

MATH203 Calculus

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Independent of path

Theorem 1

If $F(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ is continuous on an open connected region D , then the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if and only if \mathbf{F} is conservative that is $\mathbf{F}(x, y) = \nabla f(x, y)$ for some scalar function.

Theorem 2: Fundamental theorem of line integrals

Let $\mathbf{F}(x, y) = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ be continuous on an open connected region D , and let C be a piecewise smooth curve in D with endpoint $A(x_1, y_1)$ and $B(x_2, y_2)$. If $\mathbf{F}(x, y) = \nabla f(x, y)$, then

$$\int_C M(x, y)dx + N(x, y)dy = \int_{(x_1, y_1)}^{(x_2, y_2)} \mathbf{F} \cdot d\mathbf{r} = \left[f(x, y) \right]_{(x_1, y_1)}^{(x_2, y_2)}$$

Independent of path

Theorem 3

If $M(x, y)$ and $N(x, y)$ have continuous first partial derivatives on a simply connected region D , then the line integral $\int_C M(x, y)dx + N(x, y)dy$ is independent of path in D if and only if
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example 1: Let $\mathbf{F}(x, y) = (2x + y^3)\mathbf{i} + (3xy^2 + 4)\mathbf{j}$

(a) Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path.

(b) $\int_{(0,1)}^{(2,3)} \mathbf{F} \cdot d\mathbf{r}$.

Independent of path

Example 2: Show that $\int_C (e^{3y} + y^2 \sin x)dx + (3xe^{3y} + 2y \cos x)dy$ is independent of path in a simply connected region.

Example 3: Determine whether $\int_C x^2y dx + 3xy^2 dy$ is independent of path.

Example 4: Let $\mathbf{F}(x, y, z) = y^2 \cos x \mathbf{i} + (2y \sin x + e^{2z})\mathbf{j} + 2ye^{2z}\mathbf{k}$

(a) Show that $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path, and find a potential function f of \mathbf{F} .

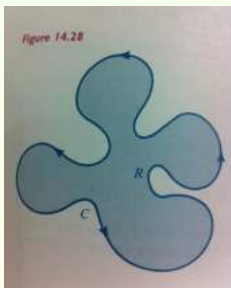
(b) If \mathbf{F} is a force field, find work done by \mathbf{F} along any curve C from $(0, 1, \frac{1}{2})$ to $(\frac{\pi}{2}, 3, 2)$.

Green's Theorem

Green's theorem

Let C be a piecewise-smooth simple closed curve, and let R be the region consisting of C and its interior. If M and N are continuous functions that have continuous first partial derivatives throughout an open region D containing R , then

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA.$$



Green's Theorem

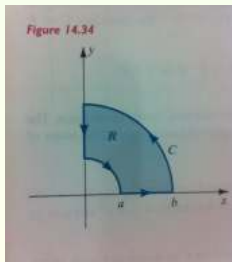
Note: Note the line integral is independent of path and hence is zero

$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed curve C .

Examples: (1) Use Green's theorem to evaluate $\oint_C 5xydx + x^3dy$, where C is the closed curve consisting of the graphs of $y = x^2$ and $y = 2x$ between the points $(0,0)$ and $(2,4)$.

(2) Use Green's theorem to evaluate $\oint_C 2xydx + (x^2 + y^2)dy$, if C is the ellipse $4x^2 + 9y^2 = 36$.

(3) Evaluate $\oint_C (4 + e^{\cos x})dx + (\sin y + 3x^2)dy$, if C the boundary of the region R between quarter-circles of radius a and b and segment on the x - and y -axes, as shown in Figure.



Green's Theorem

Theorem

If a region R in the xy -plane is bounded by a piecewise-smooth simple closed curve C , then the area A of R is

$$\iint_R dA = \oint_C x dy \quad (\text{i})$$

$$= - \oint_C y dx \quad (\text{ii})$$

$$= \frac{1}{2} \oint_C x dy - y dx. \quad (\text{iii})$$

Examples: (1) Find the area of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$.

(2) Find the area of the region bounded by the graphs of $y = 4x^2$ and $y = 16x$.