First Order Logic (Predicate Logic)

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Limitations of propositional logic

- World we want to represent and reason about consists of a number of objects with variety of properties and relations among them
- Propositional logic:
  - Represents statements about the world without reflecting this structure and without modeling these entities explicitly
- Consequence:
  - some knowledge is hard or impossible to encode in the propositional logic.
  - Two cases that are hard to represent:
    - Statements about similar objects, or relations
    - Statements referring to groups of objects.

Limitations of propositional logic

- Statements about similar objects and relations needs to be enumerated
- Example: Seniority of people domain

  For inferences we need:

  $$\textit{John is older than Mary} \;\wedge\; \textit{Mary is older than Paul}$$
  $$\Rightarrow \textit{John is older than Paul}$$

  $$\textit{Jane is older than Mary} \;\wedge\; \textit{Mary is older than Paul}$$
  $$\Rightarrow \textit{Jane is older than Paul}$$

- Problem: if we have many people and their age relations we need to represent many rules to support the inferences
- Possible solution: introduce variables

  $$\textit{PersA is older than PersB} \;\wedge\; \textit{PersB is older than PersC}$$
  $$\Rightarrow \textit{PersA is older than PersC}$$

Limitations of propositional logic

- Statements referring to groups of objects require exhaustive enumeration of objects

- Example:

Assume we want to express Every student likes vacation Doing this in propositional logic would require to include statements about every student

$$John \ likes \ vacation \quad \wedge$$

$$Mary \ likes \ vacation \quad \wedge$$

$$Ann \ likes \ vacation \quad \wedge$$

$$\ldots$$

- Solution: Allow quantification in statements

First Order Logic

- More expressive than propositional logic
- Main additions:
  - Represents explicitly objects, their properties, relations and lets us make statements about them;
  - Introduces variables that refer to arbitrary objects of certain type that can be substituted by a specific object
  - Introduces quantifiers allowing us to make statements for groups of objects without the need to represent each of them separately

First Order Logic : Syntax

- Term – a syntactic entity for representing objects
- Terms in FOL:
  - Constant symbols: represent specific objects
    - E.g. *John, France, car89*
  - Variables: represent objects of a certain type (type = domain of discourse)
    - E.g. *x,y,z*
  - Functions applied to one or more terms
    - E.g. *father-of (John)*
      father-of(father-of(John))

First Order Logic : Syntax

- Sentences in FOL:

- Atomic sentences:
  - A predicate symbol applied to 0 or more terms
  - Examples:
    - Red(car12),
    - Sister(Amy, Jane);
    - Manager(father-of(John));
  - t1 = t2 equivalence of terms
  - Example:
    - John = father-of(Peter)

First Order Logic : Syntax

- Sentences in FOL:
- Complex sentences:

  Assume ϕ, ψ are sentences in FOL. Then:

  - $(\phi \wedge \psi) \quad (\phi \vee \psi) \quad (\phi \Rightarrow \psi) \quad (\phi \Leftrightarrow \psi) \quad \neg \psi$

    and

  - $\forall x \, \phi \qquad \exists y \, \phi$

    are sentences

  Symbols $\quad \exists, \forall$

- stand for the **existential** and the **universal** quantifier

FOL Semantics: Interpretation

- An interpretation I is defined by a mapping constants, predicates and function to the domain of discourse D or relations on D

- domain of discourse: a set of objects in the world we represent and refer to.

- An interpretation I maps:
  - Constant symbols to objects in D

  $$I(John) = $$

  - Predicate symbols to relations, properties on D

  $$I(brother) = $$

  - Function symbols to functional relations on D

  $$I(father\text{-}of) = $$

FOL Semantics of sentences

- A predicate predicate(term-1, term-2, term-3, term-n) is true for the interpretation I , iff the objects referred to by term-1, term-2, term-3, term-n are in the relation referred to by predicate

$I(John) = $ $I(Paul) = $

$I(brother) = $

$brother(John, Paul) = $ in $I(brother)$

$V(brother(John, Paul), I) = $ **True**

FOL Semantics of sentences

- Equality

  $V(term\text{-}1 = term\text{-}2, I) = \textbf{True}$

  Iff $I(term\text{-}1) = I(term\text{-}2)$

- Boolean expressions: standard

  E.g. $V(sentence\text{-}1 \lor sentence\text{-}2, I) = \textbf{True}$

  Iff $V(sentence\text{-}1, I) = \textbf{True}$ or $V(sentence\text{-}2, I) = \textbf{True}$

- Quantifications

  $V(\forall x \; \phi, I) = \textbf{True}$ substitution of x with d

  Iff for all $d \in D$ $V(\phi, I[x/d]) = \textbf{True}$

  $V(\exists x \; \phi, I) = \textbf{True}$

  Iff there is a $d \in D$, s.t. $V(\phi, I[x/d]) = \textbf{True}$

Sentences with quantifiers

- Universal quantification

  All CS students are smart

- Assume the universe of discourse of x are CS students

  $\sqcup x \, smart(x)$

- Assume the universe of discourse of x are students

  $\sqcup x \, at(x, CS) \sqcup smart(x)$

- Assume the universe of discourse of x are people

  $\sqcup x \, student(x) \sqcup at(x, CS) \sqcup smart(x)$

Sentences with quantifiers

- Existential quantification

  Someone at IT is smart

- Assume the universe of discourse of x are IT students

  $\sqcup$ *x smart(x)*

- Assume the universe of discourse of x are people

  $\sqcup$ *x at(x, IT)* $\sqcup$ *smart(x)*

Sentences with quantifiers

- ($\forall$) E.g.,
  - "All cats are mammals" becomes
  - ($\forall$ x CAT(x) $\Rightarrow$ MAMMAL(x)).
  - NOTE: we can also read this as "if x is a CAT, then x is also a MAMMAL.

Nested quantifiers

- More than one quantifier may be necessary to capture the meaning of a statement in the predicate logic.

- Example:

- There is a person who loves everybody.

- Translation:
  - Assume:
    - Variables x and y denote people
    - A predicate L(x,y) denotes: "x loves y"
  - Then we can write in the predicate logic:

    $$\sqcup x \sqcup y \, L(x,y)$$

Translation exercise

- Suppose:
  - Variables x,y denote people
  - L(x,y) denotes "x loves y".
- Translate:
- Everybody loves Raymond. □*x L(x,Raymond)*
- Everybody loves somebody. □*x*□*y L(x,y)*
- There is somebody whom everybody loves. □*y*□*x L(x,y)*
- There is somebody who Raymond doesn't love. □*y¬L(Raymond,y)*
- There is somebody whom no one loves. □*y* □*x ¬L(x,y)*

Order of quantifiers

- Switching the order of multiple universal quantifiers does not change the meaning;

$$\forall x \; \forall y \; P(x,y) \Leftrightarrow \forall y \; \forall x \; P(x,y)$$

- Switching the order of multiple existential quantifiers does not change the meaning.

$$\exists x \; \exists y \; P(x,y) \Leftrightarrow \exists y \; \exists x \; P(x,y)$$

- Switching the order of a universal quantifier and an existential quantifier **DOES change meaning**.
  - E.g., if LOVES(x,y) means "x loves y" then
    - *$\forall x \; \exists y \; LOVES \;(x,y)$ means "Everyone loves someone "*
    - *$\exists y \; \forall x \; LOVES \;(x,y)$ means "Someone is loved by everyone"*

Connections between quantifiers

Everyone likes ice cream

$$\forall x \; likes \, (x, IceCream \;)$$

- Is it possible to convey the same meaning using an existential quantifier ?

There is no one who does not like ice cream

$$\neg \exists x \; \neg likes \, (x, IceCream \;)$$

- A universal quantifier in the sentence can be expressed using an existential quantifier !!!

Connections between quantifiers

Someone likes ice cream

$\exists x \; likes \; (x, IceCream \quad)$

- Is it possible to convey the same meaning using a universal quantifier ?

Not everyone does not like ice cream

$\neg \forall \; x \; \neg likes \; (x, IceCream \quad)$

- An existential quantifier in the sentence can be expressed using a universal quantifier !!!

Connections between quantifiers

- Negation of quantifiers (De Morgan's rule):

  - $\neg \exists x \, P(x)$ is equivalent to $\forall x \, \neg \, P(x)$
  - $\neg \forall x \, P(x)$ is equivalent to $\exists x \, \neg \, P(x)$

Using FOL : Kinship domain

- **Objects:** people

  $John$, $Mary$, $Jane$, …

- **Properties:** gender

  $Male\ (x)$, $Female\ (x)$

- **Relations:** parenthood, brotherhood, marriage

  $Parent\ (x, y)$, $Brother\ (x, y)$, $Spouse\ (x, y)$

- **Functions:** mother-of (one for each person x)

  $MotherOf\ (x)$

Using FOL : Kinship domain

- Relations between predicates and functions: write down what we know about them; how relate to each other.

  - Male and female are disjoint categories

    $$\forall x \; Male\,(x) \Leftrightarrow \neg Female\,(x)$$

  - Parent and child relations are inverse

    $$\forall x, y \; Parent\,(x, y) \Leftrightarrow Child\,(y, x)$$

  - A grandparent is a parent of parent

    $$\forall g, c \; Grandparent(g, c) \Leftrightarrow \exists p \; Parent(g, p) \wedge Parent(p, c)$$

  - A sibling is another child of one's parents

    $$\forall x, y \; Sibling\,(x, y) \Leftrightarrow (x \neq y) \wedge \exists p \; Parent\,(p, x) \wedge Parent\,(p, y)$$

  - And so on ....

Exercise

- Consider the following story:

"Anyone passing her artificial intelligence exam and winning the lottery is happy. But anyone who studies or is lucky can pass all her exams. Lamia did not study but is lucky. Anyone who is lucky wins the lottery".

- Write this story in FOL.

Inference in first-order logic

- Logical inference problem:
- Given a knowledge base KB (a set of sentences) and a sentence α , does the KB semantically entail α?
- In other words: In all interpretations in which sentences in the KB are true, α is also true?
- Logical inference problem in the Propositional logic
  - Computational procedures that answer:

$$KB \models \alpha \text{ ?}$$

  - **Truth-table approach**
  - **Inference rules**
  - **Resolution-refutation**
  - **Inference with Horn Clauses**
  - **Forward chaining**
  - **Backward chaining**

Inference in FOL: Truth table

- Is the Truth-table approach a viable approach for the FOL?
  - NO!
- Why?
  - It would require us to enumerate and list all possible interpretations I.
  - I = (assignments of symbols to objects, predicates to relations and functions to relational mappings)
  - Simply there are too many interpretations

Inference in FOL: Inference rules

- Is the Inference rule approach a viable approach for the FOL?
  - Yes.

- ⊔ The inference rules represent sound inference patterns one can apply to sentences in the KB.
  - What is derived by inference rules follows from the KB
  - But,
    - we need to add rules for handling quantifiers

Inference rules

- Inference rules from the propositional logic:
  - Modus ponens

$$\frac{A \Rightarrow B, \quad A}{B}$$

  - Resolution

$$\frac{A \vee B, \quad \neg B \vee C}{A \vee C}$$

  - and others: And-introduction, And-elimination, Or-introduction, Negation elimination
- Additional inference rules are needed for sentences with quantifiers and variables
  - Must involve variable substitutions

Sentences with variables

- First-order logic sentences can include variables.
  - Variable is:
    - Bound – if it is in the scope of some quantifier
    $$\Box x \; P(\, x)$$
    - Free – if it is not bound.
    $$\Box x \; P(\, y) \; \Box \; Q(\, x) \qquad y \; is \; free$$

  - Examples:
    - $\Box x \; \Box y \; Likes\,(\,x, y)$: Bound or free? **Bound**
    - $\Box x \; \Box Likes\,(\,x, y) \; \Box \; \Box y \; Likes\,(\,y, Raymond\,) \Box$: Bound or free? **Free**

Sentences with variables

- First-order logic sentences can include variables.
- Sentence (formula) is:
  - Closed – if it has no free variables

    $$\sqcup y \; \sqcup x \; P(y) \; \sqcup Q(x)$$

  - Open – if it is not closed

    $$\sqcup x \; P(\, y) \; \sqcup \; Q(\, x) \; y \; is \; free$$

  - Ground – if it does not have any variables

    $$Likes \; (John, \; Jane)$$

Variable substitutions

- Variables in the sentences can be substituted with terms.
  - (terms = constants, variables, functions)
- Substitution:
  - Is represented by a mapping from variables to terms
    {x1 / t1 , x2 / t2 ,.....}
  - Application of the substitution to sentences
    - **SUBST({x / Sam, y / Pam},Likes(x, y)) ⌐ Likes(Sam, Pam)**
    - **SUBST ({x / z, y / fatherof (John)}, Likes(x, y)) ⌐ Likes (z, fatherof (John))**

Inference rules for quantifiers

- Universal elimination

$$\frac{\forall x \; \phi(x)}{\phi(a)}$$

a - is a constant symbol

  - substitutes a variable with a constant symbol

$\forall x \; Likes(x, IceCream)$ $Likes(Ben, IceCream)$

- Existential elimination.

$$\frac{\exists x \; \phi(x)}{\phi(a)}$$

  - Substitutes a variable with a constant symbol that does not appear elsewhere in the KB

$\exists x \; Kill(x, Victim)$ $Kill(Murderer, Victim)$

Inference rules for quantifiers

- Universal instantiation (introduction)

$$\frac{\phi}{\forall x\ \phi} \qquad x - \text{is not free in } \phi$$

  - Introduces a universal variable which does not affect Φ or its assumptions.

$$Sister(Amy, Jane) \qquad \forall x\ Sister(Amy, Jane)$$

- Existential instantiation (introduction)

$$\frac{\phi(a)}{\exists x\ \phi(x)} \qquad \begin{array}{l} a - \text{is a ground term in } \phi \\ x - \text{is not free in } \phi \end{array}$$

  - Substitutes a ground term in the sentence with a variable and an existential statement

$$Likes(Ben, IceCream) \qquad \exists x\ Likes(x, IceCream)$$

Unification

- **Problem in inference: Universal elimination** gives us many opportunities for substituting variables with ground terms

$$\frac{\forall x \; \phi(x)}{\phi(a)} \qquad a \text{ - is a constant symbol}$$

- Solution: <u>make only substitutions that may help</u>
  - Use substitutions of "similar" sentences in KB

- Unification – takes two similar sentences and computes the substitution that makes them look the same, if it exists

$$UNIFY \; (p,q) = \sigma \; \text{ s.t. } SUBST(\; \sigma , p) = SUBST \; (\sigma , q)$$

Unification: Examples.

- Unification

$$UNIFY\ (p, q) = \sigma \ \ \text{s.t.}\ SUBST(\ \sigma, p) = SUBST\ (\sigma, q)$$

- Examples:

$$UNIFY\ (Knows\ (John, x), Knows\ (John, Jane)) = \{x\ /\ Jane\}$$

$$UNIFY(Knows(John, x), Knows(y, Ann)) = \ ?$$

$$UNIFY\ (Knows\ (John, x), Knows\ (y, MotherOf\ (y)))$$
$$= \ ?$$

$$UNIFY\ (Knows\ (John, x), Knows\ (x, Elizabeth)) = \ ?$$

Unification: Examples.

- Unification

$$UNIFY\ (p, q) = \sigma \quad \text{s.t.}\ SUBST(\sigma, p) = SUBST\ (\sigma, q)$$

- Examples:

$$UNIFY\ (Knows\ (John, x), Knows\ (John, Jane)) = \{x\ /\ Jane\}$$

$$UNIFY(Knows(John, x), Knows(y, Ann)) = \{x\ /\ Ann, y\ /\ John\}$$

$$UNIFY\ (Knows\ (John, x), Knows\ (y, MotherOf\ (y)))$$
$$= \{x\ /\ MotherOf\ (John), y\ /\ John\}$$

$$UNIFY\ (Knows\ (John, x), Knows\ (x, Elizabeth)) = fail$$

Generalized inference rules

- Use substitutions that let us make inferences !!
- Example: Generalized Modus Ponens
  - If there exists a substitution σ such that

$$SUBST\ (\sigma, A_i) = SUBST\ (\sigma, A_i') \quad \textbf{for all i=1,2, n}$$

$$\frac{A_1 \wedge A_2 \wedge \ldots A_n \Rightarrow B, \quad A_1', A_2', \ldots A_n'}{SUBST\ (\sigma, B)}$$

- Substitution that satisfies the generalized inference rule can be build via unification process
- Advantage of the generalized rules: they are focused – only substitutions that allow the inferences to proceed

Generalized inference rules

- Recall: Resolution inference rule is sound and complete for the propositional logic and CNF

$$\frac{A \vee B, \quad \neg A \vee C}{B \vee C}$$

- Generalized resolution rule is sound and complete for the first-order logic and CNF without equalities

$$\sigma = UNIFY \ (\phi_i, \neg \psi_j) \neq fail$$

$$\frac{\phi_1 \vee \phi_2 \ldots \vee \phi_k, \quad \psi_1 \vee \psi_2 \vee \ldots \psi_n}{SUBST(\sigma, \phi_1 \vee \ldots \vee \phi_{i-1} \vee \phi_{i+1} \ldots \vee \phi_k \vee \psi_1 \vee \ldots \vee \psi_{j-1} \vee \psi_{j+1} \ldots \psi_n)}$$

Example :

$$\frac{P(x) \vee Q(x), \quad \neg Q(John) \vee S(y)}{P(John) \vee S(y)}$$

Inference with resolution rule

- Proof by refutation:
  - Prove that KB, ⎯☐ is unsatisfiable
  - resolution is refutation-complete

- Main procedure (steps):

  1. Convert KB, ⎯☐ to CNF with ground terms and universal variables only

  2. Apply repeatedly the resolution rule while keeping track and consistency of substitutions

  3. Stop when empty set (contradiction) is derived or no more new resolvents (conclusions) follow

Conversion to CNF

1. Eliminate implications, equivalences

$$(p \Rightarrow q) \rightarrow (\neg p \vee q)$$

2. Move negations inside (DeMorgan's Laws, double negation)

$$\neg(p \wedge q) \rightarrow \neg p \vee \neg q$$
$$\neg(p \vee q) \rightarrow \neg p \wedge \neg q$$

$$\neg \forall x \; p \rightarrow \exists x \; \neg p$$
$$\neg \exists x \; p \rightarrow \forall x \; \neg p$$
$$\neg \neg \; p \rightarrow p$$

3. Standardize variables (rename duplicate variables)

$$(\forall x \; P(x)) \vee (\exists x \; Q(x)) \rightarrow (\forall x \; P(x)) \vee (\exists y \; Q(y))$$

4. Move all quantifiers left (no invalid capture possible)

$$(\forall x \; P(x)) \vee (\exists y \; Q(y)) \rightarrow \forall x \; \exists y \; P(x) \vee Q(y)$$

Conversion to CNF

5. Skolemization (removal of existential quantifiers through elimination)

- If no universal quantifier occurs before the **existential quantifier**, replace the **variable with a new constant symbol** also called **Skolem constant**

$$\exists y \; P(A) \vee Q(y) \rightarrow P(A) \vee Q(B)$$

- If a universal quantifier precedes the existential quantifier replace the variable with a function of the "universal" variable

$$\forall x \; \exists y \; P(x) \vee Q(y) \rightarrow \forall x \; P(x) \vee Q(F(x))$$

$F(x)$ - **a special function**
   - **called Skolem function**

Conversion to CNF

6. Drop universal quantifiers (all variables are universally quantified)

$$\forall x \quad P(x) \vee Q(F(x)) \rightarrow P(x) \vee Q(F(x))$$

7. Convert to CNF using the distributive laws

$$p \vee (q \wedge r) \rightarrow (p \vee q) \wedge (p \vee r)$$

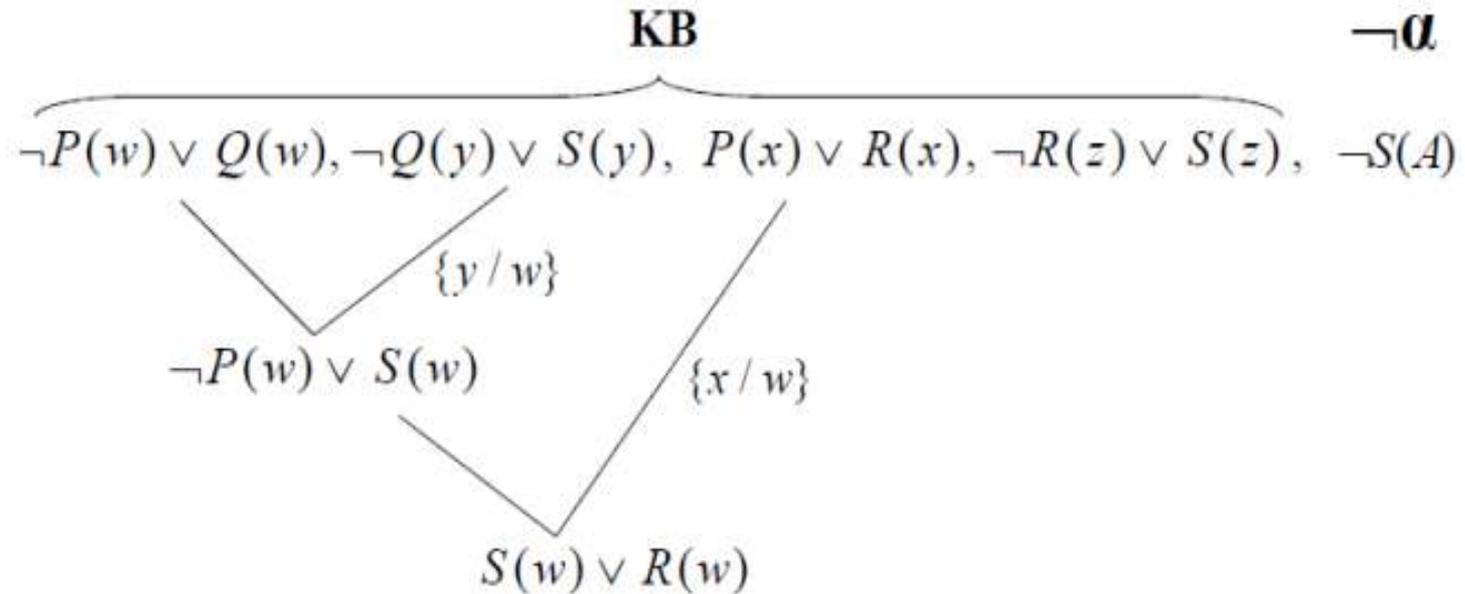**The result is a CNF with variables, constants, functions**

Resolution example

$$\overbrace{\neg P(w) \vee Q(w), \neg Q(y) \vee S(y), \ P(x) \vee R(x), \neg R(z) \vee S(z),}^{\textbf{KB}} \quad \overset{\neg \alpha}{\neg S(A)}$$

Resolution example

$$\overbrace{\neg P(w) \vee Q(w), \neg Q(y) \vee S(y), \ P(x) \vee R(x), \neg R(z) \vee S(z),}^{\textbf{KB}} \ \overbrace{\neg S(A)}^{\neg\alpha}$$

$$\{y / w\}$$

$$\neg P(w) \vee S(w)$$

Resolution example

$$\overbrace{\neg P(w) \lor Q(w), \neg Q(y) \lor S(y), \ P(x) \lor R(x), \neg R(z) \lor S(z)}^{\textbf{KB}}, \ \overset{\neg \alpha}{\neg S(A)}$$

$$\{y / w\}$$

$$\neg P(w) \lor S(w)$$

$$\{x / w\}$$

$$S(w) \lor R(w)$$

Resolution example

KB   **¬α**

$\overbrace{\neg P(w) \vee Q(w), \neg Q(y) \vee S(y), \; P(x) \vee R(x), \neg R(z) \vee S(z),}$ $\neg S(A)$

$\{y / w\}$

$\neg P(w) \vee S(w)$

$\{x / w\}$

$\{z / w\}$

$S(w) \vee R(w)$

$S(w)$

Resolution example

KB $\qquad$ $\neg\alpha$

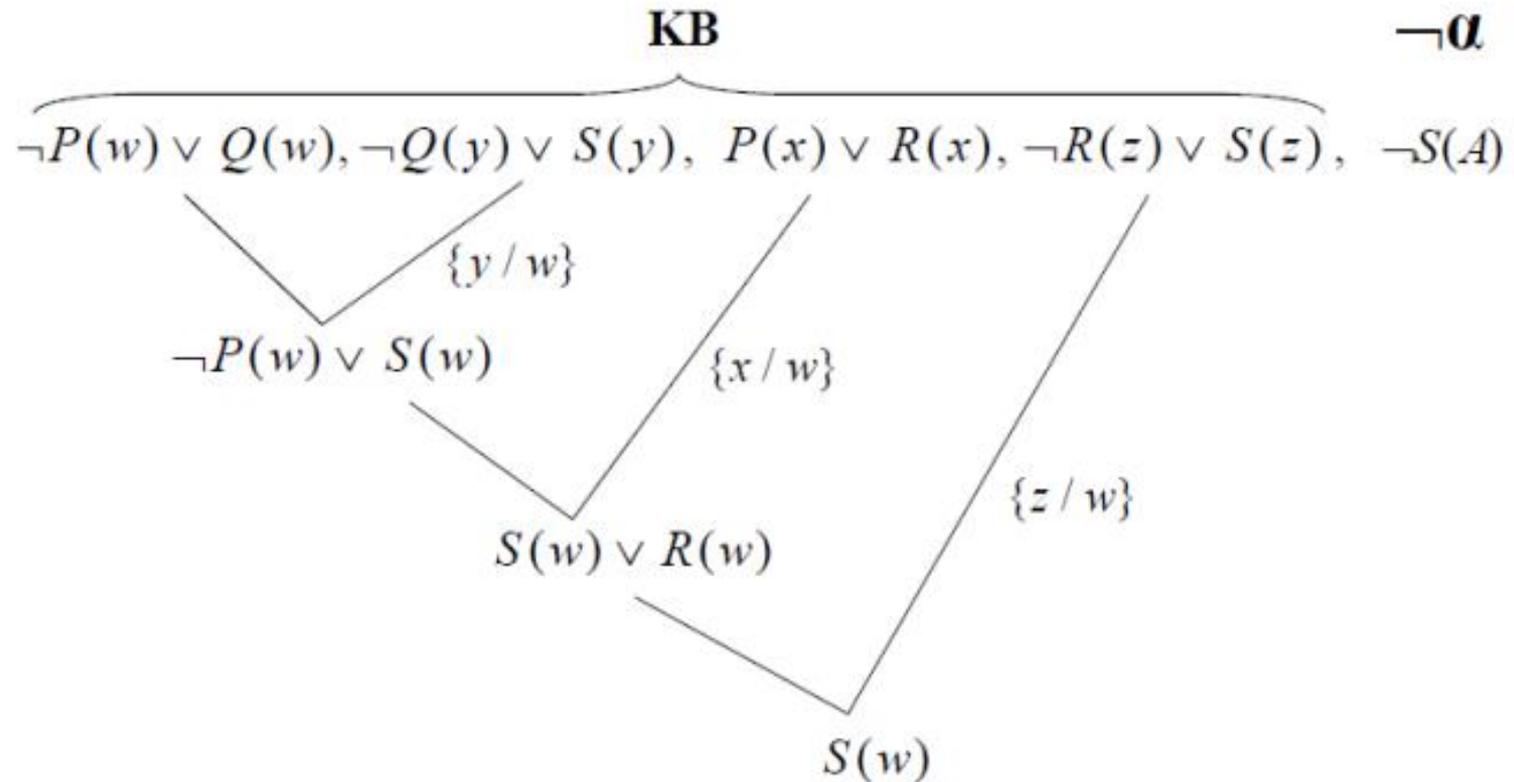$\neg P(w) \vee Q(w), \neg Q(y) \vee S(y), \ P(x) \vee R(x), \neg R(z) \vee S(z), \ \neg S(A)$

$\{y / w\}$

$\neg P(w) \vee S(w)$

$\{x / w\}$

$S(w) \vee R(w)$

$\{z / w\}$

$S(w)$ $\{w / A\}$

$KB \models \alpha$

Empty resolution
→ Contradiction
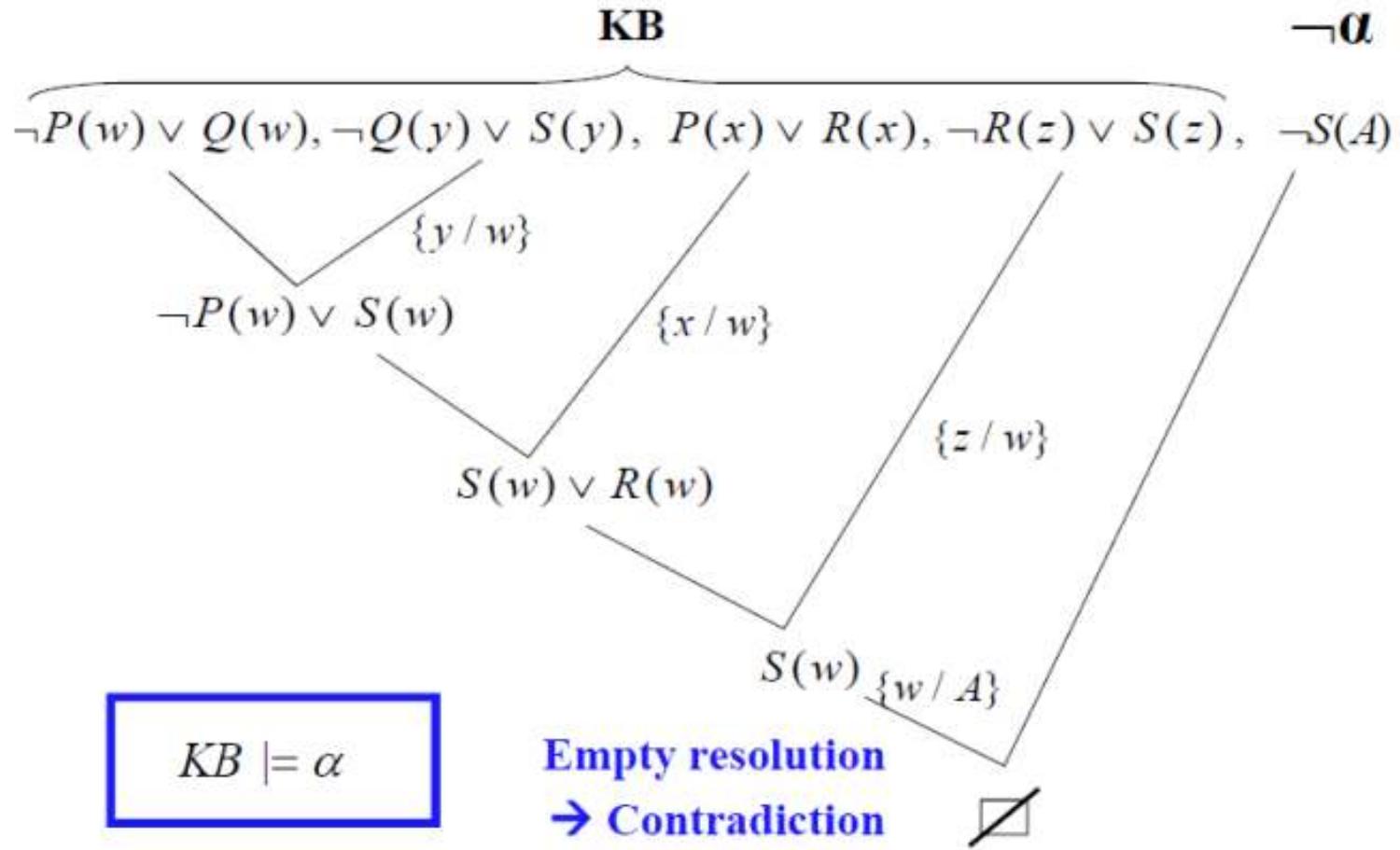
- Consider the following story:

"Anyone passing her artificial intelligence exam and winning the lottery is happy. But anyone who studies or is lucky can pass all her exams. Lamia did not study but is lucky. Anyone who is lucky wins the lottery. <u>Is Lamia happy</u>?".