3(t)$ is the input to S, then the corresponding output $y_3(t)$ may be expressed as:

$$y_3(t) = tx_3(t) = t(ax_1(t) + bx_2(t)) = atx_1(t) + btx_2(t)$$

$$= ay_1(t) + by_2(t)$$

We conclude that the system S is **linear.**

Example: Linearity - contd.

Let us now consider another system S whose input $x(t)$ and output $y(t)$ are related by

$$y(t) = x^2(t)$$

Like the previous example, to determine whether or not S is linear, we consider two arbitrary inputs $x_1(t)$ and $x_2(t)$,

$$x_1(t) \rightarrow y_1(t) = x_1^2(t)$$

$$x_2(t) \rightarrow y_2(t) = x_2^2(t)$$

Let $x_3(t)$ be a linear combination of $x_1(t)$ and $x_2(t)$. That is,

$$x_3(t) = ax_1(t) + bx_2(t)$$

where a and b are arbitrary scalars. If $x_3(t)$ is the input to S, then the corresponding output $y_3(t)$ may be expressed as:

$$y_3(t) = x_3^2(t) = \left(ax_1(t) + bx_2(t)\right)^2 = a^2x_1^2(t) + b^2x_2^2(t) + 2abx_1(t)x_2(t)$$

$$= a^2y_1(t) + b^2y_2(t) + 2abx_1(t)x_2(t)$$

Not Linear

Example: Linearity - contd.

While checking the linearity of a system, it is important to keep in mind that the system must satisfy both the additivity and homogeneity properties and that the signals as well the scaling constants are allowed to be complex.

Consider: $y[n] = \mathcal{R}e\{x[n]\}$

Additive √ Homogeneity ×

Let us assume, $x_1[n] = r[n] + js[n]$

where $r[n]$ and $s[n]$ are the real and imaginary parts of the complex signal $x[n]$, and that the corresponding output is given by $y_1[n] = r[n]$

Now we consider the scaling of the complex input with a complex number say $a = j$, i.e.

$$x_2[n] = jx_1[n] = j(r[n] + js[n]) = jr[n] - s[n]$$

Therefore, the corresponding output $y_2[n]$ is given by $y_2[n] = \mathcal{R}e\{x_2[n]\} = -s[n]$

which is not equal to the scaled version of the $y_1[n]$: $ay_1[n] = jr[n]$

We conclude that the system violates the homogeneity property, therefore it is not linear.

Example: Linearity - contd.

Find whether the following system is linear or not.

$$y[n] = 2x[n] + 3$$

If $x_1[n]$ = 2 and $x_2[n]$ = 3, then

$$x_1[n] \rightarrow y_1[n] = 2x_1[n] + 3 = 7$$

$$x_2[n] \rightarrow y_2[n] = 2x_2[n] + 3 = 9$$

However, the response to $x_3[n] = x_1[n]+x_2[n] = 5$

$$y_3[n] = 2\{x_1[n] + x_2[n]\} + 3 = 13 \neq y_1[n] + y_2[n]$$

Not linear

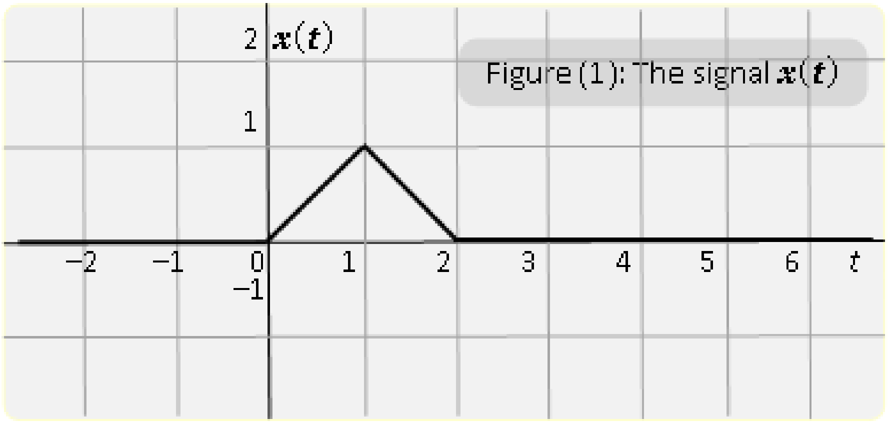
Another way:

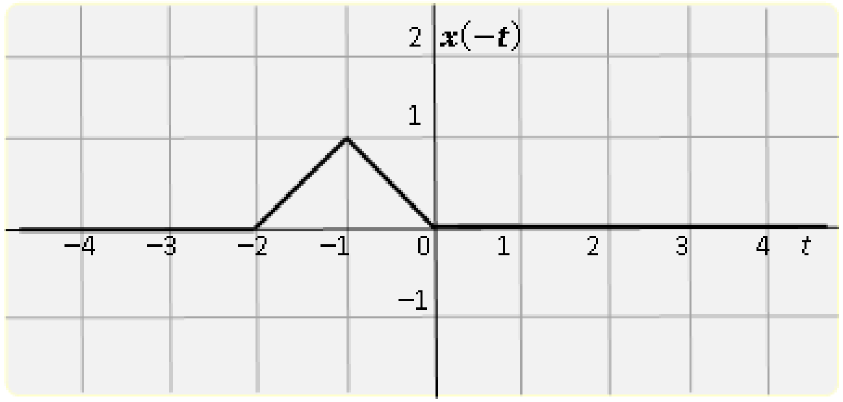
For a linear system, an input which is zero for all time results in an output which is zero for all time.

If x[n] = 0, y[n] = 3, which is not zero. So, the above system is not linear.

Workout - (14)

A continuous-time signal $x(t)$ is depicted below in Figure (1).





Draw and label the following signals:

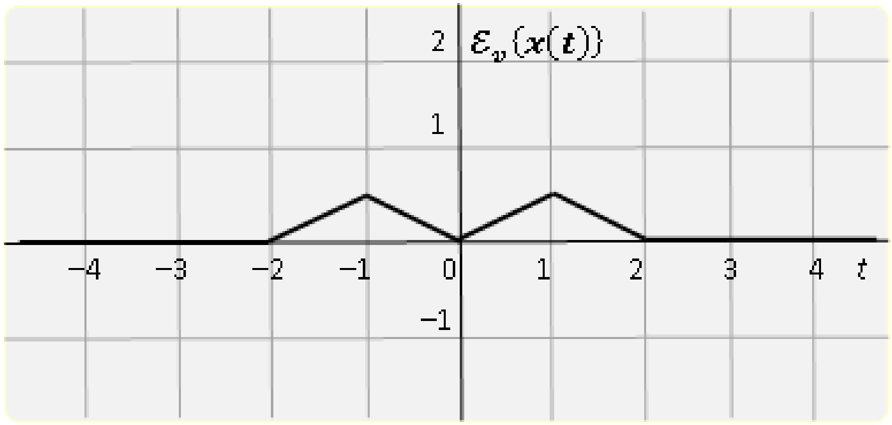
(1) $x(-t)$,

(2) Even component of $x(t)$: $\mathcal{E}_v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$

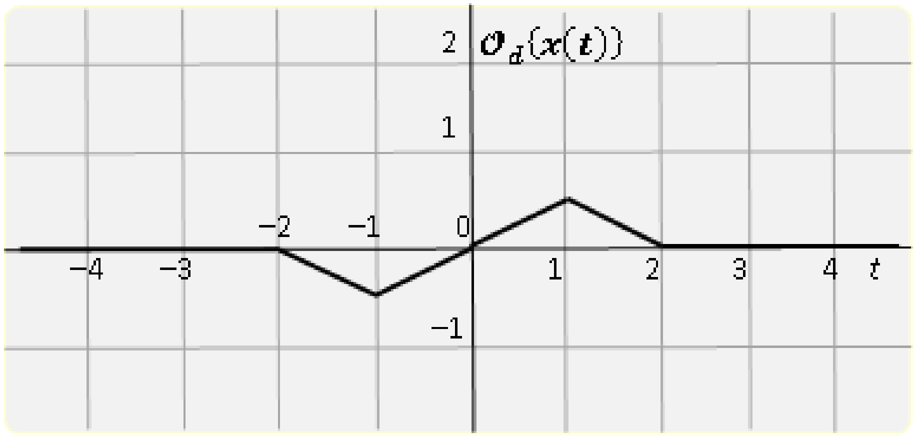
Odd component of $x(t)$: $\mathcal{O}_d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$

Workout - (14) - contd.

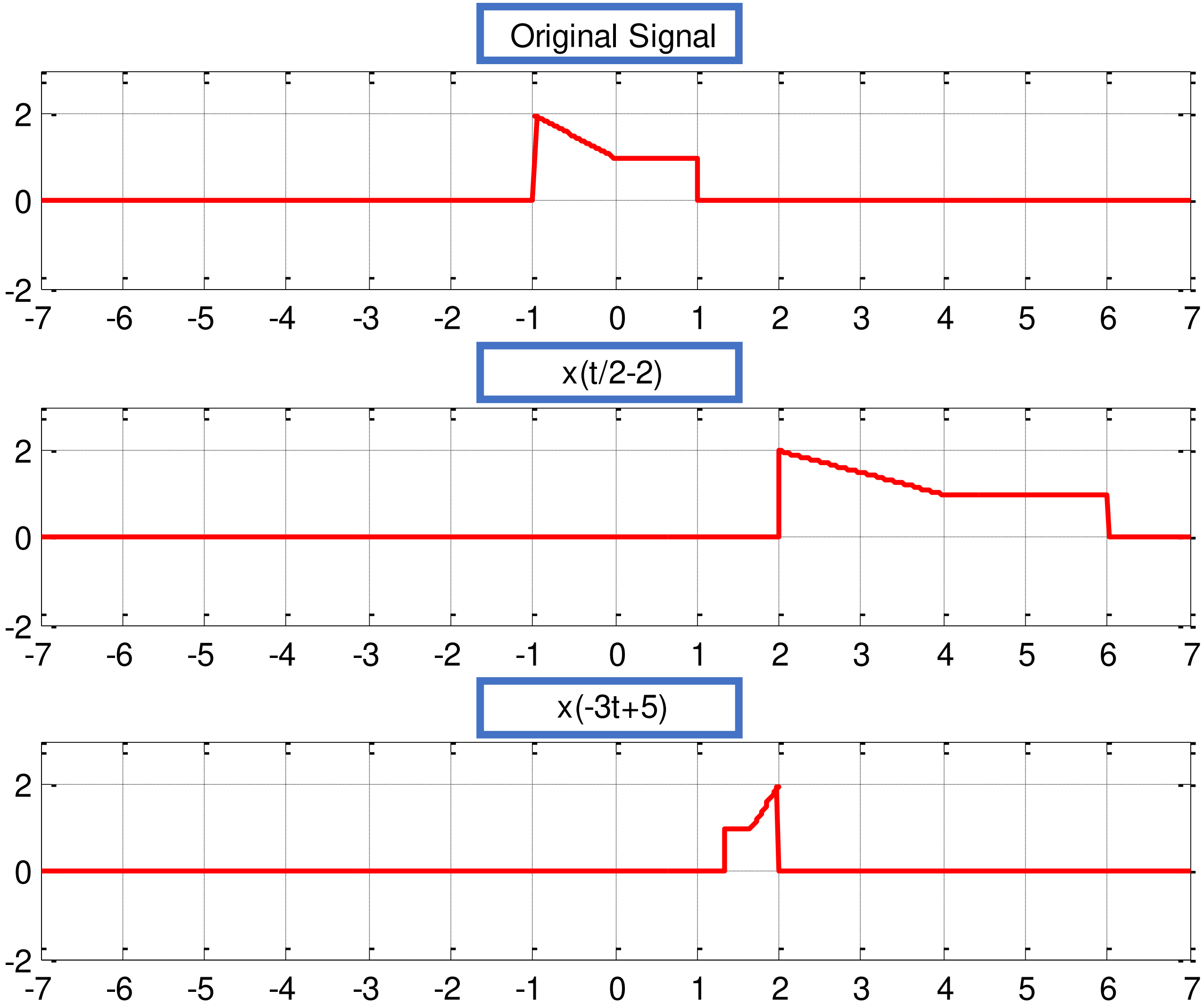
Even component of $x(t)$: $\mathcal{E}_v\{x(t)\} = \frac{1}{2}[x(t) + x(-t)]$



Odd component of $x(t)$: $\mathcal{O}_d\{x(t)\} = \frac{1}{2}[x(t) - x(-t)]$



Workout - (15)



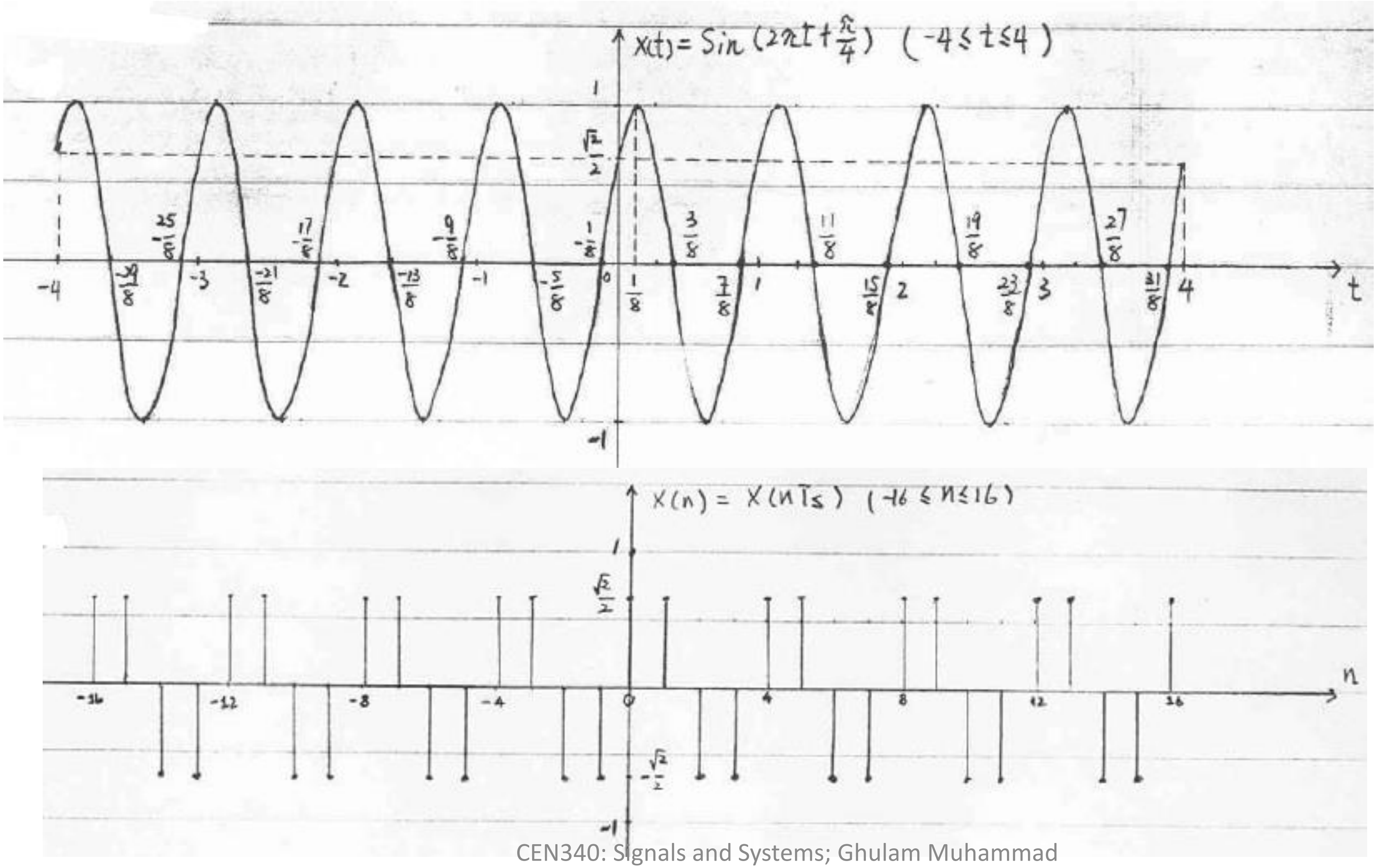
First, shift right by 2 samples. The first transition is now shifted to (-1+2 =) 1. Then each position is multiplied by 2. So, 1 becomes (1x2 =) 2.

First, shift left by 5 samples. The first transition is now shifted to (-1-5 =) -6. Then each position is multiplied by 1/3. So, -6 becomes (-6x1/3=) -2. Then, reflect. So -2 becomes 2.

Workout – (16)

Drawing Sinusoids

$$x(t) = \sin\left(2\pi t + \frac{\pi}{4}\right), \quad -4 \le t \le 4$$



Acknowledgement

The slides are prepared based on the following textbook:

- Alan V. Oppenheim, Alan S. Willsky, with S. Hamid Nawab, *Signals & Systems*, 2nd Edition, Prentice-Hall, Inc., 1997.

Special thanks to

- *Prof. Anwar M. Mirza*, former faculty member, College of Computer and Information Sciences, King Saud University
- *Dr. Abdul Wadood Abdul Waheed*, faculty member, College of Computer and Information Sciences, King Saud University