IE360: CAD/CAM

Computer Aided Design and Computer Aided Manufacturing

Lecture (6)
Projections of Geometric Models

Dr Ashraf Afifi

Industrial Engineering Department
King Saud University

Outline

- Introduction
- Orthographic Projections
- Perspective Projections
- Exercises

Introduction:

- To allow geometric models to be displayed in various views on a two-dimensional screens, ***projections*** are used to transform three-dimensional models onto two-dimensional projection plane. Various views of a model can be generated using various projection planes.

- To define a projection, a center of projection and a projection plane must be defined as shown in Figure 1.

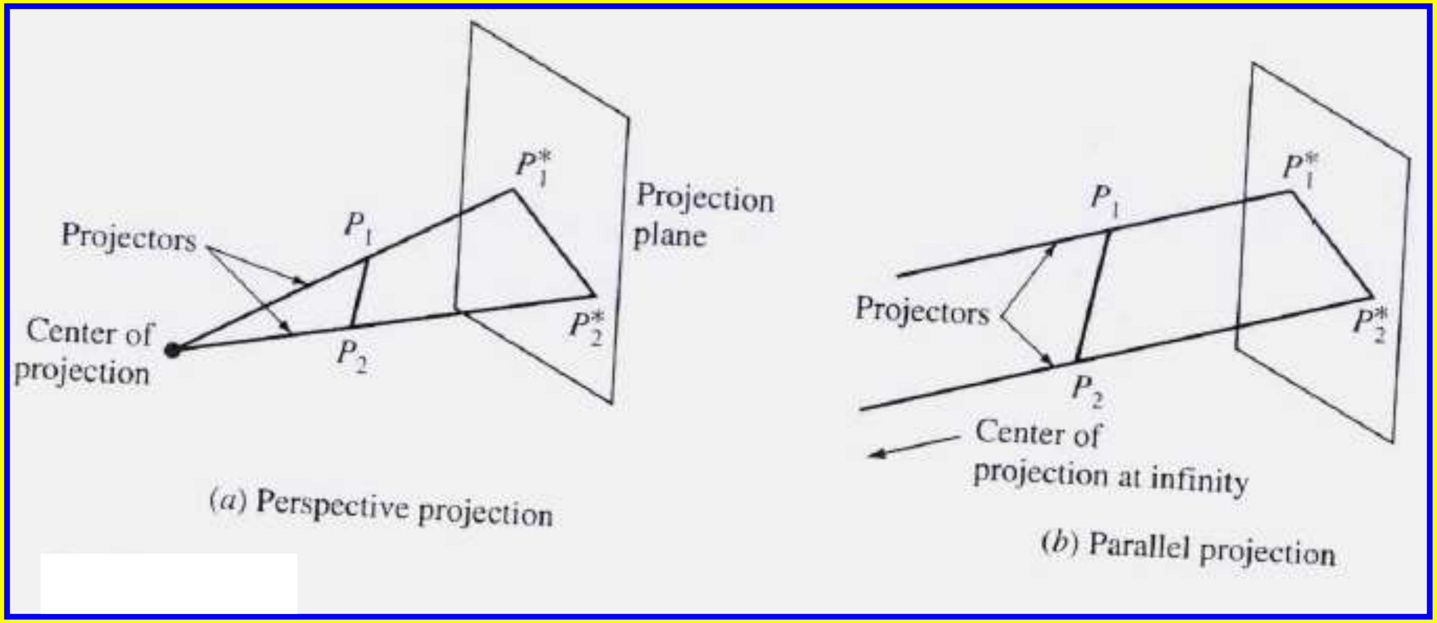


Figure 1: Projection definition.

➢ Different types of projections can be defined as shown in Figure 2.

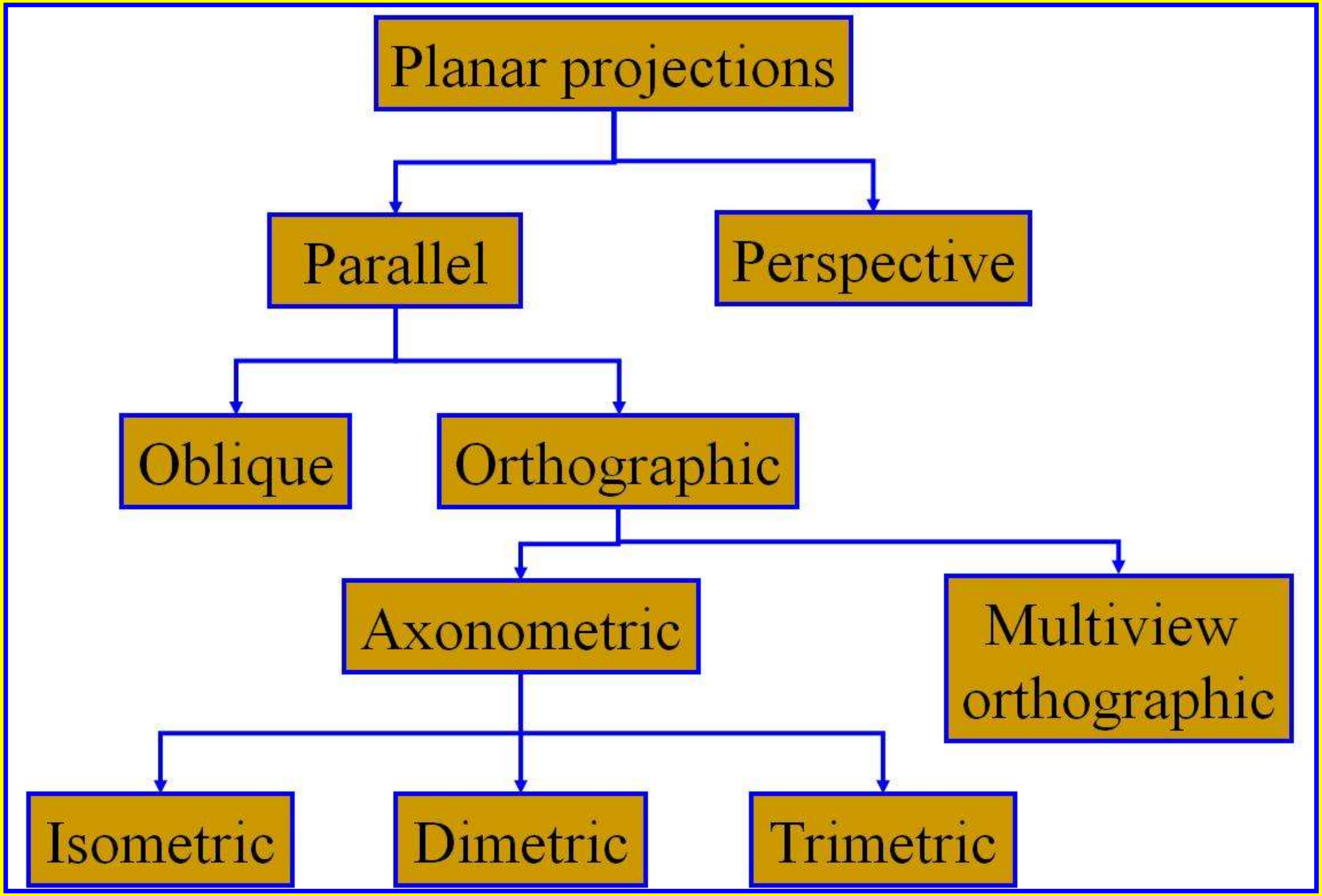


Figure 2: Types of projections.

- There are two different types of projections based on the location of the center of projection relative to the projection plane.
  - *Parallel projections*: The center of projection is located at infinity, so that all projectors are parallel to each other.
    - Preserves actual dimensions and shapes of objects,
    - Preserves parallelism,
    - Preserves angles only on faces of the object which are parallel to the projection plane.

  - *Perspective projection*: The center of projection is located at a finite distance from the projection plane.
    - Adds some realism to perspective views. As can be seen in Figure 1a, the size of an entity is inversely proportional to the distance from the center of projection; that is, the closer the entity to the center, the larger its size is.
    - Actual dimensions and angles of objects, and therefore shapes, cannot be preserved. Thus, perspective projections are not popular among engineers and draftsmen.
    - Does not preserves parallelism.

- There are two types of parallel projections based on the relation between the direction of projection and the projection plane.

  - *Orthographic*: The projectors are perpendicular to the plane of projection, as shown in Figure 3.

  - *Oblique*: The projectors are inclined with respect to the plane of projection. In addition, one of the faces of the object is kept parallel to the projection plane, as shown in Figure 4.

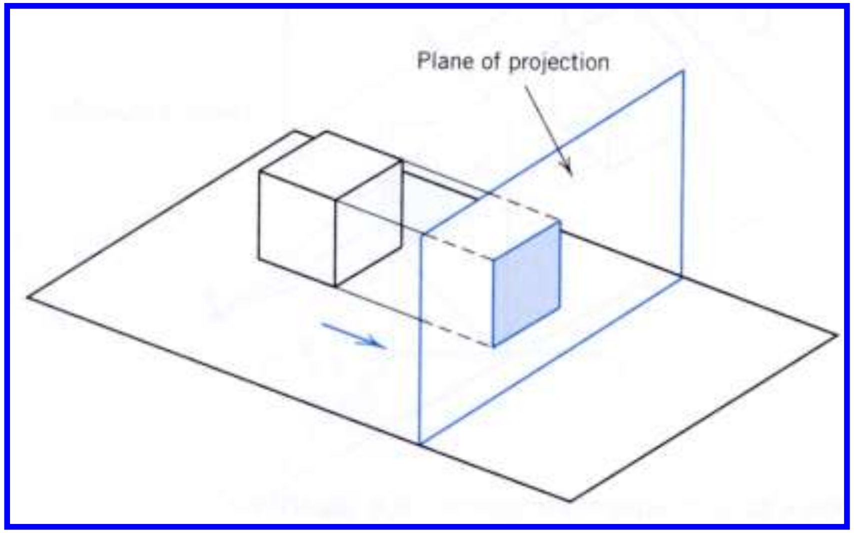


Figure 3: Example of orthographic projection.

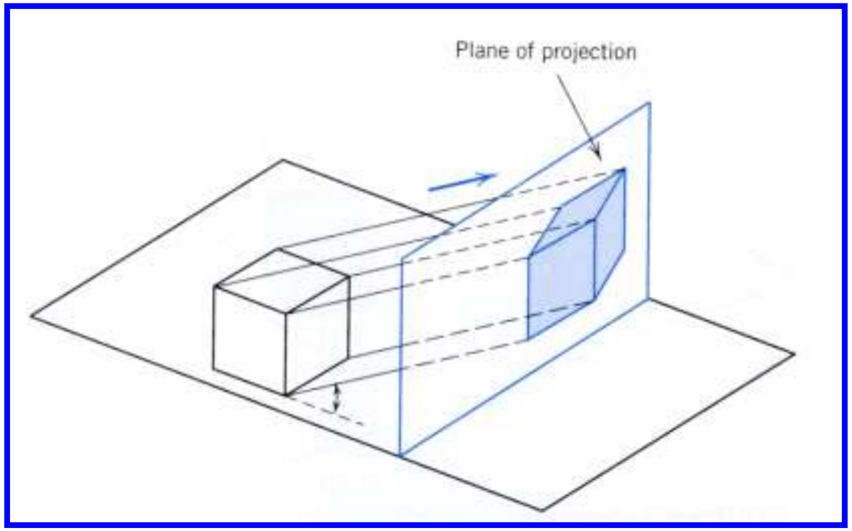


Figure 4: Example of oblique projection.

➢ There are two types of orthographic projections.

- *Multiview orthographic*: The planes of the object remain parallel to the principal planes of projection, as shown in Figure 5. The front, top, and right views that are used customarily in engineering drawings belong to this type. Other views that belong to this type are bottom, rear, and left.

- *Axonometric*: The planes of the object are inclined with respect to the projection plane, as shown in Figure 6. Axonometric projections are further divided into trimetric, diametric, and isometric.

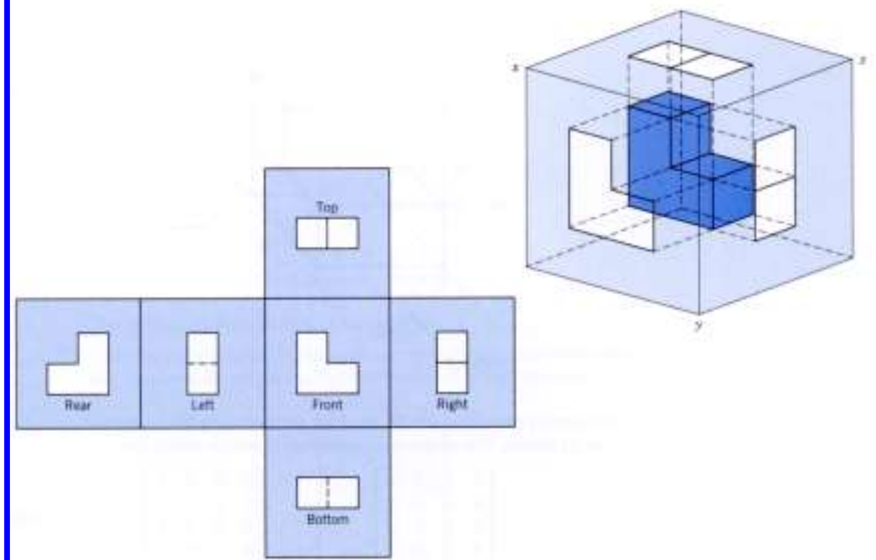


Figure 5: Example of multiview projection.

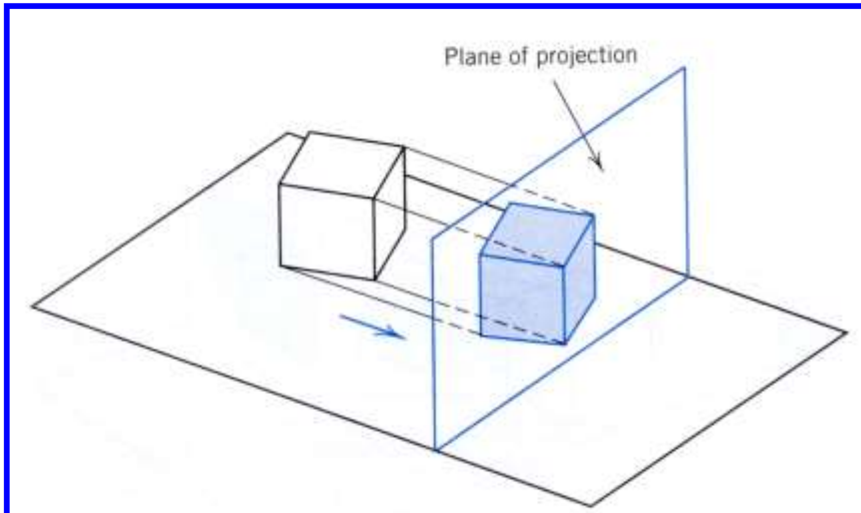


Figure 6: Example of axonometric projection.

➢ **View definition:** A view can be defined as shown in Figure 7.

- The *view origin* defines the location of the origin of the MCS of the model (to be viewed) inside the *view window*.
- The *viewing direction* is the same as the projectors shown in Figure 1b.
- The *viewing plane* is perpendicular to the viewing direction and is the same as the projection plane.
- The *viewport* (or *view window*) defines the boundaries against which the view is clipped. Displayed graphics can always be zoomed in or out to scale within the viewport.

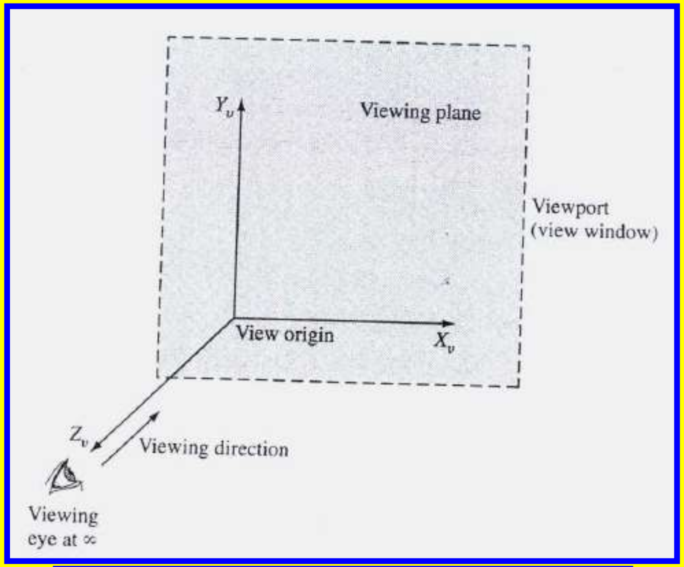


Figure 7: View definition.

- A view has a viewing coordinate system (VCS).
  - VCS is a three-dimensional system with the X_v axis horizontal pointing to the right and the Y_v axis vertical pointing upward, as shown in Figure 7. The Z_v axis defines the viewing direction.

- To obtain views of a model, the viewing plane, the X_vY_v plane, is made coincident with the XY plane of the MCS such that the VCS origin is the same as that of the MCS. Thus, a view of a model is generated in two steps:
  - Rotate the model with respect to the VCS axes until the desired model plane coincides with the viewing plane, then
  - Project the model onto that plane.

- Figure 8 shows the relationship between the MCS and VCS for typical views of a geometric model. The above two-step procedure can be applied to the figure.
  - For the front view, the XY and X_vY_v planes are identical. To obtain this view, the geometry is simply projected onto the viewing plane.
  - For the top view, the model must be rotated about the X_v axis by 90^0 so that the XY plane coincides with the X_vY_v plane.
  - The other views can be obtained in a similar fashion.

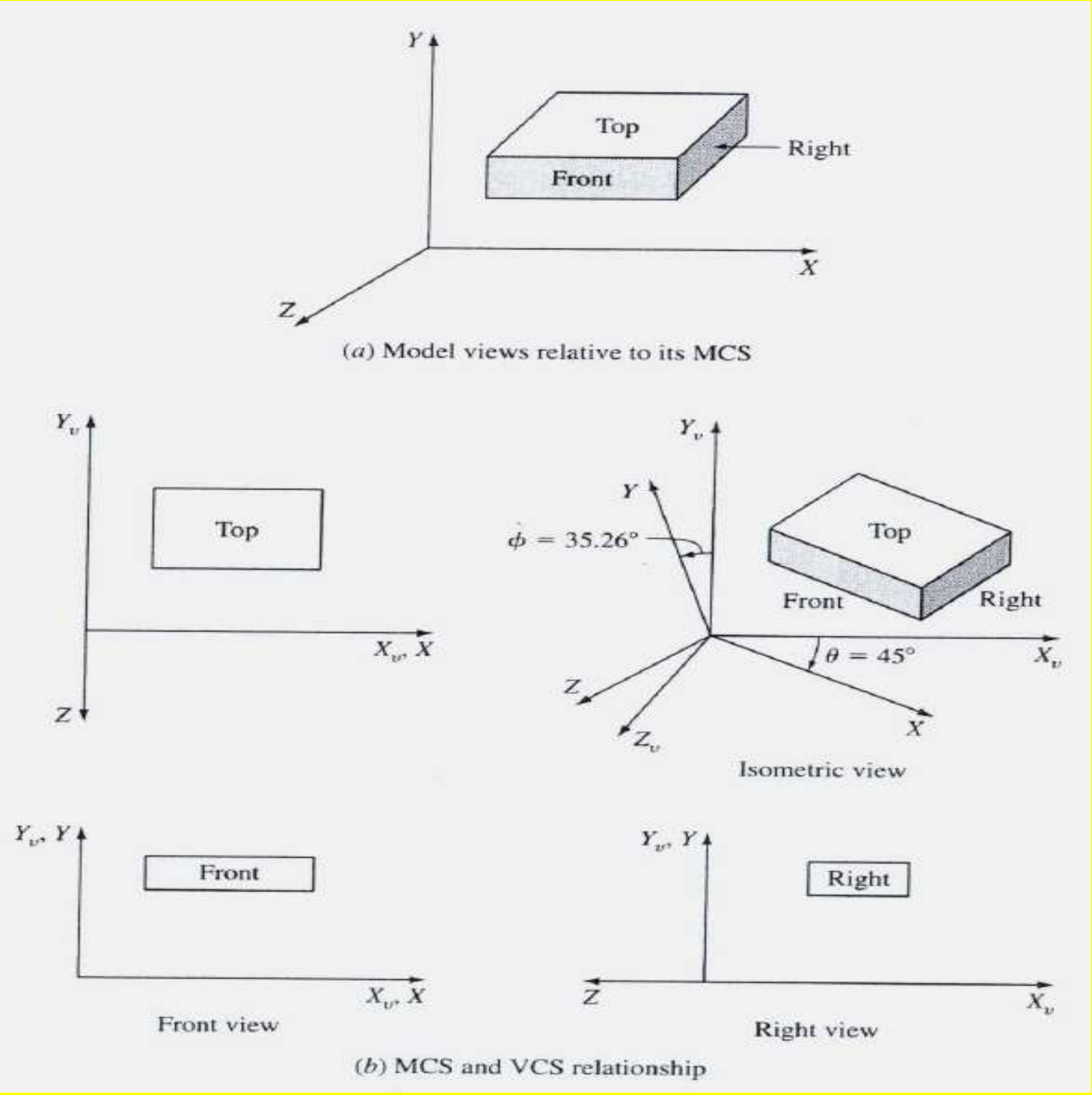


Figure 8: Relationship between MCS and VCS.

Orthographic Projections:

- An orthographic projection (view) is obtained by setting to zero the coordinate value corresponding to the MCS axis that coincides with the direction of projection (viewing) after the model rotation.
  - To obtain the front view (see Figure 8b), it is only needed to set $z = 0$ for all the key points of the model. Thus, the transformation matrix becomes

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (1)$$

and the projection of point P onto the $X_v Y_v$ plane, P_v, is given by

$$P_v = [T]\, P \qquad (2)$$

where P_v is the point expressed in the VCS system. Equation (2) gives $x_v = x$ and $y_v = y$.

- For the top view, the model and its MCS are rotated by 90^0 about the X_v axis followed by setting the y coordinate of the resulting points to zero. In this case, the transformation matrix becomes

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (3)$$

and Equation (3) gives $x_v = x$ and $y_v = -z$. This result agrees with Figure 8b.

- The right view shown in Figure 8b can be obtained by rotating the model and its MCS about the Y_v axis by - 90^0 and setting the x coordinate to zero. Thus,

$$[T] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (4)$$

which gives $x_v = -z$ and $y_v = y$.

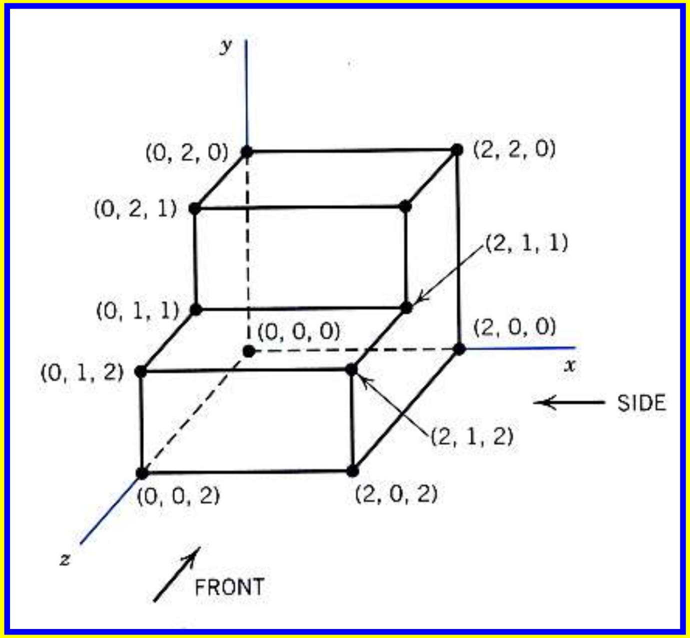
- To obtain the isometric projection or view, the model and its MCS are rotated an angle θ about the Y_v axis followed by a rotation ϕ about the X_v axis. Finally, the isometric view is projected onto the X_vY_v plane. Thus,

$$P_v = [T]Rot(x,\phi).Rot(y,\theta)P$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad (5)$$

Typical values for angles θ and ϕ are $\pm 45^0$ and $\pm 35.26^0$, respectively. The signs of the angles θ and ϕ result in four possible orientations of isometric views. Figure 8 shows the most common orientation where θ = -45 and ϕ = 35.26.

➢ **Example 1:** Given an object as shown in the following figure, find its front and side orthographic projections in the directions indicated by the arrows. Also, determine the isometric projection of the object for $\theta = -45^0$ and $\phi = 35.26^o$.

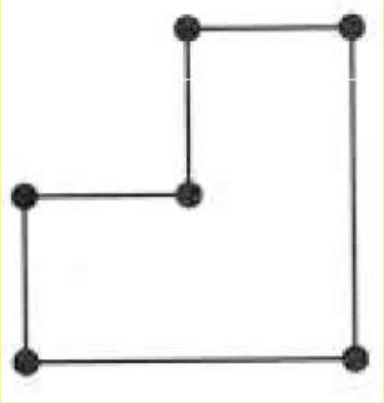


- **Answer:** The indicated front view is a projection on the *xy* plane. The projection is as shown in the following figure and the matrix of points becomes

$$[P_v] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 2 & 2 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 2 & 2 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

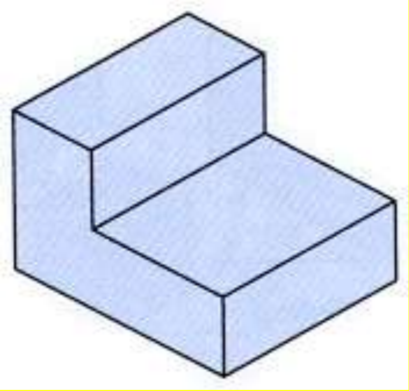
- The side view is obtained by rotating the model about the *y* axis by -90^0 and setting the *x* coordinate to zero. The projection is as shown in the following figure and the matrix of points becomes



$$[P_v] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & 2 & 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 2 & 2 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & -1 & -1 & -1 & -2 & -2 & -2 & -2 & -1 \\ 2 & 2 & 0 & 0 & 2 & 2 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

- The isometric view is obtained through rotating the model with angles -θ and ϕ about the y and x axes respectively, followed by projecting the isometric view onto the xy plane. The projection is as shown in the following figure and the matrix of points becomes



$$[P_v]=[T].Rot(x,35.26).Rot(y,-45).\begin{bmatrix}0&2&2&0&0&2&2&2&2&0&0&0\\2&2&0&0&2&2&1&1&0&0&1&1\\0&0&0&0&1&1&1&2&2&2&2&1\\1&1&1&1&1&1&1&1&1&1&1&1\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0.817&-0.577&0\\0&0.577&0.817&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0.707&0&-0.707&0\\0&1&0&0\\0.707&0&0.707&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&2&2&0&0&2&2&2&2&0&0&0\\2&2&0&0&2&2&1&1&0&0&1&1\\0&0&0&0&1&1&1&2&2&2&2&1\\1&1&1&1&1&1&1&1&1&1&1&1\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0&0\\0&0.817&-0.577&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0.707&0&-0.707&0\\0&1&0&0\\0.707&0&0.707&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&2&2&0&0&2&2&2&2&0&0&0\\2&2&0&0&2&2&1&1&0&0&1&1\\0&0&0&0&1&1&1&2&2&2&2&1\\1&1&1&1&1&1&1&1&1&1&1&1\end{bmatrix}$$

$$=\begin{bmatrix}0.707&0&-0.707&0\\-0.408&0.817&-0.408&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&2&2&0&0&2&2&2&2&0&0&0\\2&2&0&0&2&2&1&1&0&0&1&1\\0&0&0&0&1&1&1&2&2&2&2&1\\1&1&1&1&1&1&1&1&1&1&1&1\end{bmatrix}$$

$$=\begin{bmatrix}0.0&1.414&1.414&0.0&-0.707&0.707&0.707&0.0&0.0&-1.414&-1.414&-0.707\\1.632&2.448&0.861&0.0&1.224&2.040&1.224&0.816&0.0&-0.816&0.0&0.408\\0.0&0.0&0.0&0.0&0.0&0.0&0.0&0.0&0.0&0.0&0.0&0.0\\1.0&1.0&1.0&1.0&1.0&1.0&1.0&1.0&1.0&1.0&1.0&1.0\end{bmatrix}$$

Perspective Projections:

➢ The perspective projection (view) of a model can be obtained by placing the center of projection on the Z_v axis at a distance d from the origin and projecting the model onto the $Z_v = 0$ or the $X_v Y_v$ plane.

➢ Figure 9 shows the perspective projection of point P as point P_v. To find the y_v of P_v, the two similar triangles COP_2 and $CP_3 P_1$ give

$$\frac{y_v}{y} = \frac{d}{d-z} = \frac{1}{1-z/d} \qquad (6)$$

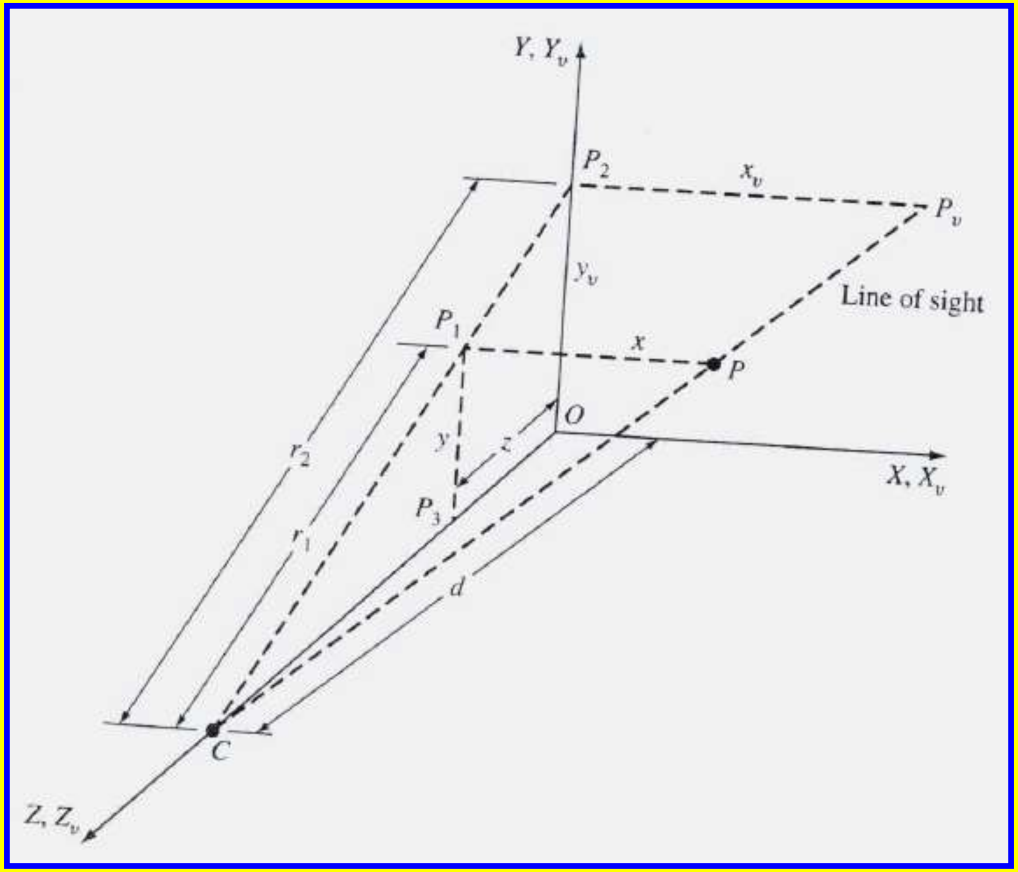


Figure 9: Perspective projection along the Z_v axis.

➢ The two similar triangles C$P_v P_2$ and CPP_1 give x_v of P_v as

$$\frac{x_v}{x} = \frac{r_2}{r_1} = \frac{d}{d-z} = \frac{1}{1-z/d} \quad (7)$$

- Rearranging Equations (6) and (7) to give y_v and x_v respectively and knowing $z_v = 0$, the result can be put in a homogeneous form as

$$P_v = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1/d & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \qquad (8)$$

- If this equation is expanded, it gives $P_v = [x \quad y \quad 0 \quad (1\text{-}z/d)]^T$. This would require the division of x and y by ($1\text{-}z/d$) to obtain the corresponding Cartesian coordinates of these homogeneous coordinates. Consequently, Equations (6) and (7) result.

- **Example 2:** Find the perspective projection of the tetrahedron defined by the coordinates of its vertices as follows: P_1(3,4,0), P_2(1,0,4), P_3(2,0,5), and P_4(4,0,3) onto a projection plane at $z = 0$. The center of projection is at distance $d = 5$ from the origin along the z axis.

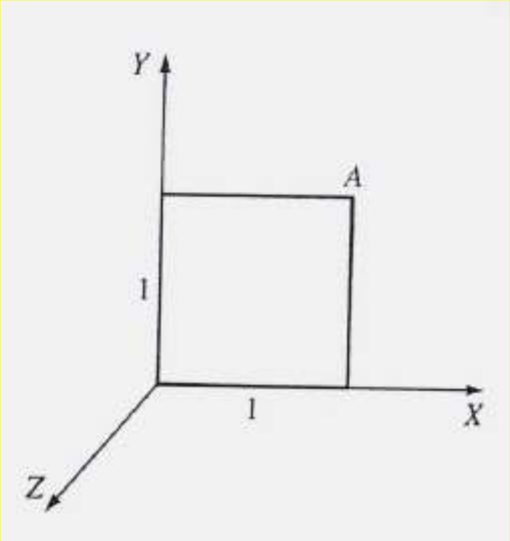
- **Solution:**

$$[P_v]=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & -1/5 & 1\end{bmatrix}\begin{bmatrix}3 & 1 & 2 & 4\\ 4 & 0 & 0 & 0\\ 0 & 4 & 5 & 3\\ 1 & 1 & 1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}3 & 1 & 2 & 4\\ 4 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 0.2 & 0 & 0.4\end{bmatrix}=\begin{bmatrix}3 & 5 & 2 & 10\\ 4 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 1\end{bmatrix}$$

Exercises:

➢ **Exercise 6:** The square shown lies in the *XY* plane. It is rotated CW by 30^0 about the *Y* axis, followed by 45^0 CW rotation about the *X* axis, followed by translation of 2, 4, and -3 in the *X*, *Y*, and *Z* directions, respectively. Find the coordinates of corner *A* in the final position of the square. Also, find the coordinates of *A* if the square is projected (in its final position) onto the *XY* plane (front view), *XZ* plane (top view), and *YZ* plane (right view).



➢ **Exercise 7:** Consider the 3D object shown in the following figure. The coordinates of the vertices are given as follows: *A* = [3, 5, 3], *B* =[7, 5, 3], *C* =[7, 5, 5], *D* = [3, 5, 5], *E* = [3, 6, 5], and *F* = [3, 6, 3].

- Obtain the front, top, and right orthographic projections (views) of the object.
- Calculate the perspective projection of the object if the center of projection is at distance $d = 10$ from the origin along the z axis.
- Determine the isometric projection of the object if the angles of rotation about the y and x axes are -45^0 and 35.26^o, respectively.