IE360: CAD/CAM

Computer Aided Design and Computer Aided Manufacturing

Lecture (4)
Geometric Transformations

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Introduction:

- Geometric transformations play a central role in model construction and viewing.
  - They are used in modeling to express locations of objects relative to others.
  - In generating a view of an object, they are used to achieve the effect of different viewing positions and directions.
- Typical CAD/CAM construction commands to translate, rotate, zoom, and mirror entities are all based on geometric transformations.
- Once the model construction is complete, its viewing in its modeling space is achieved via geometric transformations.
  - Orthographic views for engineering drawings can be obtained by projecting the model onto the proper plane.
  - In addition, the model itself can be rotated or scaled up and down to view it in its three-dimensional space.
- Geometric transformations can also be used to create animated files of geometric models to study their motion.

Coordinate Systems:

- Three types of coordinate systems are needed in order to input, store, and display model geometry and graphics. These are the *world coordinate system* (WCS), the *model coordinate system* (MCS), and the *screen coordinate system* (also named device coordinate system) (SCS).

- The WCS system:
  - WCS is the reference space of the model with respect to which all the model geometrical data is stored.
  - It is a Cartesian system which forms the default coordinate system.
  - WCS is the only coordinate system that the CAD/CAM software recognizes when storing or retrieving geometrical information in or from a model database.

- In order for the user to communicate properly and effectively with a model, the relationships between the WCS orthogonal planes and the model views must be understood by the user.
- Typically, there are two possible orientations of the WCS in space, as shown in Figure 1.

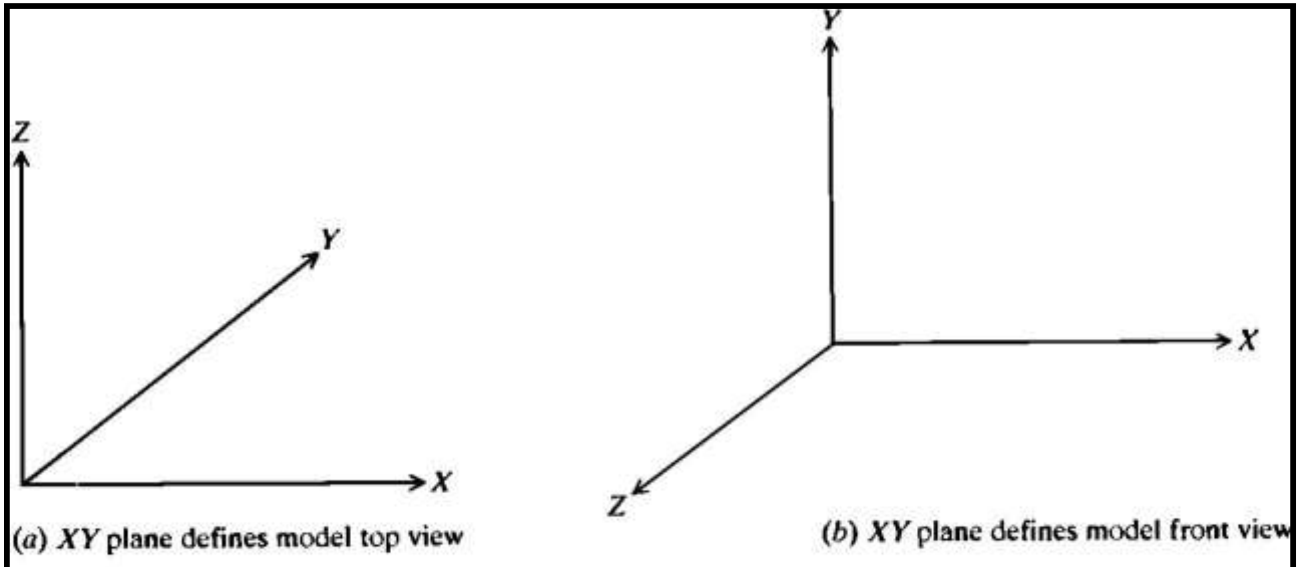


Figure 1: Possible orientations of the WCS in space.

- Figure 1a shows that the *XY* plane is the horizontal plane and defines the model top view. The front and right side views are consequently defined by the *XZ* and *YZ* planes, respectively.

- Figure 1b shows the other possible orientation where the *XY* plane is vertical and defines the front view. As a result, the *XZ* and *YZ* planes define the top and right side views, respectively.

- In both orientations, the *XY* plane is the default construction plane. If the user utilizes such a plane, the first face to be constructed of a model becomes the top or front view depending on which WCS is used.

➢ **Example 1:** Figure 2 shows a geometric model that is to be utilized for design and manufacturing applications. The model is constructed by using point P2 as the origin of the WCS and the front face is created first. What are the top, front, right side, and isometric views a designer obtains when using the WCS shown in Figure 1b.

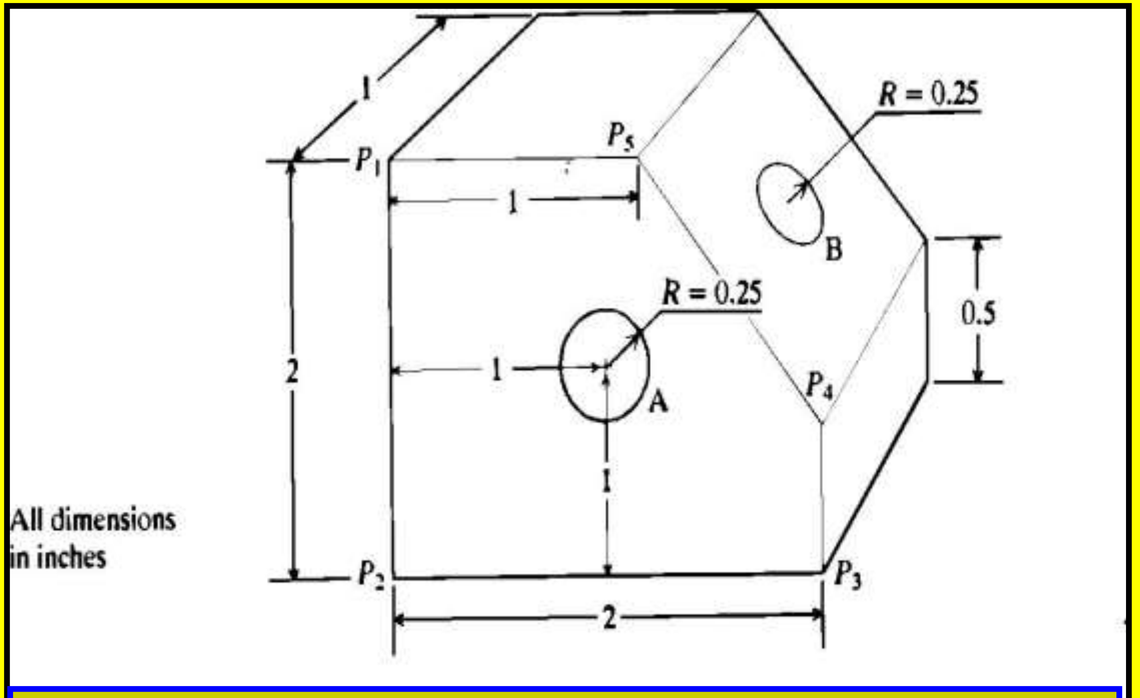


Figure 2: Geometric model of an object.

➢ **Answer:** Under the above construction conditions, the orientation of the WCS relative to the model and the corresponding views are shown in Figure 3.

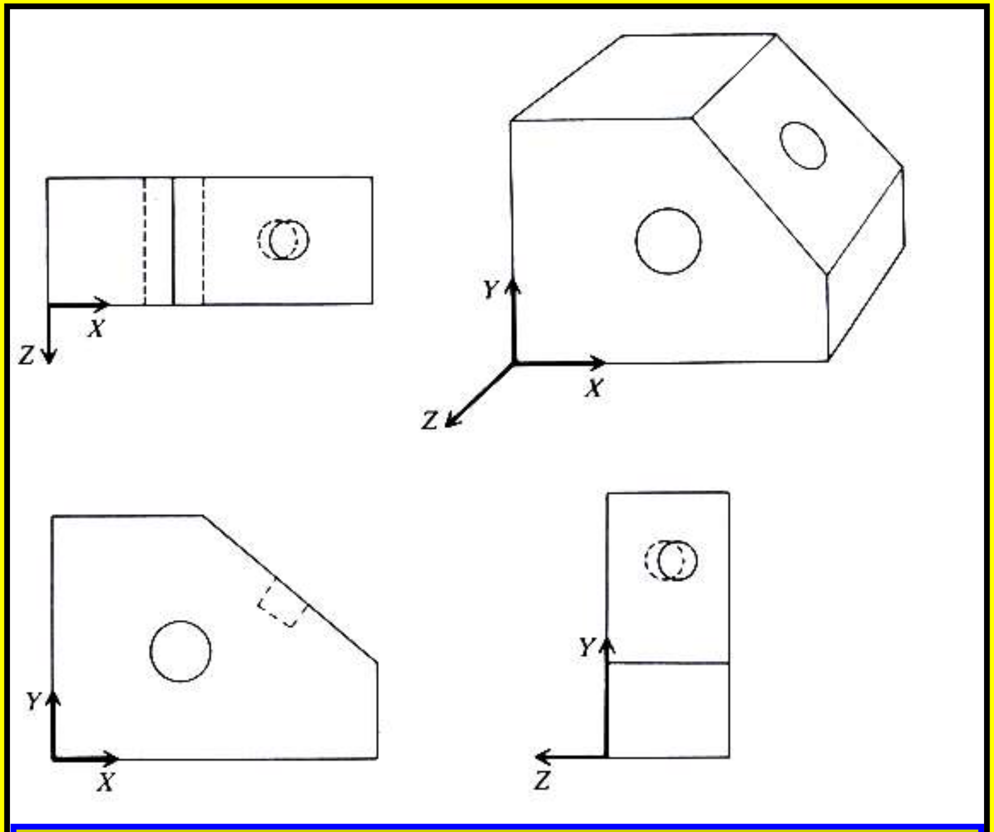


Figure 3: Views of the object shown in Figure 2, utilizing the WCS of Figure 1a.

➢ The MCS system:

- It is often convenient in the development of geometric models and the input of geometrical data to refer to an auxiliary coordinate system instead of the WCS.

- This is usually useful when a desired plane (face) of construction is not easily defined as one of the WCS orthogonal planes, as in the case of inclined faces of a model.

- The user can define a Cartesian coordinate system whose *XY* plane is coincident with the desired plane of construction. That system is the *model coordinate system* (MCS)

- The MCS is a convenient user-defined system that facilitates geometric construction.

- It can be established at any position and orientation in space that the user desires.

- While the user can input data in reference to the MCS, the CAD/CAM software performs the necessary transformations to the WCS before storing the data.

➢ The SCS system:

- The SCS is a two-dimensional device-dependent coordinate system whose origin is usually located ate the lower left corner of the graphic display, as shown in Figure 4.

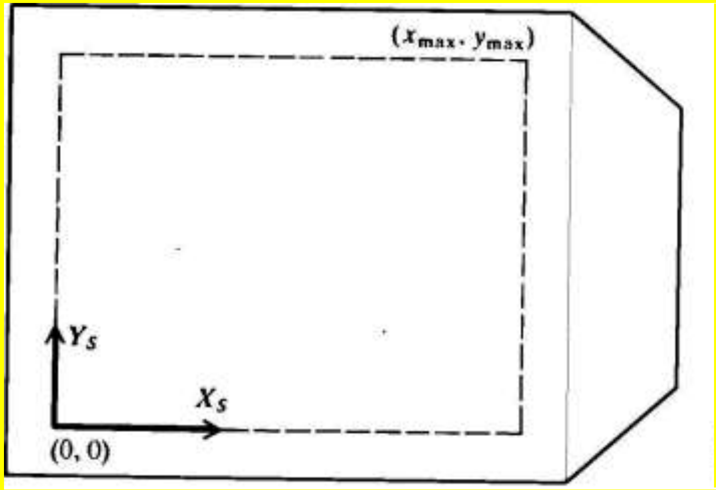


Figure 4: Typical SCS.

Transformations of Geometric Models:

- As previously mentioned, the CAD/CAM software interprets user coordinate inputs in reference to the MCS. At the meantime, the software calculates the corresponding homogeneous transformation matrix between the MCS and the WCS to convert these inputs into coordinates relative to the WCS before storing them in the database.

- Geometric transformations typically include *translation*, *rotation*, *scaling* and any combination of them. They can be easily implemented using matrix notations.

- **Translation:**
  - When an object is translated by a, b, and c in the x, y, and z directions, respectively, from its initial position at which its MCS coincides with the WCS (see Figure 5), the world coordinates of a point on the object at the new position, (X_w, Y_w, and Z_w) are obtained as follows:

$$X_w = X_m + a$$
$$Y_w = Y_m + b \qquad (1)$$
$$Z_w = Z_m + c$$

- In Equation (1), X_m, Y_m, and Z_m also are the model coordinates of the same point. Equation (1) can be expressed in the following form, using matrix operations:

$$\begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_m \\ Y_m \\ Z_m \\ 1 \end{bmatrix} \qquad (2)$$

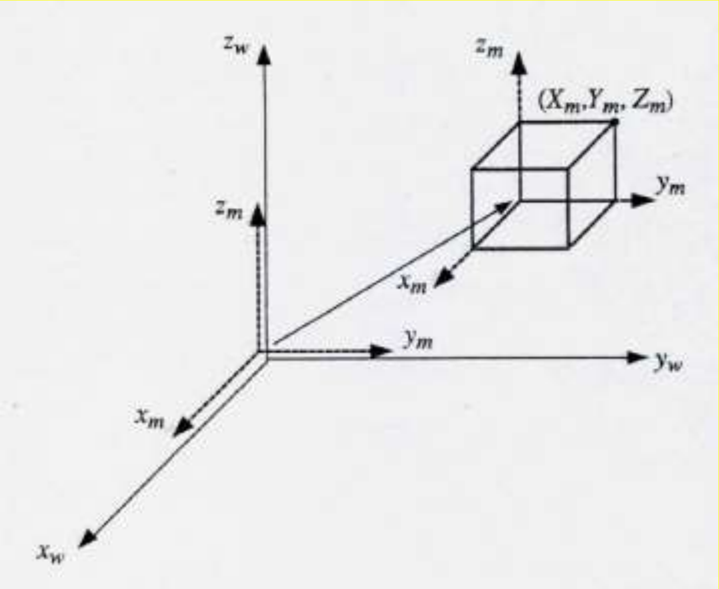


Figure 5: Translation of an object.

- The addition operation in Equation (1) could be expressed as the multiplication operation in Equation (2) by using homogeneous coordinates that represent a three-dimensional vector by four scalars instead of three.

- The matrix used to transform the homogeneous coordinates is called the *homogeneous transformation matrix*.

- Therefore, the transformation matrix on the right-hand side of Equation (2), denoted $Trans(a,b,c)$, is a *homogeneous transformation matrix* for a translation.

➢ **Rotation:**

- Suppose that an object is rotated by θ about the x axis of the WCS together with its MCS (which again coincides the WCS at its initial position, as illustrated in Figure 6).

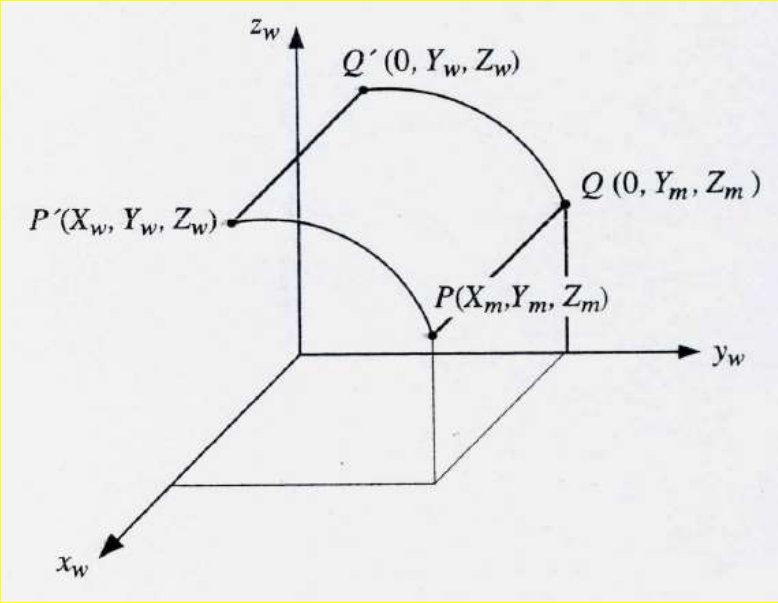


Figure 6: Rotation about the *x* axis.

- The world coordinates of a point on the object at the new position, (X_w, Y_w, Z_w), can be obtained from its original coordinates, (X_m, Y_m, Z_m), as follows.

- Here, (X_m, Y_m, Z_m) are the coordinates of the point with respect to the MCS, and thus are equal to its WCS before rotation.

▪ The relation between (X_w, Y_w, Z_w) and (X_m, Y_m, Z_m) becomes clear when Figure 6 is projected onto the *yz* plane, as shown in Figure 7.

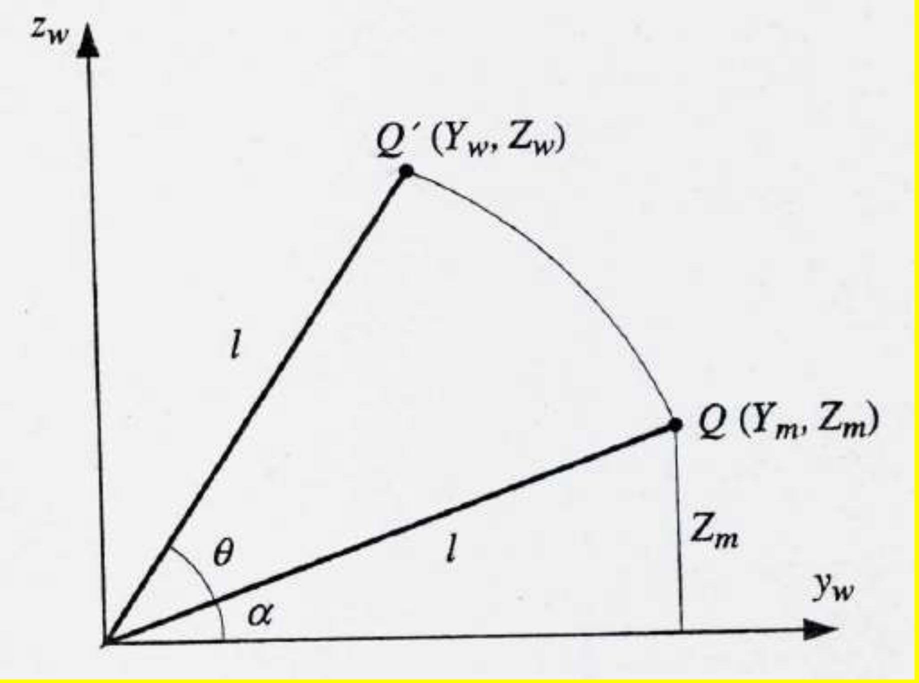


Figure 7: Projection onto the *yz* plane.

▪ From Figure 7, the following equations can easily be obtained.

$$X_{w'} = X_m \tag{3}$$

$$\begin{aligned} Y_w &= l\cos(\theta + \alpha) \\ &= l(\cos\theta \cos\alpha - \sin\theta \sin\alpha) \\ &= l \cos\alpha \cos\theta - l \sin\alpha \sin\theta \\ &= Y_m \cos\theta - Z_m \sin\theta \end{aligned} \tag{4}$$

$$\begin{aligned} Z_w &= l\sin(\theta + \alpha) \\ &= l(\sin\theta \cos\alpha + \cos\theta \sin\alpha) \\ &= l \cos\alpha \sin\theta + l \sin\alpha \cos\theta \\ &= Y_m \sin\theta + Z_m \cos\theta \end{aligned} \tag{5}$$

- Equations (3), (4), and (5) can be expressed in matrix form as

$$\begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_m \\ Y_m \\ Z_m \\ 1 \end{bmatrix} \tag{6}$$

- The matrix on the right-hand side of Equations (6) is a homogeneous transformation matrix for the rotation about the *x* axis and thus is denoted $Rot(x, \theta)$.

- The homogeneous transformation matrix for the rotation about the *y* or *z* axis can be derived similarly and expressed as

$$Rot(y, \theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (7)$$

$$Rot(z, \theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (8)$$

- The transformation matrices given in Equations (6), (7), and (8) assumes positive rotations about the coordinate axes as shown in Figure 8. In the figure, the coordinate system is right-handed and CCW rotations are assumed positive when looking along the axis toward the origin.

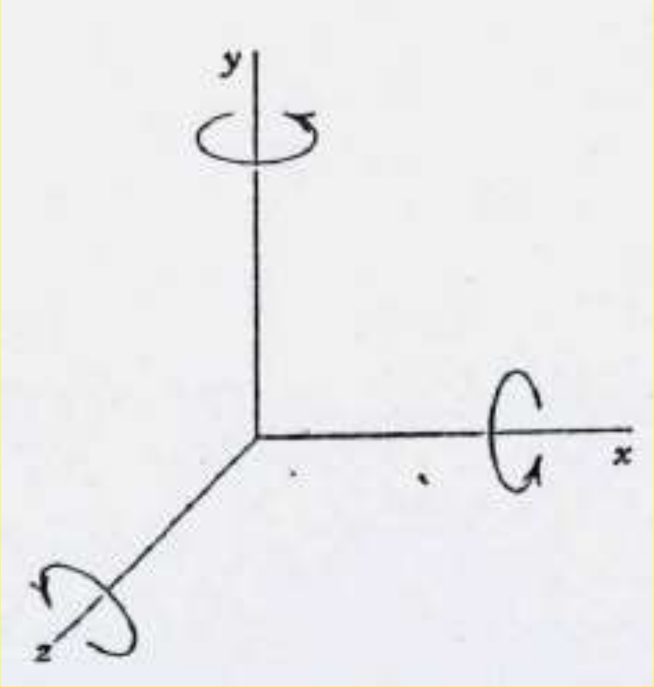


Figure 8: Positive rotations about the coordinate axes.

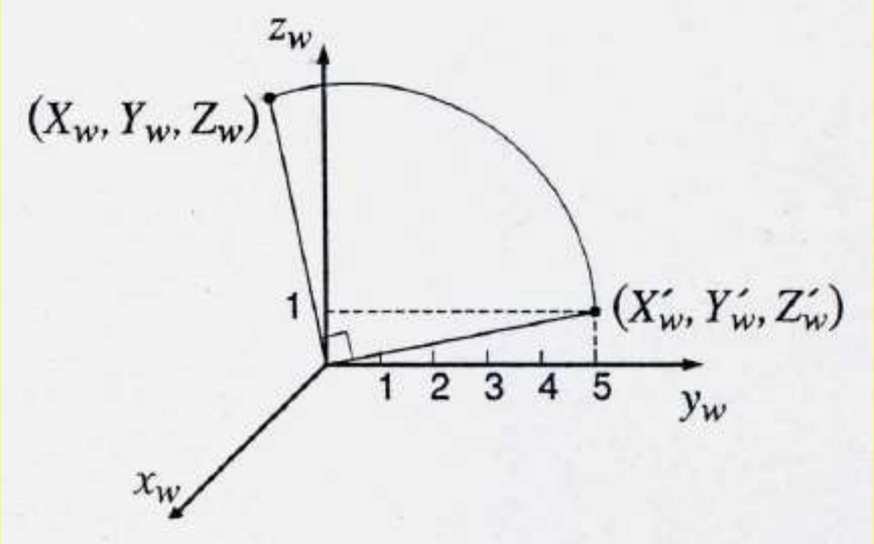
- If the rotations are CW, the matrices are expressed as follows:

$$Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (9)$$

$$Rot(y,\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (10)$$

$$Rot(z,\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (11)$$

➢ **Example 2:** An object in space is translated by 5 units in the *y* direction of the WCS and then rotated by 90 degrees about the *x* axis of the WCS. If a point on the object has the coordinates (0, 0, 1) with respect to the MCS, what will be the world coordinates of the same point after the translation and the rotation?



➢ **Answer:** The coordinates (X'_w, Y'_w, Z'_w) after translation can be obtained by:

$$\begin{bmatrix} X'_w \\ Y'_w \\ Z'_w \\ 1 \end{bmatrix} = Trans(0,5,0).\begin{bmatrix} X_m \\ Y_m \\ Z_m \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 1 \\ 1 \end{bmatrix}$$

- Then a rotation is applied:

$$\begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} = Rot(x,90^0)\begin{bmatrix} X'_w \\ Y'_w \\ Z'_w \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 90 & -\sin 90 & 0 \\ 0 & \sin 90 & \cos 90 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 5 \\ 1 \end{bmatrix}$$

➢ Rotations about Arbitrary Axes:

- Rotations about Arbitrary Axes in space are frequently found in engineering. Figure 9 illustrates an aspect of this fact – the robot grip rotates about axis *AB*, inclined with respect to the principal axes and not through the origin.

- This type of rotation is obtained by first using a sequence of translations and simple rotations that will make the arbitrary axis coincide with one of the principal coordinate axes; second, by performing the desired rotation; and third, by returning the axis to its original position. These steps are clarified in the following example.

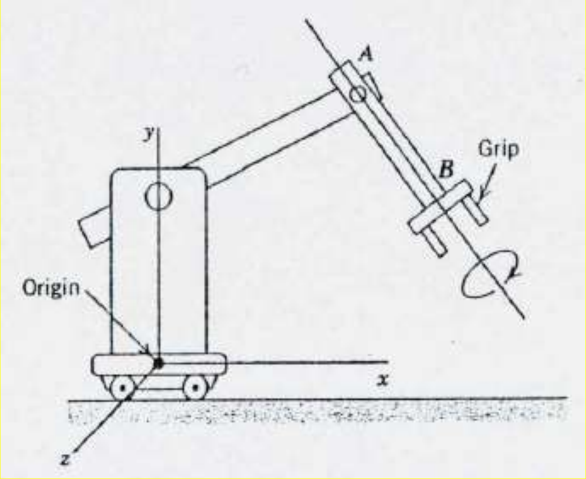
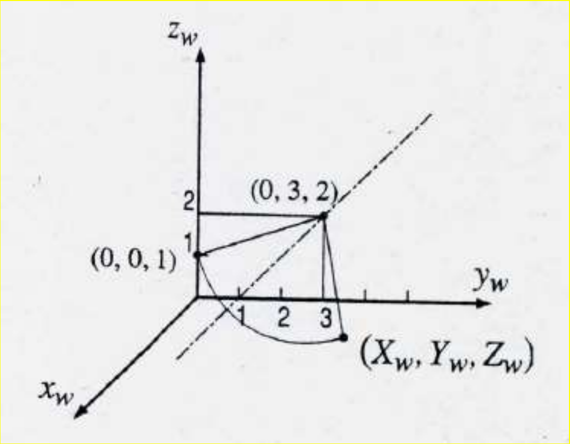


Figure 9: Robot grip rotates about axis *AB*, not one of the principal axes.

➢ **Example 3:** An object in space is rotated by 90 degrees about an axis that is parallel to the *x* axis of the WCS and passes through a point having world coordinate (0, 3, 2). If a point on the object has model coordinates (0, 0, 1), what will be the world coordinates of the same point after rotation?



➢ **Answer:** First, the rotation axis must pass through the origin and coincide with one of the principal coordinate axes.

- Thus, we have to translate the object by (0, -3, -2) together with the rotation axis so that the rotation axis coincides with the *x* axis of the WCS.
- Then, we have to rotate the object about the *x* axis by 90 degrees.
- Now, the object is translated again, by (0, 3, 2), to return to the original position.

- These operations can be expressed as

$$\begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} = Trans(0,3,2).Rot(x,90^0).Trans(0,-3,-2)\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 90^0 & -\sin 90^0 & 0 \\ 0 & \sin 90^0 & \cos 90^0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 5 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 1 \end{bmatrix}$$

- This result reflects the operations illustrated in the accompanying figure.

- In the above example, the concatenation of the transformations into a single matrix is a much more convenient expression, especially when the coordinates of numerous points need to be calculated.
    - In that case the transformation matrices *Trans*(0,3,2). *Rot*(*x*,90). *Trans*(0,-3,-2) can be multiplied in advance to give an equivalent transformation matrix, and then the resulting matrix is applied to all the points involved.
    - This process of calculating the equivalent transformation matrix by multiplying the associated transformation matrices in the proper sequence is called ***concatenation***.

- In general, if n transformations are applied to a point starting with transformation 1, with $[T_1]$, and ending with transformation n, with $[T_n]$, then the concatenated transformation of the point is given by

$$P^* = [T_n]\,[T_{n-1}]\,\ldots\,[T_2]\,[T_1]P \qquad (12)$$

➢ Scaling:

- To scale an object up or down s_x times in x, s_y times in y, and s_z times in z direction, the following transformation matrix is applied:

$$\begin{bmatrix} X' \\ Y' \\ Z' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \qquad (13)$$

- The transformation matrix in Equation (13) is used when the object is scaled with respect to the origin.

- However, it may often be desirable to scale an object with respect to a point P on the object represented by (X_P, Y_P, Z_P).

- In this case, a translational transformation, $Trans(-X_P,-Y_P,-Z_P)$, is applied first so that the reference point for the scaling is moved to the origin, then the scaling matrix in Equation (13) is applied, and finally $Trans(X_P,Y_P,Z_P)$ is applied to move the object back to its original position.

➢ **Mirroring:**

- With the *xy* plane as a mirror, the mirror reflection can be accompanied with the following transformation matrix because only the sign of the *z* coordinate has to be reversed:

$$\begin{bmatrix} X' \\ Y' \\ Z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \qquad (14)$$

- The transformation matrix for other mirror reflections, with the *yz* plane or the *xz* plane as a mirror, can be derived in the same way.

➢ **Example 4:** Consider a square object lies in the *xy* plane. The coordinates of its vertices are as follows: A = [0, 6], B =[0, 0], C =[6, 0], and D = [6, 6]. Find the coordinates of the square if it is shrinked to one third of its size, and then mirrored about the *yz* plane.

➢ **Answer:** The coordinates of the square after shrinking and mirroring can be obtained as follows:

$$P^* = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 & 6 \\ 6 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 & 6 \\ 6 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -2 & -2 \\ 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Mapping:

- Mapping involves calculating the coordinates of a point with respect to a coordinate system from known coordinates of the same point with respect to another coordinate system.

- Considering the two coordinate systems illustrated in Figure 10, the coordinates (X_2, Y_2, Z_2) of the point P with respect to the $x_2y_2z_2$ coordinate system can be calculated by applying the transformation matrix $T_{1\text{-}2}$ to (X_1, Y_1, Z_1), which are the coordinates of the same point with respect to the $x_1y_1z_1$ coordinate system.

$$[X_2 \;\; Y_2 \;\; Z_2 \;\; 1]^T = T_{1-2}.[X_1 \;\; Y_1 \;\; Z_1 \;\; 1]^T \tag{15}$$

- Replacing $T_{1\text{-}2}$ with its elements allows Equation (15) to be expressed as

$$\begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \\ 1 \end{bmatrix} = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \\ 1 \end{bmatrix} \tag{16}$$

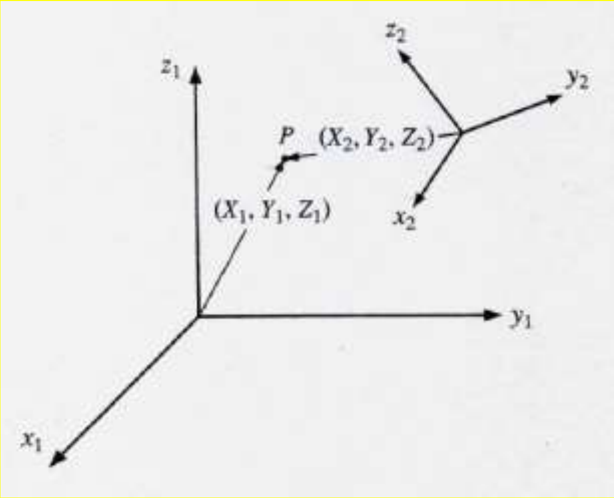


Figure 10: Mapping between two coordinate systems.

- The values n_x, n_y, and n_z are, respectively, the x_2, y_2, and z_2 components of a unit vector along the x_1 axis of the $x_1y_1z_1$ coordinate system. Thus, n_x, n_y, and n_z are easily derived from the relative orientation between the two coordinate systems involved.

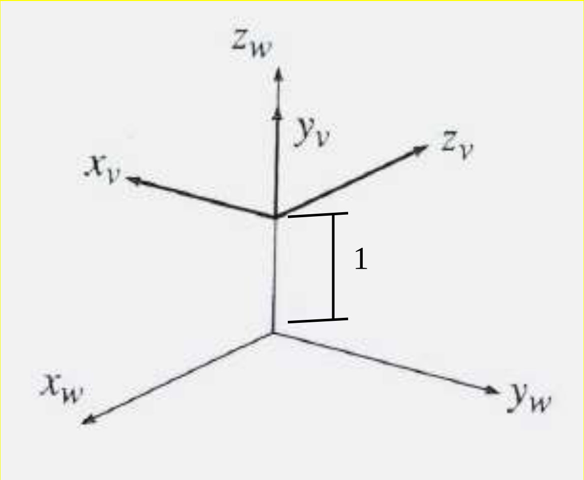
- Similarly, o_x, o_y, and o_z are the x_2, y_2, and z_2 components of the y_1 axis and a_x, a_y, and a_z are those of the z_1 axis .

- The values p_x, p_y, and p_z are obtained as the coordinates of the origin of the $x_1y_1z_1$ coordinate system with respect to the $x_2y_2z_2$ coordinate system.

➢ **Example 5:** From the relative position between the viewing coordinate system and the world coordinate system shown in the following figure,

(i) calculate the mapping transformation $T_{w\text{-}v}$, and

(ii) calculate the coordinate of a point in viewing coordinates if it has world coordinates (5, 0, 1).



➢ **Answer:**

- The first three numbers in the first column of $T_{w\text{-}v}$ (i.e., n_x, n_y, and n_z), are (0, 0, -1) because they are the x_v, y_v, and z_v components of the x_w axis.
- Similarly, o_x, o_y, and o_z, which are the x_v, y_v, and z_v components of the y_w axis, are (-1, 0, 0), and a_x, a_y, and a_z are (0, 1, 0).

- Because p_x, p_y, and p_z are, respectively, the x_v, y_v, and z_v coordinates of the origin of the $x_w y_w z_w$ coordinate system, and their values are 0, -1, and 0, respectively. Therefore, T_{w-v} can be driven as follows

$$T_{w-v} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

➢ We obtain the viewing coordinates of (5, 0, 1) by applying T_{w-v} as follows:

$$\begin{bmatrix} X_v \\ Y_v \\ Z_v \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

➢ Thus, it can be concluded that the viewing coordinates of (5, 0, 1) are (0, 0, -5), as can be guessed from the figure.

Exercises:

- **Exercise 3:** Consider the 3D object shown in the following figure. The coordinates of the vertices are given as follows: A = [3, 5, 3], B =[7, 5, 3], C =[7, 5, 5], D = [3, 5, 5], E = [3, 6, 5], and F = [3, 6, 3]. Find the coordinates of the object if it is rotated by 30^0 in counterclockwise direction at point *D* about the *y* axis.