Review

Definition:

Reliability is the probability that a component or system will perform a required function for a given period of time when used under stated operating conditions.

Notes:

- Reliability is concerned with the life of a system from a success/failure point of view.
- Reliability is a "time" oriented quality characteristic.
- Reliability is a probability which is a function of time.
- The random variable used to measure reliability is the "time"-to-failure random variable, *T*.

Next…

How to measure reliability?

Failure Distributions
-- Reliability Measures

Overview

1. Probability Functions Representing Reliability
    - 1.1 Reliability Function
    - 1.2 Cumulative Distribution Function (CDF)
    - 1.3 Probability Density Function (PDF)
    - 1.4 Hazard Function
    - 1.5 Relationships Among $R(t)$, $F(t)$, $f(t)$, and $h(t)$
2. Bathtub Curve
    - -- How population of units age over time
3. Summary Statistics of Reliability
    - 3.1 Expected Life (Mean time to failure)
    - 3.2 Median Life and B_α Life
    - 3.3 Mode
    - 3.4 Variance

Probability Functions Representing Reliability

1. Reliability Function
2. Cumulative Distribution Function (CDF)
3. Probability Density Function (PDF)
4. Hazard Function
5. Relationships Among $R(t)$, $F(t)$, $f(t)$, and $h(t)$

Reliability Function

Definition:

Reliability function is the probability that an item is functioning at any time *t*.

Let *T* = "time"-to-failure random variable, reliability at time *t* is

$$R(t) = P(T \geq t), \quad t \geq 0$$

For example, reliability at time t=100s is

R(100) = P(T >= 100)

Properties:

$$0 \leq R(t) \leq 1; \; R(0) = 1; \; \lim_{t \to \infty} R(t) = 0,$$

$$\text{monotonically decreasing}$$

Reliability Function

Two interpretations:

- *R(t)* is the probability that an individual item is functioning at time *t*
- *R(t)* is the expected fraction of the population that is functioning at time *t* for a large population of items.

Other names:

- Survivor Function -- Biostatistics
- Complementary CDF

CDF and PDF

Cumulative Distribution Function, *F(t)*

F(t)= *P(T<t)* =*1-R(t)*

- Properties:

$$0 \le F(t) \le 1;\ F(0) = 0;\ \lim_{t\to\infty} F(t) = 1$$

- Interpretation:

  F(t) is the probability that an item fails before time *t*.

Probability Density Function, *f(t)*

$$f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt}$$

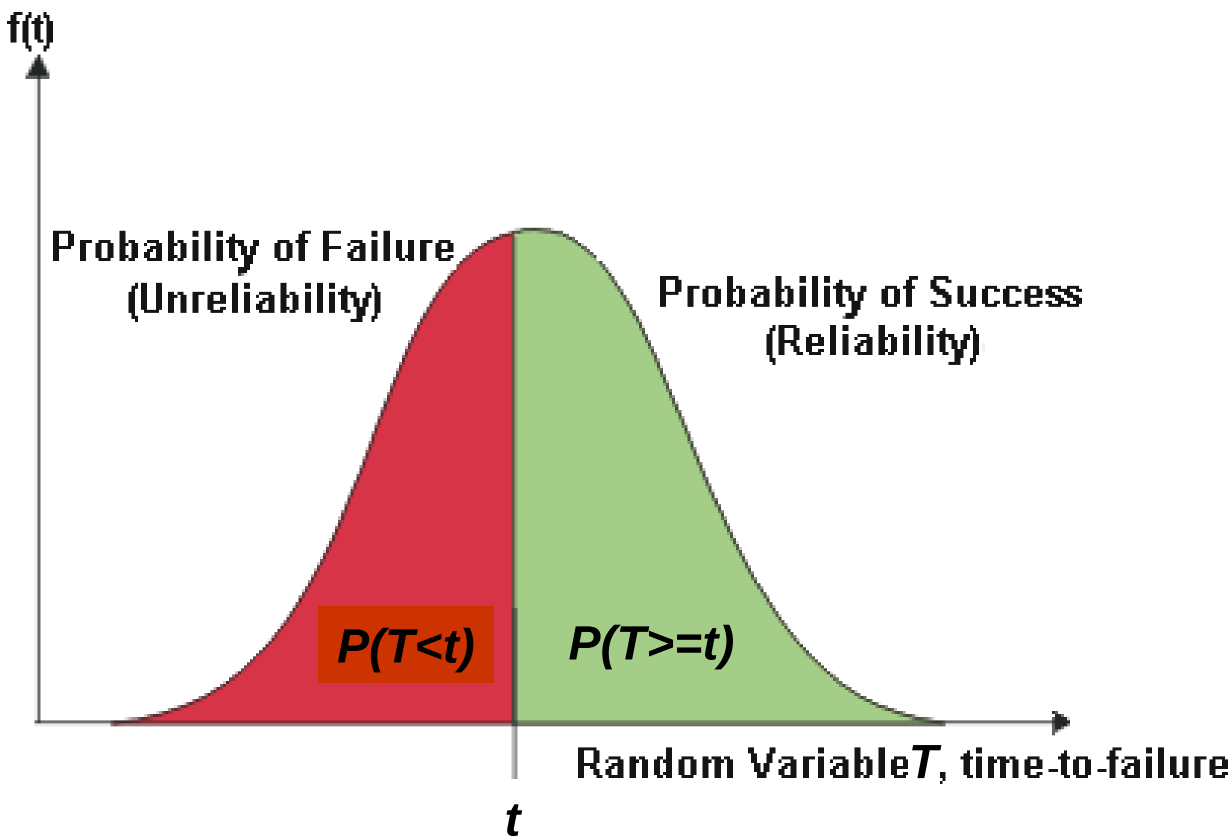
- <u>Properties:</u>

$$f(t) \ge 0;\ \int_0^\infty f(t)dt = 1$$

- Interpretation:

  f(t) indicates the likelihood of failure for any *t*, and it describes the shape of the failure distribution.

Relationships Among *R(t), F(t),* and *f(t)*



$$R(t) = P(T \geq t) = 1 - P(T < t), \quad t \geq 0$$

Relationships Among *R(t)*, *F(t)*, and *f(t)*

$$f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt}$$

$$R(t) = \int_{t}^{\infty} f(u)du$$

$$R(t) = 1 - \int_{0}^{t} f(u)du$$

$$R(t) = 1 - F(t)$$

Examples

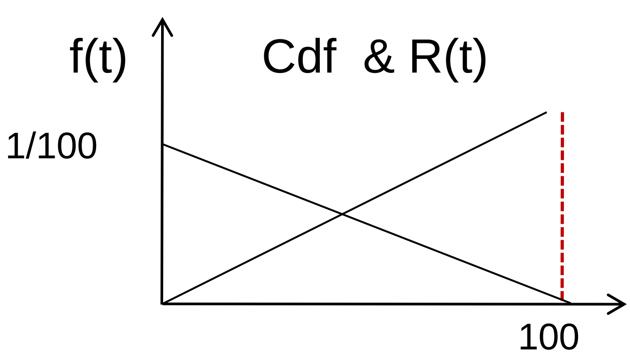
Example 1

Consider the pdf for the uniform random variable given below:

$$f(t) = \frac{1}{100},\ 0 \le t \le 100$$

where *t* is time-to-failure in hours. Draw the pdf, cdf and the reliability function.

Solution



$$F(t) = \int_0^t f(t)dt = \int_0^t \frac{1}{100} dt = \frac{t}{100}$$

$$R(t) = 1 - F(t) = 1 - \frac{t}{100}$$

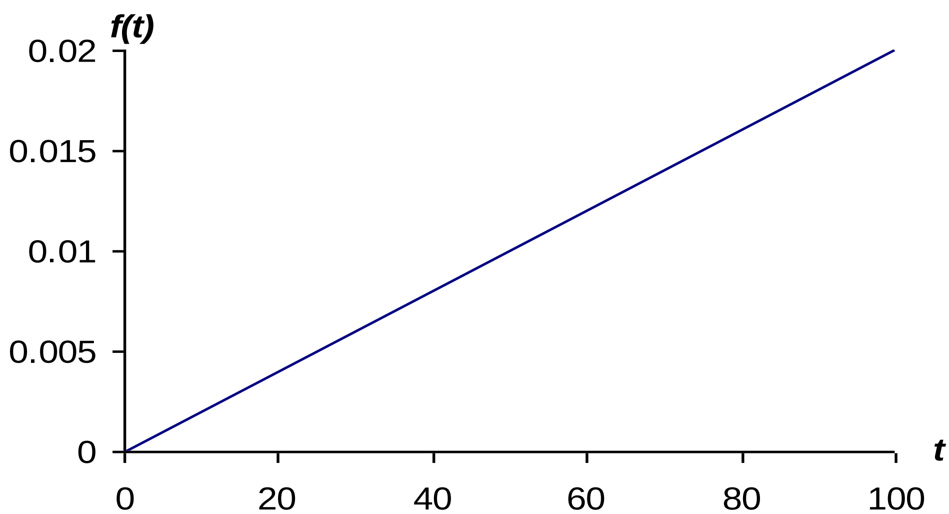
Examples

Example 2

Given the probability density function

$$f(t) = \frac{1}{5000} t, \; 0 \le t \le 100$$

where t is time-to-failure in hours and the pdf is shown below:



Graph the cdf and the reliability function.

Solution

$$F(t) = \int_0^t f(t)dt = \int_0^t \frac{1}{5000} t \, dt = \frac{t^2}{10000}$$

$$R(t) = 1 - F(t) = 1 - \frac{t^2}{10000}$$

Examples

Example 3

For the reliability function

$$R(t) = e^{-(t/800)^2}, t \geq 0$$

where t is time-to-failure in hours.

(1) What is the 200 hr reliability?

(2) What is the 500 hr reliability?

(3) If this item has been working for 200 hrs, What is the reliability of 500 hrs?

Solution

$$R(200) = e^{(200/800)^2} =$$

$$R(500) = e^{(500/800)^2} =$$

$$R(500/200) = \frac{p(T \geq 500)}{p(T \geq 200)} = \frac{R(500)}{R(200)} =$$

Examples

Example 4

Given the following time to failure probability density function (pdf):

$$f(t) = 0.01e^{-0.01t}, t \geq 0$$

where t is time-to-failure in hours. What is the reliability function?

Solution

$$R(t) = e^{-0.01t}$$

Examples

Example 5

Given the cumulative distribution function (cdf):

$$F(t) = 1 - e^{-(t/800)^3}, t \geq 0$$

where t is time-to-failure in hours.

(1) What is the reliability function?

(2) What is the probability that a device will survive for 70 hr?

Hazard Function

Motivation for Hazard Function

$$f(t) = \frac{dF(t)}{dt} = -\frac{dR(t)}{dt} = \lim_{\Delta t \to 0} \frac{R(t) - R(t+\Delta t)}{\Delta t}$$

It is often more meaningful to normalize with respect to the reliability at time *t*, since this indicates the failure rate for those surviving units. If we add *R(t)* to the denominator, we have the **hazard function** or **"instantaneous" failure rate function** as

$$h(t) = \lim_{\Delta t \to 0} \frac{R(t) - R(t+\Delta t)}{R(t)\Delta t}$$

$$= \frac{1}{R(t)} \lim_{\Delta t \to 0} \frac{R(t) - R(t+\Delta t)}{\Delta t}$$

$$= \frac{f(t)}{R(t)}$$

Hazard Function

Notes:

- For small Δt values,

$$h(t)\Delta t = \frac{R(t) - R(t+\Delta t)}{R(t)}$$

$$= \frac{P[t \le T \le t+\Delta t]}{P[T \ge t]}$$

$$= P[t \le T \le t+\Delta t \mid T \ge t]$$

which is the conditional probability of failure in the time interval from *t* to *t+ Δt* given that the system has survived to time *t*.

Hazard Function

Notes (cont.):

- The shape of the hazard function indicates how population of units is aging over time
  - Constant Failure Rate (CFR)
  - Increasing Failure Rate (IFR)
  - Decreasing Failure Rate (DFR)
- Some reliability engineers think of modeling in terms of h(t)

Other Names for Hazard Function

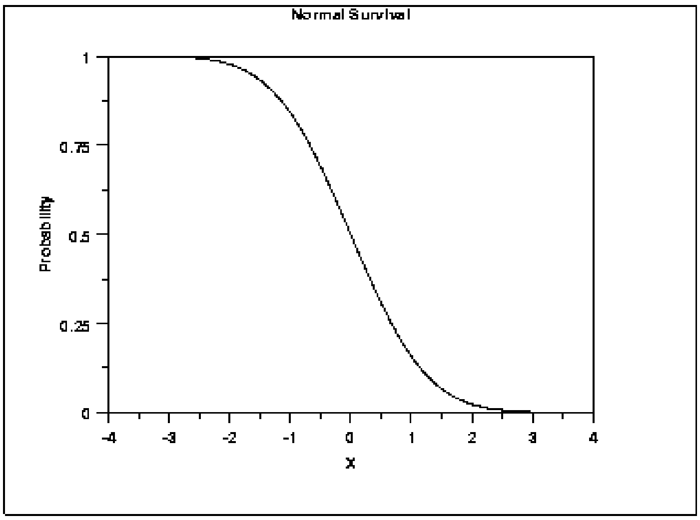
- Reliability: hazard function/hazard rate/failure rate
- Actuarial science: force of mortality/force of decrement
- Vital statistics: age-specific death rate

Hazard Function

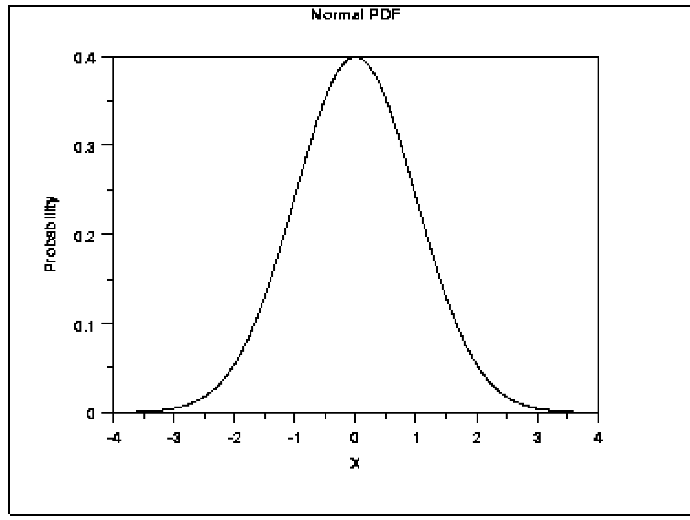
Various Shapes of Hazard Functions and Their Applications

Shapes of Hazard Functions	Applications
Constant Failure Rate (CFR)	• Failures due to completely random or chance events • It should dominate during the useful life period
Increasing Failure Rate (IFR)	• The most likely situation • Items wear out or degrade with time
Decreasing Failure Rate (DFR)	• Less common situation • Burn-in period of a new product
Bathtub-shape Failure Rate (BT)	• Typical shape of many products

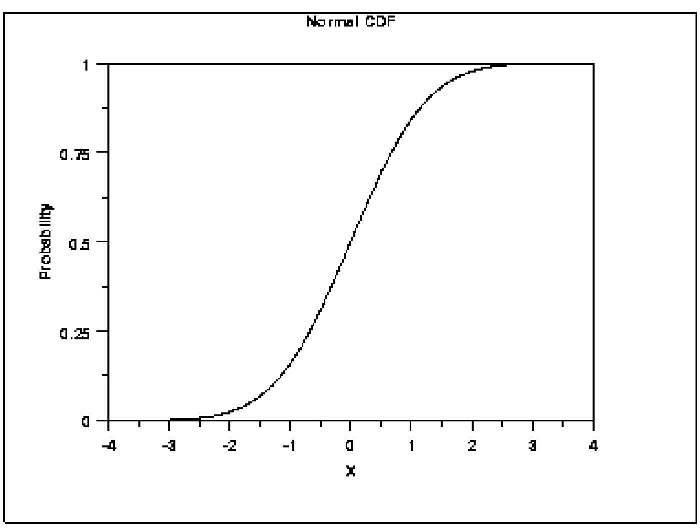
Plots of *R(t)*, *F(t)*, *f(t)*, *h(t)* for the normal distribution



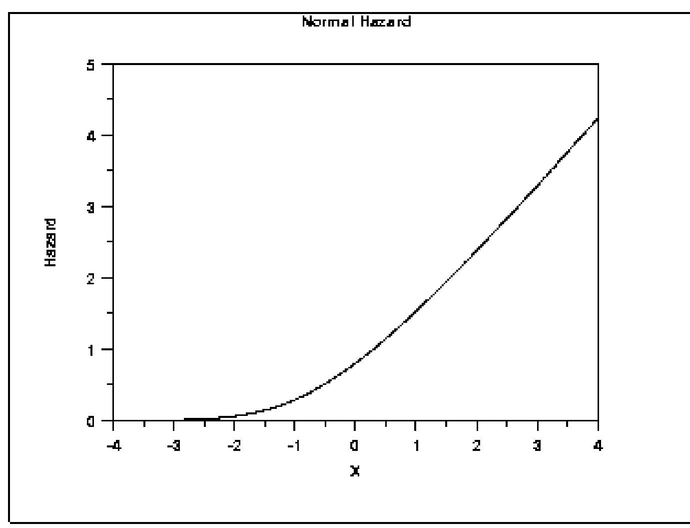
R(t)



f(t)



F(t)



h(t)

Relationships Among *R(t)*, *F(t)*, *f(t)*, and *h(t)*

$$h(t) = \frac{f(t)}{R(t)}$$

$$R(t) = \exp\left[-\int_0^t h(u)\,du\right]$$

$$f(t) = h(t)\exp\left[-\int_0^t h(u)\,du\right]$$

$$F(t) = 1 - R(t)$$

Relationships Among *R(t)*, *F(t)*, *f(t)*, and *h(t)*

One-to-One Relationships Between Various Functions

	f(t)	*R(t)*	*F(t)*	*h(t)*
f(t)	.	$\int_t^\infty f(u)du$	$\int_0^t f(u)du$	$\dfrac{f(t)}{\int_t^\infty f(u)du}$
R(t)	$-\dfrac{dR(t)}{dt}$	.	$1-R(t)$	$-\dfrac{dR(t)}{R(t)dt}$
F(t)	$\dfrac{dF(t)}{dt}$	$1-F(t)$	.	$\dfrac{dF(t)}{\left(1-F(t)\right)dt}$
h(t)	$h(t)e^{-\int_0^t h(u)du}$	$e^{-\int_0^t h(u)du}$	$1-e^{-\int_0^t h(u)du}$	.

Notes: The matrix shows that any of the three other probability functions (given by the columns) can by found if one of the functions (given by the rows) is known.

Examples

Example 6

Consider the pdf used in Example 2 given by

$$f(t) = \frac{1}{5000} t, 0 \leq t \leq 100$$

Calculate the hazard function.

Solution

$$h(t) = \frac{f(t)}{R(t)} = \frac{(1/5000)t}{1 - \frac{t^2}{10000}}$$

Examples

Example 7

Given *h(t)=18t*, find *R(t), F(t),* and *f(t).*

Solution

$$R(t) = e^{-\int_0^t 18t\,dt} = e^{-9t^2}$$

$$F(t) =$$

$$f(t) =$$

Bathtub Curve

The failure of a population of fielded products is due to

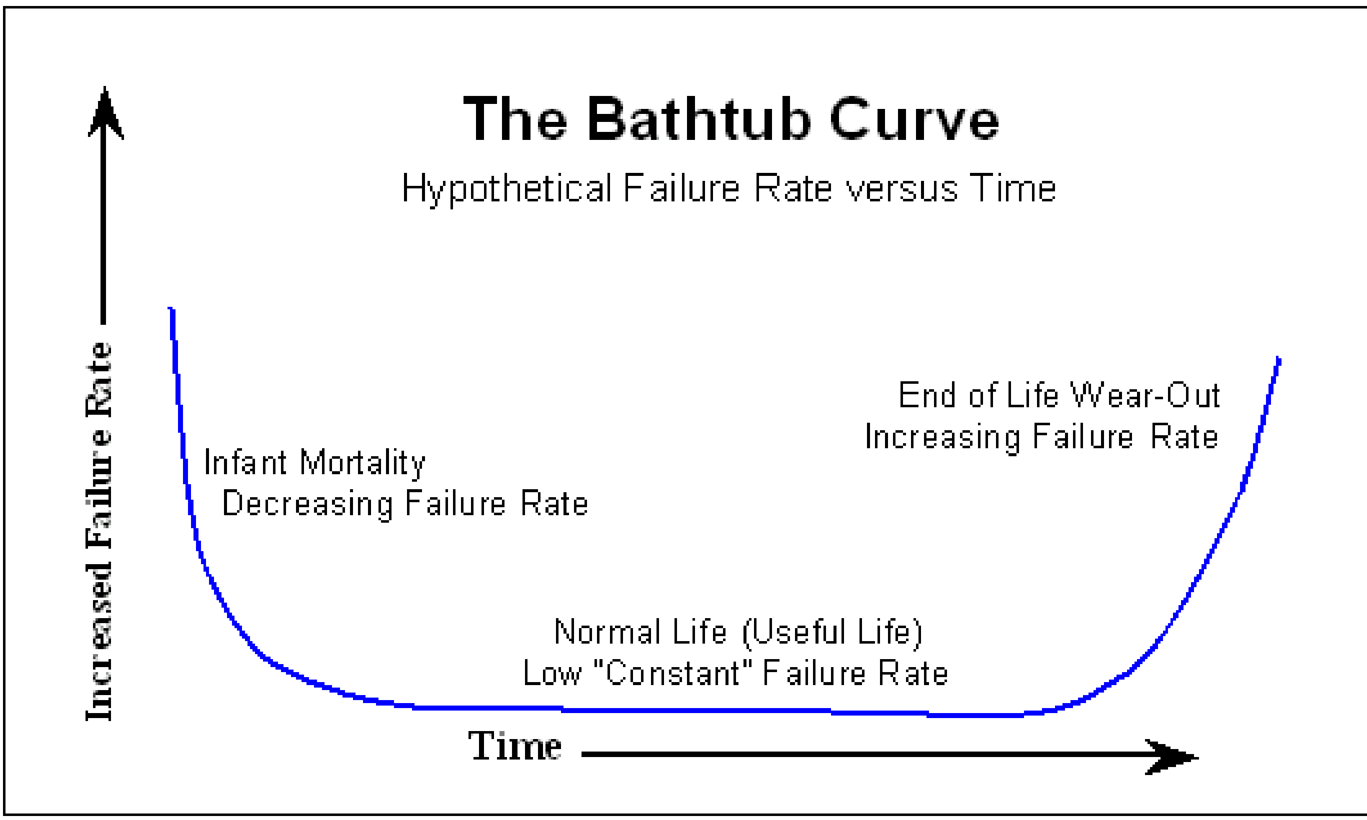
- Problems due to inherent design weakness.
- The manufacturing and quality control related problems.
- The variability due to the customer usage.
- The maintenance policies actually practiced by the customer and improper use or abuse of the product.

Bathtub Curve

Over many years, and across a wide variety of mechanical and electronic components and systems, people have calculated empirical population failure rates as units age over time and repeatedly obtained a bathtub shape:

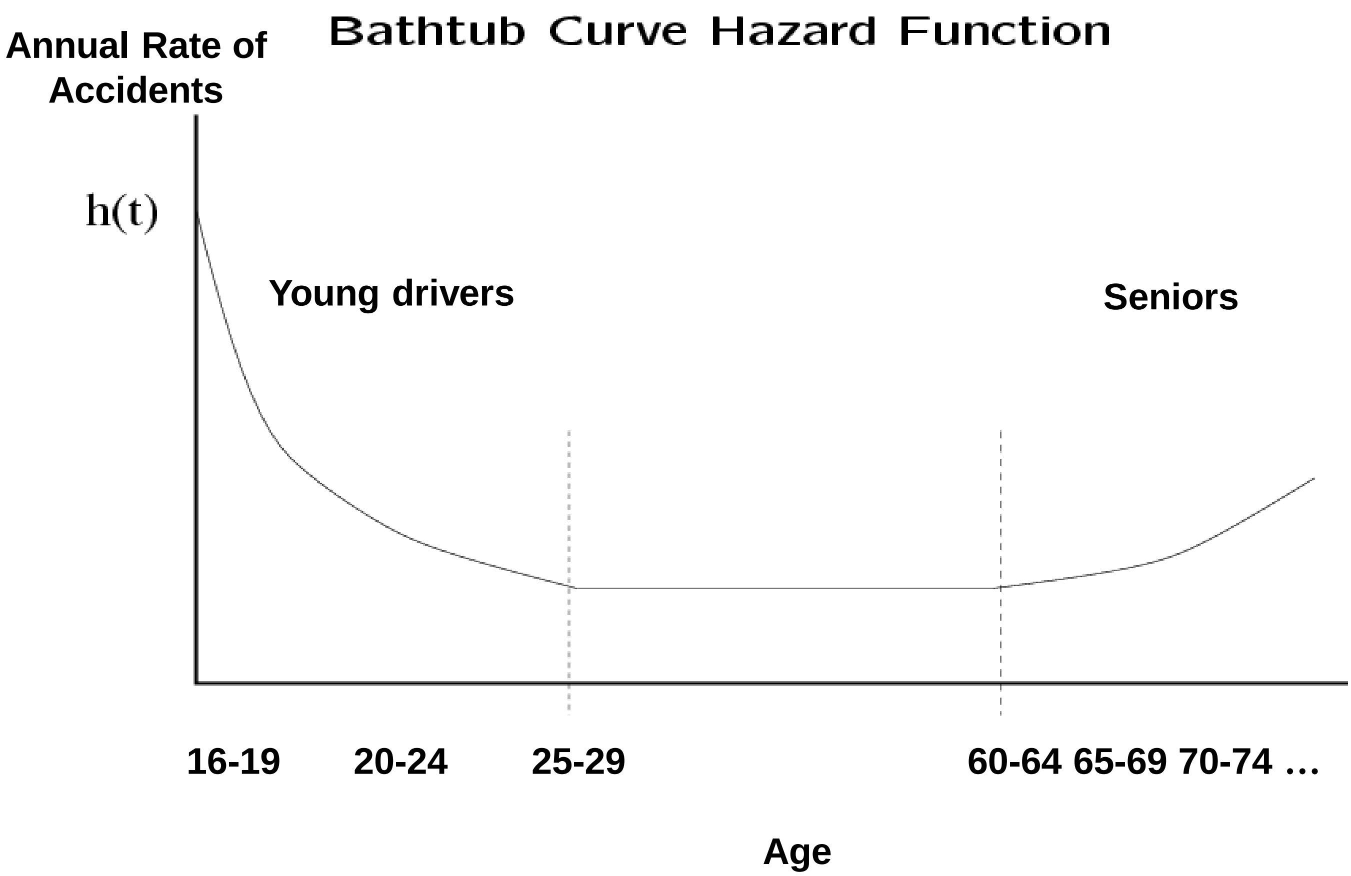
- **Infant mortality (burn-in) period**: decreasing failure rate early in the life cycle
- **Constant failure rate (useful life) period**: nearly constant failure rate
- **Wear-out period**: the failure rate begins to increase as materials wear out and degradation failures occur at an ever increasing rate.

Bathtub Curve



Bathtub Curve

An Example: Accident Rates and Age



Bathtub Curve

Typical information for components of a PC

Component	Infant Mortality Rate	Typical useful life period (years)	Likelihood of Failure Before Wearout
Power Supply	Low	3-6	Moderate
Motherboard	Moderate	4-7	Low
Processor	Low	7+	Very Low
System Memory	Moderate to High	7+	Very Low
Video Card	Low to Moderate	5-7	Low
Monitor	Low to Moderate	5-7+	Moderate to High
Hard Disk Drive	Moderate to High	3-5	Moderate to High
Floppy Disk Drive	Low	7+	Low
CD-ROM Drive	Moderate	3-5	Moderate
Modem	Low	5-7+	Low
Keyboard	Very Low	3-5	Moderate
Mouse	Very Low	1-4	Moderate to High

Summary Statistics of Reliability

1. Expected Life (Mean time to failure)
2. Median Life and B_α Life
3. Mode
4. Variance

Expected Life

$$E[T] = \mu = \int_0^\infty t f(t)dt, \quad t \geq 0$$

E[T] is also called:

- Mean Time to Failure (MTTF) for nonrepairable items
- Mean Time between Failure (MTBF) for repairable items that can be completely renewed by repair

E[T] is a measure of the central tendency or average value of the failure distribution, and it is known as the center of gravity in physics.

Expected Life

Relationship between E[T] and *R(t):*

- Show that E(T) can be re-expressed as

$$E[T]=\int_0^{\infty} R(t)dt, \quad t \geq 0$$

- Sometimes, one expression is easier to integrate than the other → exponential example, how?

Examples

Example 8

Given that

$$R(t) = 0.01e^{-0.01t}, t \geq 0$$

What is the MTTF?

Solution

$$E(t) = \int_0^{\infty} 0.01e^{-0.01t} dt = 1$$

Examples

Example 9

Given that

$$R(t) = e^{-(t/100)^2}, t \geq 0$$

what is the MTTF?

Median Life and B_α Life

The median life, *B_{50}*, divides the distribution into two equal halves, with 50% of the failure occurring before and after *B_{50}*.

$$R(B_{50}) = 0.5 = P[T \geq B_{50}]$$

B_α Life is the time by which *α* percent of the items fail.

$$R(B_\alpha) = 1 - \alpha / 100 = P[T \geq B_\alpha]$$

For example, *B_{10}* life can be calculated by

$$R(B_{10}) = 1 - 10/100$$

$$P[T \geq B_{10}] = 0.9$$

Mode and Variance

Mode is the time value at which the probability density function achieves a maximum.

$$f(t_{\text{mode}}) = \max_{0 \le t \le \infty} f(t)$$

Mode is the most likely observed failure time.

Variance

$$Var[T] = \sigma^2 = \int_0^\infty (t-\mu)^2 f(t)dt, \quad t \ge 0$$

$$Var[T] = E[T^2] - \left(E[T]\right)^2$$

Examples

Example 9

Consider the pdf used in Example 2 (triangle distribution) given by

$$f(t) = \frac{1}{5000} t, \; 0 \leq t \leq 100$$

Calculate the MTTF, the B_{50} life (the median life), and the mode.

Examples

Example 10

For the exponential distribution with mean=100, calculate the B_{50} life (the median life), and the mode.