IE360: CAD/CAM

Computer Aided Design and Computer Aided Manufacturing

Lecture (5)
Representation and Manipulation of Curves

Dr Ashraf Afifi

Industrial Engineering Department
King Saud University

Outline

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- Analytic Curves
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Introduction:

- The need to study the mathematical basis of geometric modeling is many-fold.
  - It provides a good understanding of terminology encountered in the CAD/CAM systems.
  - It enables users to decide intelligently on the types of entities necessary to use in a particular model to meet certain geometric requirements such as slopes and/or curves.
  - It enables users to interpret any unexpected results they may encounter from using a particular CAD/CAM system.
  - It equips those who are involved in the decision-making process and evaluations of CAD/CAM systems with better evaluation criteria.
  - It provides engineers and designers with new sets of tools and capabilities that they can use in their engineering applications.

Types of Curve Equations:

- Curve equations can be classified as two groups.
  - *Parametric equation*: It relates the x, y, and z coordinates of the points on a curve with a parameter.
  - *Nonparametric equation*: It directly relates the x, y, and z coordinates with a function.

- Example: Consider a circle of radius R centered at the origin of the reference coordinate system. If the circle is located in the xy plane, the parametric equation of the circle can be expressed as

$$x = R\cos\theta \quad y = R\sin\theta \quad z = 0 \quad (0 \le \theta \le 2\pi) \qquad (1)$$

However, the same circle may be expressed without using the parameter θ as

$$x^2 + y^2 - R^2 = 0, \qquad z = 0. \qquad (2)$$

Or

$$y = \pm\sqrt{R^2 - x^2}, \qquad z = 0. \qquad (3)$$

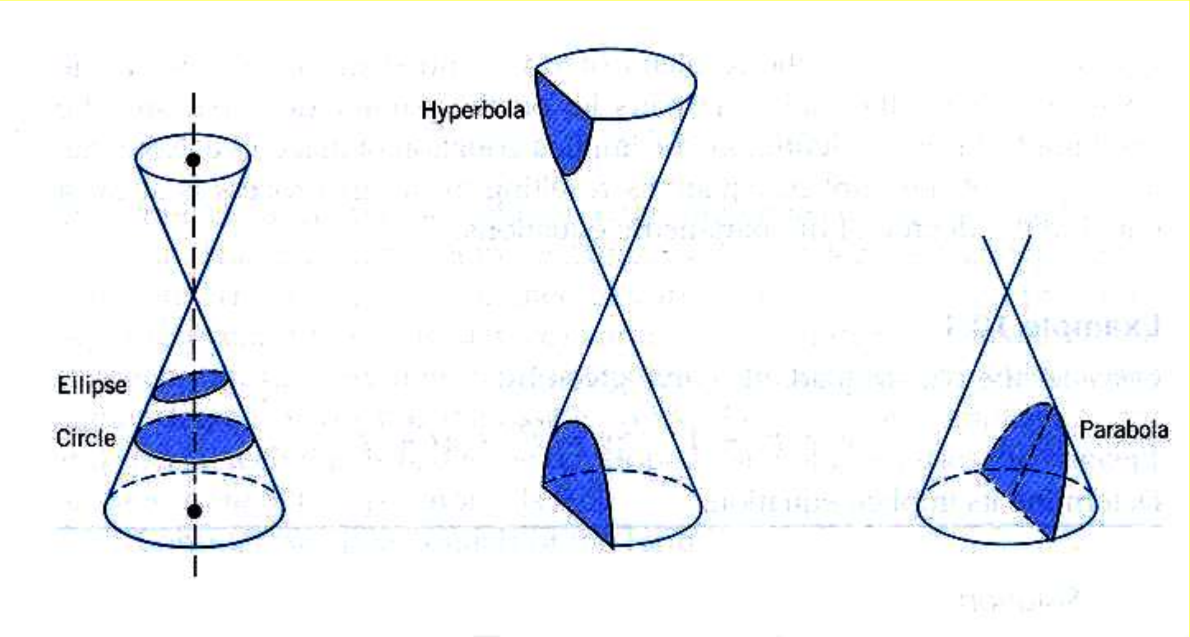
- The parametric equation is the most popular form for representing curves and surfaces in CAD systems.

Analytic Curves:

➢ *Conic sections* are the most widely used analytic curves in CAD/CAM systems.

- They are obtained by cutting a cone with a plane, as shown in the accompanying figure.

- Based on the location and orientation of the cutting plane with respect to the cone, the section curve may be a *circle*, an *ellipse*, a *parabola* or a *hyperbola*, as shown in the following figure.

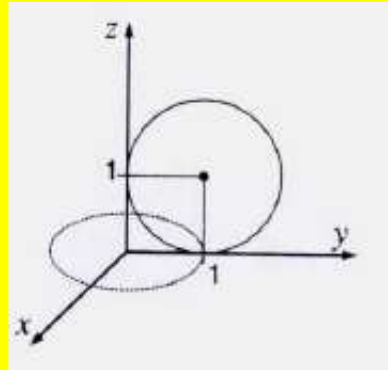


- **Circle or Circular Arc:**
  - A circle or its portion on the *xy* plane with radius R and center at (X_c, Y_c) can be represented by the equations:

$$X = R\cos\theta + X_c, \qquad y = R\sin\theta + Y_c \tag{4}$$

  - In Equation (4), the value of θ is incremented to 2π for a complete circle or to a certain value for a circular arc.
  - The equation of a circle or circular arc lying on planes other than the *xy* plane can be derived by applying the proper transformation matrices to Equations (1) or (4).

- **Example 1:** A circle of unit radius is centered at (0, 1, 1) and located on the *yz* plane, as illustrated in the accompanying figure. Derive the parametric equation of the circle by applying the proper transformation matrices to Equation (1).



➢ **Answer:** The circle of interest can be obtained by drawing a unit circle on the *xy* plane at the origin, rotating this circle by -90 degrees about the *y* axis, and then translating it by 1 in the *y* and 1 in the *z* directions. If the *x*, *y*, and *z* of the points on the circle of interest are denoted x^*, y^*, and z^* and those of the reference circle are denoted *x*, *y*, and *z*, the following equations will hold.

$$[x^* \; y^* \; z^* \; 1]^T = Trans(0,1,1).Rot(y,-90).[x \quad y \quad z \quad 1]^T$$

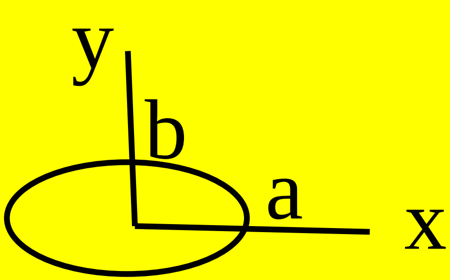
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(-90) & 0 & \sin(-90) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(-90) & 0 & \cos(-90) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = [-z \quad y+1 \quad x+1 \quad 1]^T$$

- Therefore

$$x^* = -z = 0$$
$$y^* = y + 1 = R \sin\theta + 1$$
$$z^* = x + 1 = R \cos\theta + 1 \qquad (0 \le \theta \le 2\pi)$$

- **Ellipse or Elliptic Arc:**
  - The parametric equation of an ellipse centered at the origin and located in the *xy* plane of the reference coordinate system can be represented by the following equations. It is assumed that the major axis is in the *x* direction with length *a* and the minor axis is in the *y* direction with length *b*.

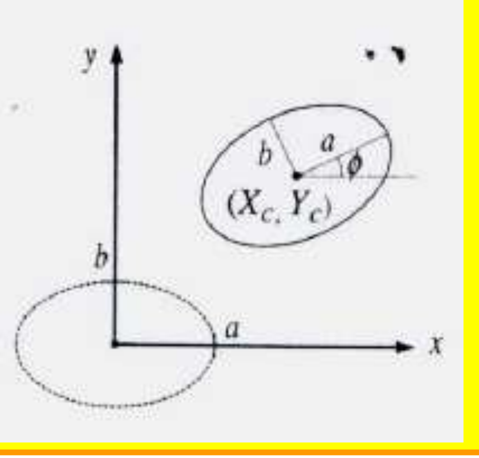
$$x = a\cos\theta \qquad y = b\sin\theta \qquad z = 0 \qquad (5)$$



▪ In Equation (5), the range of the parameter θ would be from 0 to 2π for an ellipse and from 0 to the value corresponding to the end point for an elliptic arc.

▪ A general ellipse on an arbitrary plane with arbitrary directions of the major and minor axes can be obtained by applying the proper transformation matrices to Equation (5), as was done for a circle.

➢ **Example 2:** Derive the parametric equation of an ellipse in the *xy* plane, which has ata center (X_c, Y_c) and the major and minor axes are as illustrated in the accompanying figure.



➢ **Answer:** The ellipse of interest can be obtained by rotating the reference ellipse at the origin by ϕ about the z axis and translating it by X_c in the x direction and by Y_c in the y direction. If the x, y, and z of the points on the ellipse of interest are denoted x^*, y^*, and z^* and those of the reference ellipse are denoted x, y, and z, the following equations will hold.

$$[x^*\ y^*\ z^*\ 1]^T = Trans(X_c, Y_c, 0).Rot(z, \phi).[x\ y\ z\ 1]^T$$

$$= \begin{bmatrix} 1 & 0 & 0 & X_c \\ 0 & 1 & 0 & Y_c \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$= [x\cos\phi - y\sin\phi + X_c \quad x\sin\phi + y\cos\phi + Y_c \quad 0 \quad 1]^T$$

➢ Therefore

$$x^* = x\cos\phi - y\sin\phi + X_c = a\cos\theta\cos\phi - b\sin\theta\sin\phi + X_c$$
$$y^* = x\sin\phi + y\cos\phi + Y_c = a\cos\theta\sin\phi + b\sin\theta\cos\phi + Y_c$$
$$z^* = 0 \qquad (0 \le \theta \le 2\pi)$$

Synthetic Curves:

- Analytic curves are usually not sufficient to meet geometric design requirements of mechanical parts. Products such as car bodies, ship hulls, airplane wings, propeller blades and bottles are a few examples that require free-form, or synthetic, curves.

- Mathematically, synthetic curves represent a curve fitting problem to construct a smooth curve that passes through given data points, usually called *control points*. Therefore, polynomials are the typical form of these curves.

- Major CAD/CAM systems provide several types of synthetic curves, of which the most important are Hermite, and Bezier curves.

- Figure 1 shows examples of Hermite, and Bezier curves.

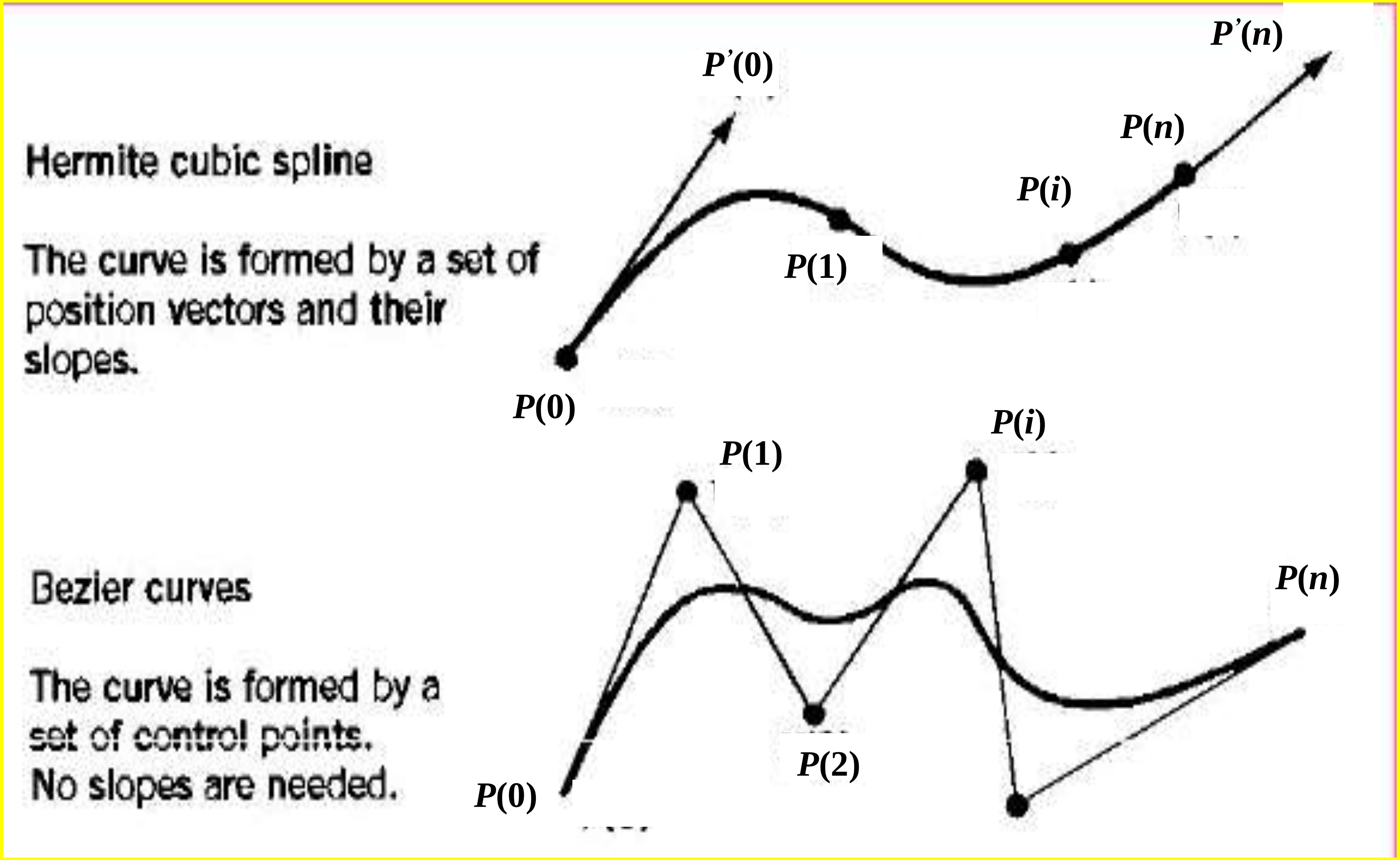


Figure 1: Synthetic curves.

➢ Hermite Curves:

- Hermite curves are used to interpolate to given data points, not to design free-form curves as Bezier curves do. They can be represented by 3-degree polynomial equations as follows:

$$P(u) = \sum_{i=0}^{3} a_i u^i \qquad (10)$$

$$= a_0 + a_1 u + a_2 u^2 + a_3 u^3 \qquad (0 \le u \le 1)$$

- The tangent vector to the curve at any point is given by differentiating Equation (10) with respect to *u* to give

$$P'(u) = \sum_{i=0}^{3} a_i i u^{i-1}$$

$$= a_1 + 2a_2 u + 3a_3 u^2 \qquad (0 \le u \le 1) \qquad (11)$$

- In order to find the algebraic coefficients a_0, a_1, a_2, and a_3, consider the Hermite curve with the two endpoints shown in Figure 2. Applying the boundary conditions $P_0, P_0^{'}$ at $u = 0$ and $P_1, P_1^{'}$ at $u = 1$, Equations (10) and (11) give

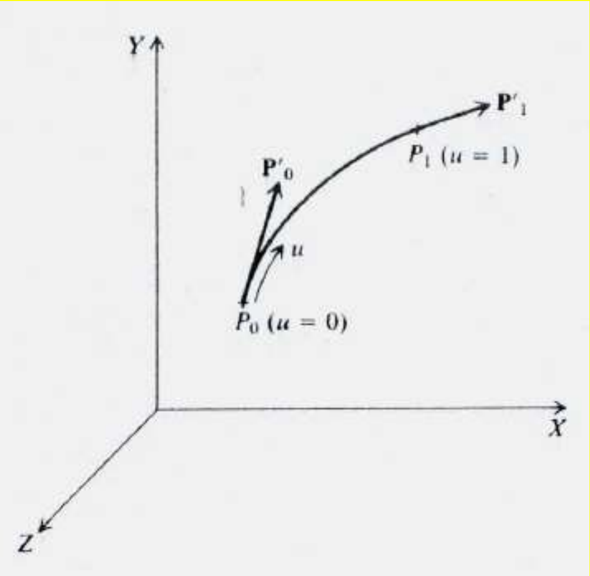


Figure 2: Hermite curve.

$$
\begin{aligned}
P_0 &= P(0) = a_0 \\
P_1 &= P(1) = a_0 + a_1 + a_2 + a_3 \\
P_0' &= P'(0) = a_1 \\
P_1' &= P'(1) = a_1 + 2a_2 + 3a_3
\end{aligned}
\qquad (12)
$$

- Equation (12) can be solved for a_0, a_1, a_2, and a_3 to obtain

$$
\begin{aligned}
a_0 &= P_0 \\
a_1 &= P_0' \\
a_2 &= -3P_0 + 3P_1 - 2P_0' - P_1' \\
a_3 &= 2P_0 - 2P_1 + P_0' + P_1'
\end{aligned}
\qquad (13)
$$

▪ Substituting Equation (13) into Equation (10) gives the new expression of the curve equation:

$$P(u) = (1-3u^2+2u^3)P_0 + (3u^2-2u^3)P_1 + (u-2u^2+u^3)P_0^{'} + (-u^2+u^3)P_1^{'} \quad (0 \le u \le 1) \qquad (14)$$

▪ Note that the curve equation no longer includes the algebraic coefficients; instead it contains $P_0, P_1, P_0^{'},$ and $P_1^{'}$. These new coefficients are called *geometric coefficients* and Equation (14) is called the *Hermite curve*. Its advantage is that the change in curve shape can be anticipated intuitively from changes in the geometric coefficients.

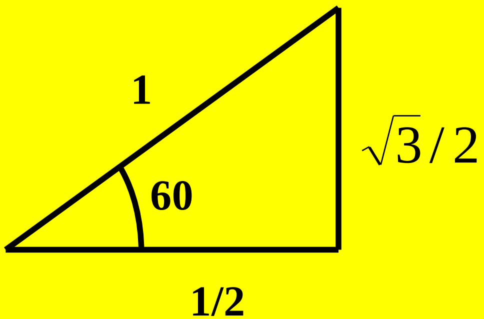
▪ The tangent vectors become

$$P^{'}(u) = (-6u+6u^2)P_0 + (6u-6u^2)P_1 + (1-4u+3u^2)P_0^{'} + (-2u+3u^2)P_1^{'} \quad (0 \le u \le 1) \qquad (15)$$

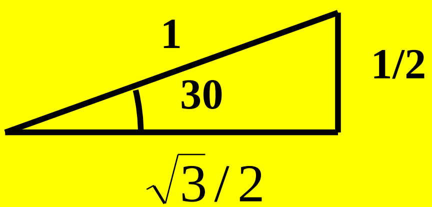
➢ **Example 3:** Consider the Hermite curve with the two endpoints $P_0 = [1 \quad 2 \quad 0]^T$, and $P_1 = [3 \quad 1 \quad 0]^T$. The slopes of the position vectors at the given data points are 60 and 30 degrees respectively, and the magnitude of each vector is 1. Determine and plot the equation of the resulting Hermite curve.

➢ **Answer:** From the opposite triangles, $P_0^{'}$ and $P_1^{'}$ can be calculated as follows:

$$P_0^{'} = \begin{bmatrix} \cos 60 \\ \sin 60 \end{bmatrix} = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix}$$



$$P_1^{'} = \begin{bmatrix} \cos 30 \\ \sin 30 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

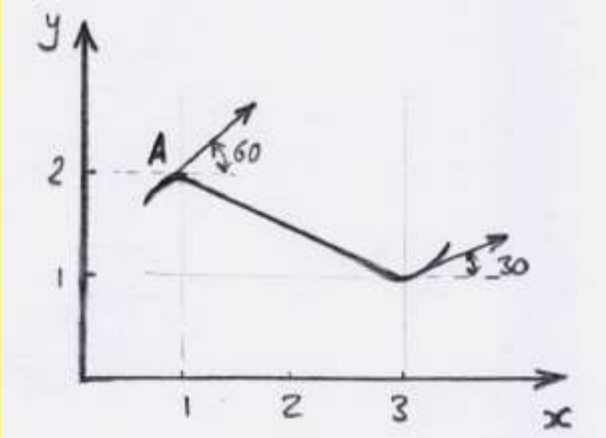


➢ Using Equation (14), the Hermite curve and its plot can be expressed as follows:

$$P(u) = (1-3u^2+2u^3)\begin{bmatrix}1\\2\end{bmatrix} + (3u^2-2u^3)\begin{bmatrix}3\\1\end{bmatrix} + (u-2u^2+u^3)\begin{bmatrix}1/2\\\sqrt{3}/2\end{bmatrix} + (-u^2+u^3)\begin{bmatrix}\sqrt{3}/2\\1/2\end{bmatrix}$$

$$= \begin{bmatrix}1*(1-3u^2+2u^3)+3*(3u^2-2u^3)+1/2*(u-2u^2+u^3)+\sqrt{3}/2*(-u^2+u^3)\\2*(1-3u^2+2u^3)+1*(3u^2-2u^3)+\sqrt{3}/2*(u-2u^2+u^3)+1/2*(-u^2+u^3)\end{bmatrix}$$

$$= \begin{bmatrix}1+1/2u+(5-\sqrt{3}/2)u^2+(\sqrt{3}/2-7/2)u^3)\\2+\sqrt{3}/2u+(-7/2-\sqrt{3})u^2+(5/2+\sqrt{3}/2)u^3)\end{bmatrix}$$



- Answer verification:

$$P_0 = P(u=0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad P_1 = P(u=1) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$P(u) = \begin{bmatrix} 1/2 + (10-\sqrt{3})u + (3\sqrt{3}/2 - 21/2)u^2) \\ \sqrt{3}/2 + (-7-2\sqrt{3})u + (15/2 + 3\sqrt{3}/2)u^2) \end{bmatrix} \rightarrow \begin{bmatrix} P_x'(u) \\ P_y'(u) \end{bmatrix}$$

$$P_0' = P'(u=0) = \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} \qquad P_1' = P'(u=1) = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

$$Slope(A) = \frac{P_y'(u=0)}{P_x'(u=0)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

$$Slope(B) = \frac{P_y'(u=1)}{P_x'(u=1)} = \frac{1/2}{\sqrt{3}/2} = 1/\sqrt{3}$$

➢ Bezier Curves:

- As previously mentioned, Hermite curves are based on *interpolation techniques*. Curves resulting from these techniques pass through the given data points.

- Another alternative to create curves is to use *approximation techniques* which produce curves that do not pass through the given data points. Instead these points are used to control the shape of the resulting curve.

- The Bezier curve is an example based on approximation techniques.

- The Bezier curve is defined in terms of the locations of $n + 1$ points. These points are called data or *control points*.

- The Bezier curve has the following properties:
    - The curve passes only through the first and the last control points or vertices of the polygon. The other vertices define the shape of the curve.

- The curve is always tangent to the first and last polygon segments.
- The curve shape tends to follow the polygon shape.

▪ The above properties enable the user to sketch or predict the curve shape once its control points are given as illustrated in Figure 3.

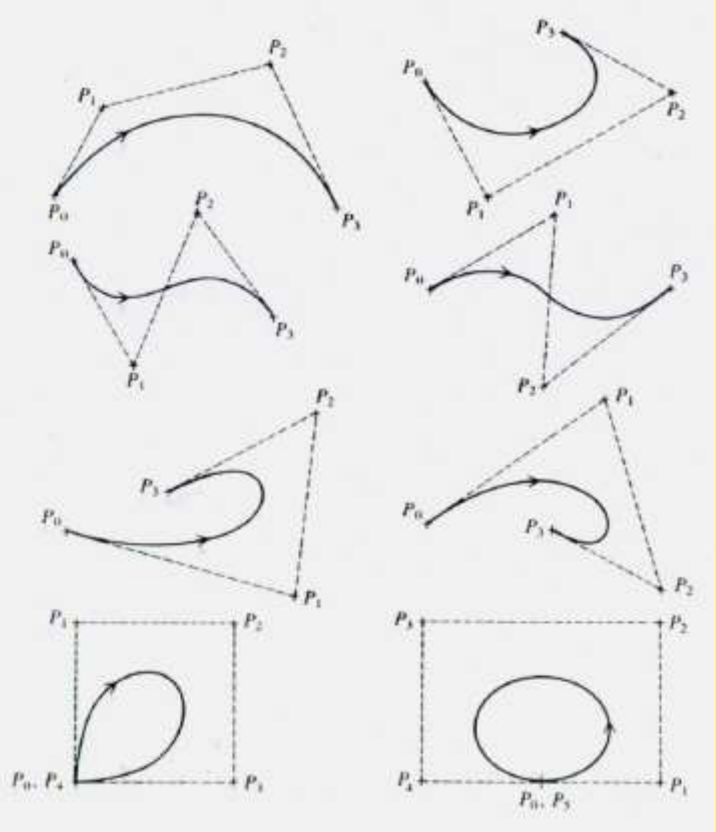


Figure 3: Bezier curves for various control points.

- Mathematically, for $n + 1$ control points, the Bezier curve is defined by the following polynomial of degree n:

$$P(u) = \sum_{i=0}^{n} P_i B_{i,n}(u), \qquad 0 \le u \le 1 \tag{16}$$

where $P(u)$ is any point on the curve and P_i is a control point. $B_{i,n}$ are the Bernstien polynomials, which are given by

$$B_{i,n}(u) = C(n,i)u^i(1-u)^{n-i} \tag{17}$$

where $C(n, i)$ is the binomial coefficient

$$C(n,i) = \frac{n!}{i!(n-i)!} \tag{18}$$

Substituting Equations (17) and (18) into Equation (16) gives

$$P(u) = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} u^i (1-u)^{n-i} P_i, \qquad 0 \le u \le 1 \tag{19}$$

Expanding Equations (19) and observing that $C(n, 0) = C(n, n) = 1$ give

$$P(u) = P_0(1-u)^n + P_1C(n,1)u(1-u)^{n-1} + P_2C(n,2)u^2(1-u)^{n-2} + \\ + \ \dots \ + P_{n-1}C(n,n-1)u^{n-1}(1-u) + P_nu^n \qquad 0 \le u \le 1 \tag{20}$$

- It should be noted that when substituting $u = 0$ and 1 in Equation (20), $P(0)$ and $P(1)$ will be equal to P_0 and P_n respectively, which proves the first property of the Bezier curve.

- Also, the second property of the Bezier curve can be proven by noting that, using Equation (20), the first derivatives at the endpoints are given by

$$P'(0) = n(P_1 - P_0) \\ P'(1) = n(P_n - P_{n-1}) \tag{21}$$

where $(P_1 - P_0)$ and $(P_n - P_{n-1})$ define the first and last segments of the curve polygon.

➢ Example 4: The coordinates of four control points are given by $P_0 = [2 \quad 2 \quad 0]^T$, $P_1 = [2 \quad 3 \quad 0]^T$, $P_2 = [3 \quad 3 \quad 0]^T$, and $P_3 = [3 \quad 2 \quad 0]^T$. Find the equation of the resulting Bezier curve. Also find points on the curve for u = 0, ¼, ½, ¾, and 1.

➢ Solution: As the number of control points is 4, n is set to 3. Substituting the n value into Equation (19) or (20) gives

$$P(u) = P_0(1-u)^3 + 3P_1u(1-u)^2 + 3P_2u^2(1-u) + P_3u^3 \qquad 0 \le u \le 1$$

Substituting the u values in this equation gives

$$P(0) = P_0 = [2 \quad 2 \quad 0]^T$$

$$P(1/4) = \frac{27}{64}P_0 + \frac{27}{64}P_1 + \frac{9}{64}P_2 + \frac{1}{64}P_3 = [2.156 \quad 2.563 \quad 0]^T$$

$$P(1/2) = \frac{1}{8}P_0 + \frac{3}{8}P_1 + \frac{3}{8}P_2 + \frac{1}{8}P_3 = [2.5 \quad 2.75 \quad 0]^T$$

$$P(3/4) = \frac{1}{64}P_0 + \frac{9}{64}P_1 + \frac{27}{64}P_2 + \frac{27}{64}P_3 = [2.844 \quad 2.563 \quad 0]^T$$

$$P(1) = P_3 = [3 \quad 2 \quad 0]^T$$

Figure 4 shows the curve and the points.

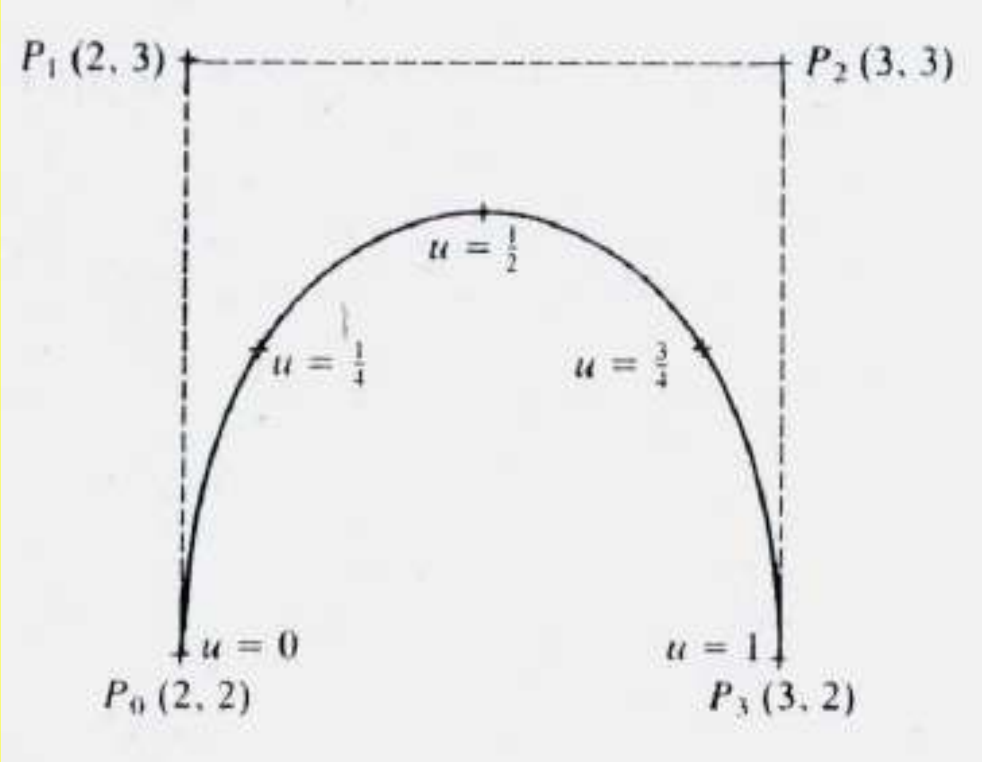


Figure 4: Bezier curve and generated points.

Exercises:

- **Exercise 4:** Find the equation of a Hermite cubic spline that passes through points (1, 2) and (3, 4), and whose tangent vectors are the two lines connecting these two points with point (2, 7). The Hermite curve is defined as:

$$P(u) = (1 - 3u^2 + 2u^3)P_0 + (3u^2 - 2u^3)P_1 + (u - 2u^2 + u^3)P_0' + (-u^2 + u^3)P_1' \qquad (0 \le u \le 1)$$

- **Exercise 5:** The coordinates of three control points are given by [0, 0, 0], [12, 3, 0], and [8, 5, 0]. Find the equation of the resulting Bezier curve. Assuming *u* is incrementing by 0.2, plot the curve along with the envelop of the control points. For $n + 1$ control points, the Bezier curve is defined by the following polynomial of degree *n*:

$$P(u) = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} u^i (1-u)^{n-i} P_i, \qquad 0 \le u \le 1$$