MATH203 Calculus

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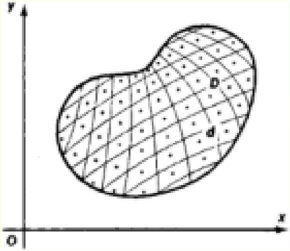
Double Integrals

Riemann Sum

Let f be a function of two variables defined on region R, and Let $P = \{R_k\}$ be an inner partition of R. A Riemann sum of f for P is any sum of the form

$$\sum_k f(u_k, v_k)\Delta A_k \tag{1}$$

where u_k, v_k is a point in R_k and ΔA_k is the area of R_k



Double Integrals

Remarks

1- The summation (1) extends over all the subregions $R_1, R_2, \ldots, R_n$ of P.

2- $\lim\limits_{\|P\|\to 0} \sum\limits_{k} f(u_k, v_k) = C \qquad C \in \mathbb{R}$, if f is continuous on R.

Double Integrals of f over R

If f is a function of two variables that is defined on a region R. The double integral of f over R is

$$\iint\limits_{R} f(x,y)\mathrm{d}A = \lim_{\|P\|\to 0} \sum_{k} f(x_k, y_k)\Delta A_k \tag{2}$$

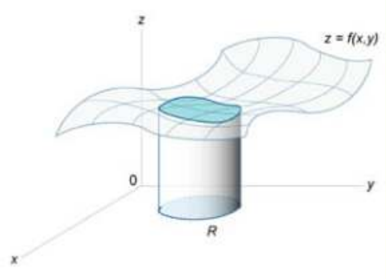
provide the limit exists.

Double Integrals

Remarks

1- If $f(x, y) \geqslant 0$ and continuous throughout the region R, then the double integral $\iint\limits_R f(x, y)\mathrm{d}A$ may be used to find the Volume V of the solid Q that lies under the graph of $z = f(x, y)$ and over R, i.e.

$$V = \iint\limits_R f(x, y)\mathrm{d}A, \quad f(x, y) \geqslant 0 \text{ on } R$$



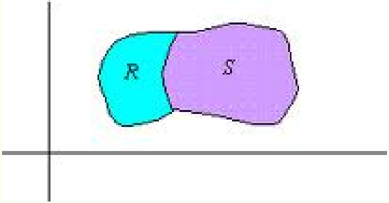
2- If the region R describes the base of a mountain and $f(x, y)$ is the height at point (x, y), then the double integral $\iint\limits_R f(x, y)\mathrm{d}A$ is the Volume of the mountain.

3- If the region R describes the surface of a lake and $f(x, y)$ is the depth of the water at point (x, y), then the double integral $\iint\limits_R f(x, y)\mathrm{d}A$ is the Volume of the water in the lake.

4- If $f(x, y) \leqslant 0$ and continuous throughout the region R, then the double integral $\iint\limits_R f(x, y)\mathrm{d}A$ is the negative of the Volume V of the solid Q that lies over the graph of $z = f(x, y)$ and under R.

Properties of double integrals

- $\iint\limits_R \mathbf{c}\, f(x,y)\mathrm{d}A = \mathbf{c} \iint\limits_R f(x,y)\mathrm{d}A$ for every real number $\mathbf{c}$.
- $\iint\limits_R [f(x,y) \pm g(x,y)]\mathrm{d}A = \iint\limits_R f(x,y)\mathrm{d}A \pm \iint\limits_R g(x,y)\mathrm{d}A$.
- If Q is the union of two non-over lapping regions R and S, $\iint\limits_Q f(x,y)\mathrm{d}A = \iint\limits_R f(x,y)\mathrm{d}A + \iint\limits_S f(x,y)\mathrm{d}A$



- If $f(x,y) \leqslant 0$ throughout the region R, then $\iint\limits_R f(x,y)\mathrm{d}A \leqslant 0$

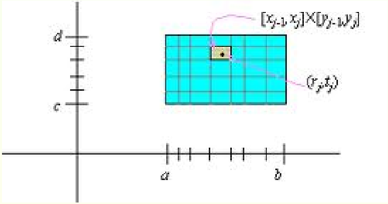
Evaluation theorem (1) Rectangular Regions

Let f be continuous function on a closed rectangular region R, then $\iint_R f(x,y)\mathrm{d}A$ can be evaluated by using an **iterated integral** of the following type

$$\int_a^b \Big[\int_c^d f(x,y)\mathrm{d}y\Big]\mathrm{d}x = \int_a^b \int_c^d f(x,y)\mathrm{d}y\mathrm{d}x$$

or

$$\int_c^d \Big[\int_a^b f(x,y)\mathrm{d}x\Big]\mathrm{d}y = \int_c^d \int_a^b f(x,y)\mathrm{d}x\mathrm{d}y$$



Double Integrals

Remarks

1- $\int_c^d f(x,y)\mathrm{d}y$ is partial inegration w.r.t y, regarding x as a constant.

2- $\int_a^b f(x,y)\mathrm{d}x$ is partial inegration w.r.t x, regarding y as a constant.

Examples

Evaluate the following integrals:

(1) $\displaystyle\int_1^2\int_{-1}^2 (12xy^2 - 8x^3)\mathrm{d}y\mathrm{d}x.$

(2) $\displaystyle\int_{-1}^2\int_1^2 (12xy^2 - 8x^3)\mathrm{d}x\mathrm{d}y.$

(3) $\displaystyle\int_1^3\int_2^4 (40 - 2xy)\mathrm{d}x\mathrm{d}y.$

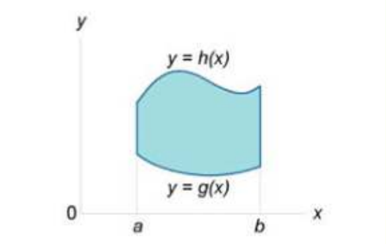
(4) $\displaystyle\int_1^2\int_{1-x}^{\sqrt{x}} x^2y\mathrm{d}x\mathrm{d}y.$

Double Integrals

Non-Rectangular Regions

CASE 1 An iterated integral may be defined over the region R_x as shown below

$$\int_a^b \left[\int_{g(x)}^{h(x)} f(x,y)\mathrm{d}y \right] \mathrm{d}x = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\mathrm{d}y\mathrm{d}x$$

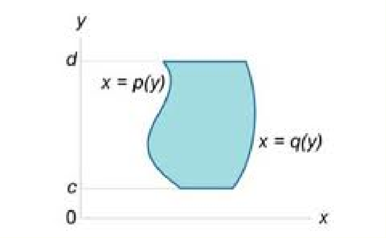


Double Integrals

Non-Rectangular Regions

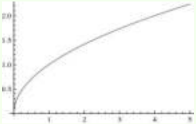
CASE 2 An iterated integral may be defined over the region R_y as shown below

$$\int_c^d \Big[\int_{p(y)}^{q(y)} f(x,y)\mathrm{d}x\Big]\mathrm{d}y = \int_c^d \int_{p(y)}^{q(y)} f(x,y)\mathrm{d}x\mathrm{d}y$$

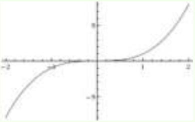


Double Integrals

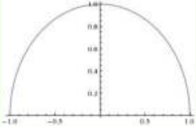
Some important graphs



(a) $y = \sqrt{x}$

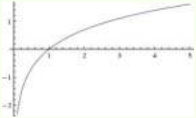


(b) $y = x^3$

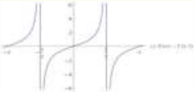


(c) $y = \sqrt{1 - x^2}$

(d) $y = e^x$



(e) $y = \ln(x)$



(f) $y = \tan x$

Figure: Some inportant graphs

Double Integrals

Examples

Sketch the region bounded by the graphs of :
(1) $y = \sqrt{x}$ and $y = x^3$.
(2) $y = \sqrt{1 - x^2}$ and $y = 0$.
for $f(x, y) = x - y$.