MATH203 Calculus

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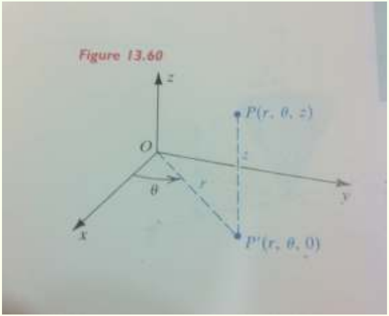
Cylindrical coordinates

Theorem

The rectangular coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) of a point P are related as follows:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad \tan\theta = \frac{y}{x},$$
$$r^2 = x^2 + y^2, \quad z = z$$



Cylindrical coordinates

Example 1

Change the equation to cylindrical coordinates.
(i) $z^2 = x^2 + y^2$, (ii) $x^2 - y^2 - z^2 = 1$

Triple Integral using cylindrical coordinates

$$\iiint\limits_Q f(r,\theta,z)\mathrm{d}V = \lim_{\|P\|\to 0}\sum_{k=1}^{n} f(r_k,\theta_k,z_k)\Delta V_k$$

where $\Delta V_k = \overline{r}\Delta r_k \Delta\theta_k \Delta z_k$

Nice Region

If $Q = \{(r,\theta,z) : a \leqslant r \leqslant b, c \leqslant \theta \leqslant d, m \leqslant z \leqslant n\}$, then

$$\iiint\limits_Q f(r,\theta,z)\mathrm{d}V = \int_m^n \int_c^d \int_a^b f(r,\theta,z) r dr d\theta dz$$

Cylindrical coordinates

Complicated Region

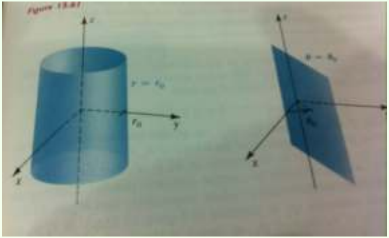
If $Q = \{(r, \theta, z) : (r, \theta) \in R, K_1(r, \theta) \leqslant z \leqslant K_2(r, \theta)\}$, then

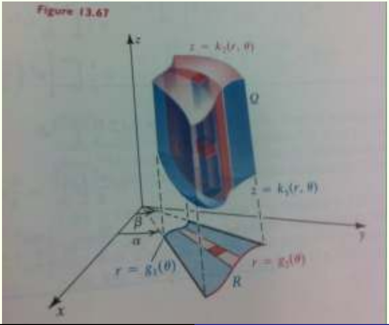
$$\iiint\limits_Q f(r, \theta, z)\mathrm{d}V = \iint\limits_R \Big[\int_{k_1(r,\theta)}^{k_2(r,\theta)} f(r, \theta, z) dz \Big] \mathrm{d}A$$

then,

$$\iiint\limits_Q f(r, \theta, z)\mathrm{d}V = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{k_1(r,\theta)}^{k_2(r,\theta)} f(r, \theta, z) r dz dr d\theta$$

Cylindrical coordinates

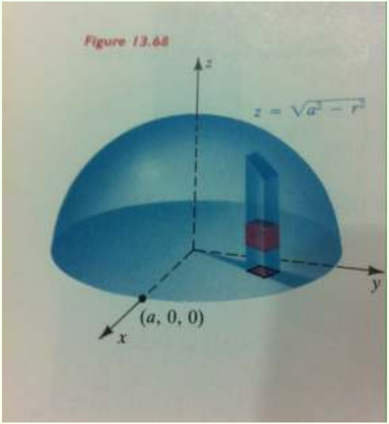




Cylindrical coordinates

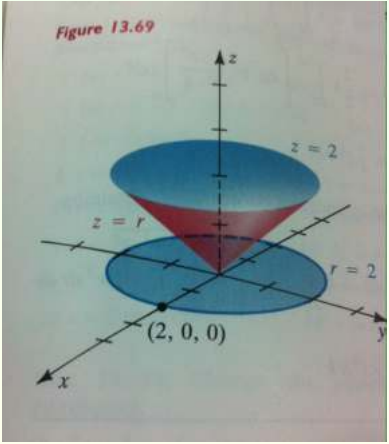
Examples

(1) Find the centroid of solid Q as shown in Figure, where $z = \sqrt{a^2 - r^2}$.



Cylindrical coordinates

(2) By using cylindrical coordinates, find the mass of solid Q bounded by the cone $z = \sqrt{x^2 + y^2}$ and $z = 2$, where the density at (x, y, z) is $\delta = k(x^2 + y^2 + z^2)$.



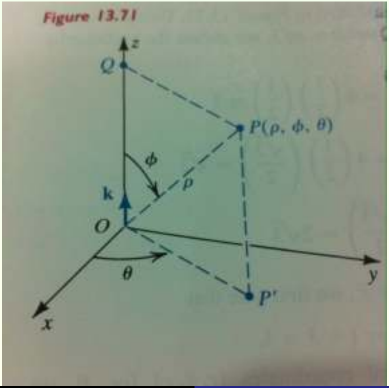
Cylindrical coordinates

(3) A solid has the shape of the region Q that lies inside the cylinder $r = a$, within the sphere $r^2 + z^2 = 4a^2$ and above the $xy-$plane. The density at a point $\delta(x, y, z) = kz$. Find the mass and the moment of inertia I_z of the solid.

Spherical coordinates

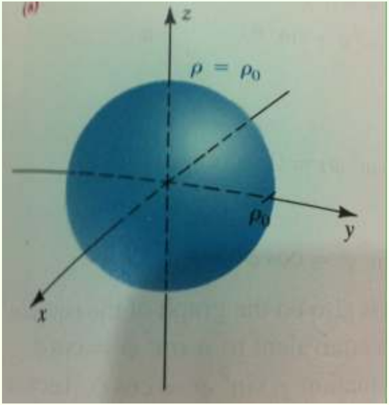
Theorem

The rectangular coordinates (x, y, z) and the spherical coordinates (ρ, ϕ, θ) of a point P are related as follows:
(1) $x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$
(2) $\rho^2 = x^2 + y^2 + z^2$



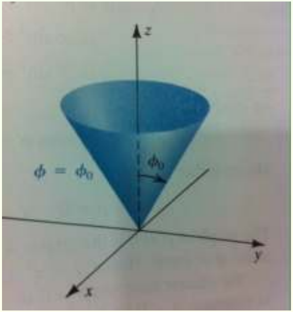
Some important graphs

(1) $\rho = c, c > 0$
graph is a sphere of radius $\rho = c$, with center $(0, 0, 0)$.



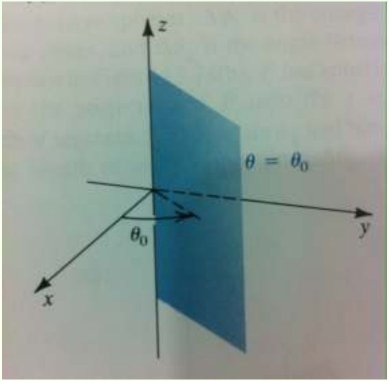
Some important graphs

(2) $\phi = c, 0 < c < \frac{\pi}{2}$
graph is a half-cone opening upward, with center $(0, 0, 0)$.
(i) $\phi = 0$, graph is non-negative part of z-axis.
(ii) $\phi = \frac{\pi}{2}$, graph is xy-plane.
(iii) $\phi = \pi$, graph is non-positive half of z-axis.
(iv) $\frac{\pi}{2} < \phi < \pi$ graph is a half-cone opening downwards, with center $(0, 0, 0)$.



Some important graphs

(1) $\theta = c$
graph is a half plane containing the $z-$axis.



Spherical coordinates

Example 1

1- Change the spherical coordinates to (a) rectangular coordinates (b) cylindrical coordinates.

(i) $(4, \frac{\pi}{6}, \frac{\pi}{3})$, (ii) $(1, \frac{3\pi}{4}, \frac{2\pi}{3})$

2- Change the cylindrical coordinates to (a) rectangular coordinates (b) spherical coordinates.

(i) $(4, -\frac{\pi}{6}, 6)$

Example 2

Find an equation in the spherical coordinates, whose graph is the paraboloid $z = x^2 + y^2$.

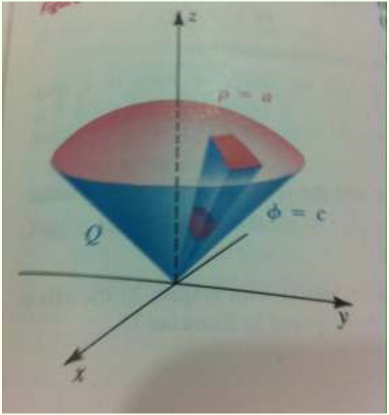
Triple Integral using spherical coordinates

$$\iiint_Q f(\rho, \phi, \theta)\mathrm{d}V = \int_m^n \int_c^d \int_a^b f(\rho, \phi, \theta)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Spherical coordinates

Examples

(1) Find the volume and the centroid of solid Q, as shown in Figure, that bounded above by the sphere $\rho = a$ and below by cone $\phi = c$, where $0 < c < \frac{\pi}{2}$.



Spherical coordinates

Examples

(2) Use spherical coordinates to drive the formula for the volume of sphere centered at the origin and with radius a.

(3) Find the volume of sphere that lies inside the sphere $x^2 + y^2 + z^2 = 2$ and outside $z^2 = x^2 + y^2$.

(4) Use spherical coordinates to express region between the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

(5) Find the integral of $f(x, y, z) = e^{(x^2+y^2+z^2)^{\frac{3}{2}}}$ in the region $R = \{x \geqslant 0, y \geqslant 0, z \geqslant 0, x^2 + y^2 + z^2 = 1\}$.