MATH203 Calculus

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Triple Integrals

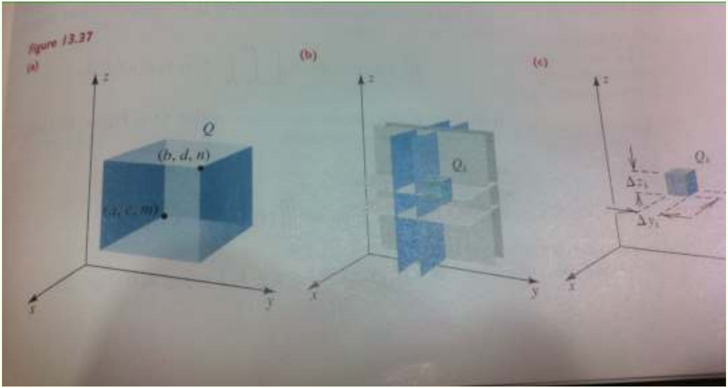
Definition

If f is a continuous function defined over a bounded solid Q, then the **triple integral of f over Q** is defined as

$$\iiint_Q f(x, y, z)\mathrm{d}V = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k, y_k, z_k)\Delta V_k \tag{1}$$

provided the limit exists, where Q_k is the $k-$th subregion of Q, V_k is the volume of Q_n, (x_k, y_k, z_k) is a point, $\|P\|$ is length of the longest diagonal of all the Q_k.

Triple Integrals



Triple Integrals

Application of a triple integral is the volume of the solid region Q is given by

$$\text{Volume of} \quad Q = \iiint_Q \mathrm{d}V$$

Example:

Evaluate the iterated integral $\iiint_Q \mathrm{d}z\mathrm{d}x\mathrm{d}y.$, where

$Q = \{(x, y, z) : -1 \leqslant x \leqslant 1, 3 \leqslant y \leqslant 4, 0 \leqslant z \leqslant 2\}.$

Notes

Note 1:

To evaluate a triple integral in order $dzdydx$, hold both x and y constant for inner most integral, then hold x constant for the second integration.

Note 2:

The symbol on the right-hand side of the equation is an iterated triple integral.

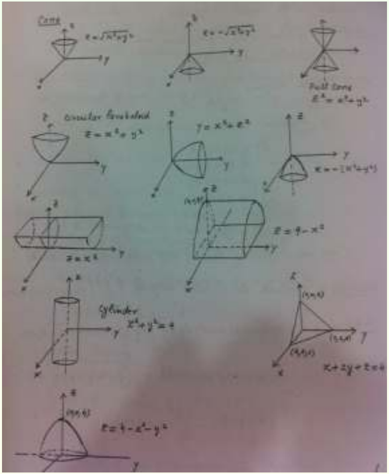
Note 3:

A triple integral $\iiint\limits_Q \mathrm{d}V$ can be evaluated in six different orders, namely

$dV = dzdydx = dydxdz = dxdzdy = dzdxdzy = dxdydz = dydzdx$.

Triple Integrals

Some important graphs



Triple Integrals

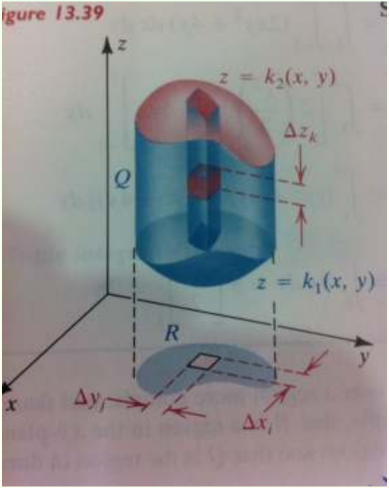
Evaluation theorem:

Triple integrals can be defined over a region more complicated han a rectangular box. Suppose that R is a region in the $xy-$plane that can be divided into R_x and R_y regions and that Q is the region in three dimensions defined by

$Q = \{(x, y, z) : (x, y)\text{is in} \quad R \quad \text{and} \quad k_1(x, y) \leqslant z \leqslant k_2(x, y)\}$, where k_1 and k_2 are continuous functions, then triple integral defines as

$$\iiint\limits_{Q} f(x, y, z)\mathrm{d}V = \iint\limits_{R} \Big[\int_{k_1(x,y)}^{k_2(x,y)} f(x, y, z)dz \Big]\mathrm{d}A \qquad (2)$$

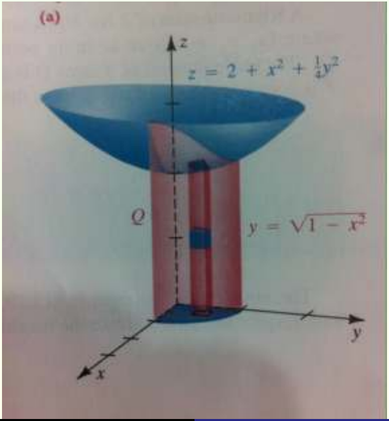
Triple Integrals



Triple Integrals

Example 1

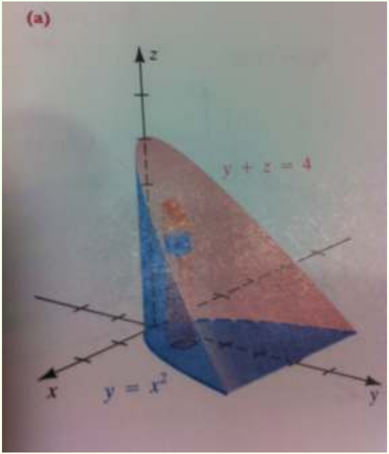
Express the iterated integral $\iiint\limits_Q \mathrm{d}V.$, if Q is the region in the first octant bounded by the coordinate plane, paraboloid $z = 2 + x^2 + \frac{1}{4}y^2$ and the cylinder $x^2 + y^2 = 1$.



Triple Integrals

Example 2

Find the volume V of the solid that is bounded by cylinder $y = x^2$ and by the plane $y + z = 4$ and $z = 0$.



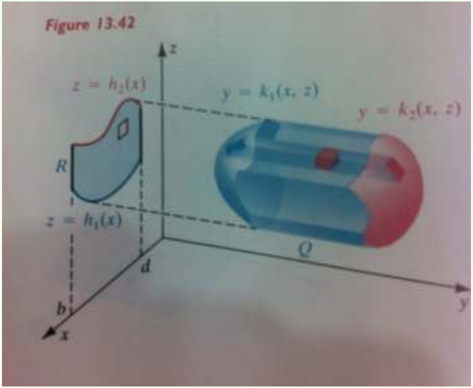
Triple Integrals

Evaluation theorem:

Let f be a continuous functions on the solid region Q defined by $b \leqslant x \leqslant d$, $h_1 \leqslant y \leqslant h_2$ and $k_1 \leqslant z \leqslant k_2$, where h_1, h_2, k_1 and k_2 are continuous functions, then

$$\iiint_Q f(x,y,z)\mathrm{d}V = \int_b^d \int_{h_1(x,y)}^{h_2(x,y)} \int_{k_1(x,y)}^{k_2(x,y)} f(x,y,z)dydzdx \qquad (3)$$

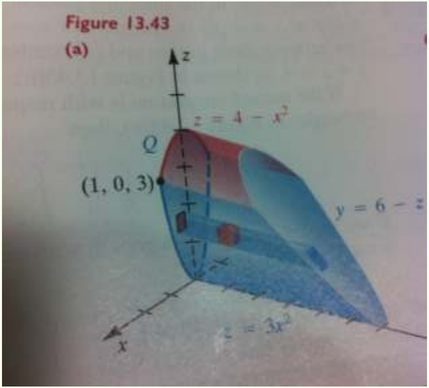
Triple Integrals



Triple Integrals

Example 3

Find the volume of the region Q bounded by graphs of $z = 3x^2$, $z = 4 - x^2, y = 0$ and $z + y = 6$.



Triple Integrals

Definition of mass

$m = \delta V$, where δ is mass density and V is Volume.

Mass of Solid

$$m = \iiint_Q \delta(x, y, z)\mathrm{d}V.$$

Mass of Lamina

$$m = \iint_R \delta(x, y)\mathrm{d}A.$$

Triple Integrals

Examples

(1) A lamina having area mass density $\delta(x,y) = y^2$ and has the shape of the region bounded by the graphs of $y = e^{-x}, x = 0, x = 1, y = 0$. Set up an iterated double integral that can be used to find the mass of the lamina.

(2) A solid having density $\delta(x,y,z) = z + 1$ has the shape of the region bounded by the graphs of $z = 4 - x^2 - y^2$, $z = 0$. set up an iterated triple integral that can be used to find the mass of the solid.

(3) A solid having density $\delta(x,y,z) = x^2 + y^2$ has the shape of the region bounded by the graphs of $x + 2y + z = 4, x = 0, y = 0, z = 0$. set up an iterated triple integral that can be used to find the mass of the solid.

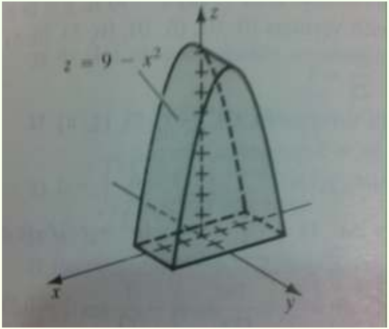
Triple Integrals

Examples

(1) Sketch and find the volume of the region Q bounded by graphs of $z = 9 - x^2$, $z = 0, y = -1$ and $y = 2$.

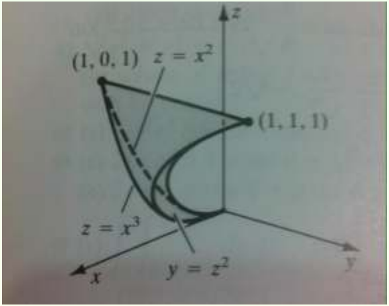
(2) Sketch and find the volume of the region Q bounded by graphs of $z = x^2$, $z = x^3, y = z^2$ and $y = 0$.

Sketch 1



Triple Integrals

Sketch 2



Center of mass and Moment of inertia

Definition

Let L be a lamina that has the shape of region R in the $xy-$plane. If the area mass density at (x, y) is $\delta(x, y)$ and if δ is continuous on R, then the mass m, the moments M_x and M_y, and the center of mass $(\overline{x}, \overline{y})$ are

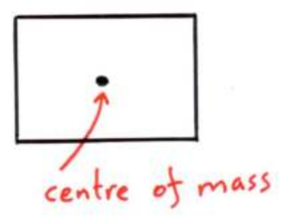
(i) $m = \iint\limits_R \delta(x, y) \mathrm{d}A$.

(ii) $M_x = \iint\limits_R y\delta(x, y) \mathrm{d}A$, $M_y = \iint\limits_R x\delta(x, y) \mathrm{d}A$

(iii) $\overline{x} = \frac{M_y}{m} = \dfrac{\iint\limits_R x\delta(x, y) \mathrm{d}A}{\iint\limits_R \delta(x, y) \mathrm{d}A}$, $\overline{y} = \frac{M_x}{m} = \dfrac{\iint\limits_R y\delta(x, y) \mathrm{d}A}{\iint\limits_R \delta(x, y) \mathrm{d}A}$.

Center of mass and Moment of inertia

Note: If L is homogeneous with constant mass density, the center of mass is also called the centroid



Moments of inertia of a Lamina

$I_x = \iint\limits_R y^2 \delta(x, y) \mathrm{d}A$ about the $x-$axis.

$I_y = \iint\limits_R x^2 \delta(x, y) \mathrm{d}A$ about the $y-$axis.

$I_O = I_x + I_y = \iint\limits_R (x^2 + y^2) \delta(x, y) \mathrm{d}A$ about the origin.

Center of mass and Moment of inertia

Moments and Center of mass in $3D$

(i) $m = \iiint\limits_Q \delta(x,y,z)\mathrm{d}V.$

(ii) $M_{xy} = \iiint\limits_Q z\delta(x,y,z)\mathrm{d}V$, $M_{xz} = \iiint\limits_Q y\delta(x,y,z)\mathrm{d}V$

$M_{yz} = \iiint\limits_Q x\delta(x,y,z)\mathrm{d}V$

(iii) $\overline{x} = \frac{M_{yz}}{m} = \dfrac{\iiint\limits_Q x\delta(x,y,z)\mathrm{d}V}{\iiint\limits_Q \delta(x,y,z)\mathrm{d}V}$, $\overline{y} = \frac{M_{xz}}{m} = \dfrac{\iiint\limits_Q y\delta(x,y,z)\mathrm{d}V}{\iiint\limits_Q \delta(x,y,z)\mathrm{d}V}$.

$\overline{z} = \frac{M_{xy}}{m} = \dfrac{\iiint\limits_Q z\delta(x,y,z)\mathrm{d}V}{\iiint\limits_Q \delta(x,y,z)\mathrm{d}V}$.

Center of mass and Moment of inertia

Note: If L is homogeneous with constant mass density, the center of mass is also called the centroid

Moments of inertia of solids

$I_z = \iiint\limits_Q (x^2 + y^2)\delta(x,y,z)\mathrm{d}V$ moment of inertia about the $z-$axis.

$I_x = \iiint\limits_Q (y^2 + z^2)\delta(x,y,z)\mathrm{d}V$ moment of inertia about the $x-$axis.

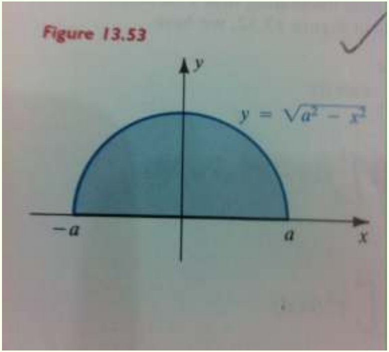
$I_y = \iiint\limits_R (x^2 + z^2)\delta(x,y,z)\mathrm{d}V$ moment of inertia about the $y-$axis.

Center of mass and Moment of inertia

Examples

(1) A lamina having area mass density $\delta(x, y) = kx$ and has the shape of the region R in the $xy-$plane bounded by the parabola $x = y^2$ and the line $x = 4$. Find the center of mass.

(2) A lamina having area mass density $\delta(x, y) = ky$ and has the semicirclar illustrated in Figure. Find the moment of inertia with respect to the $x-$axis.



Center of mass and Moment of inertia

Examples

(3) Set up an iterated integral that can be used to find the center of mass of the solid Q bounded by the paraboloid $x = y^2 + z^2$ and the palne $x = 4$ and density $\delta(x, y, z) = x^2 + y^2$.

(4) Let Q be the solid in the first octant bounded by the coordinates planes and the graphs of $z = 9 - x^2$ and $2x + y = 6$. Set up iterated integrals that can be used to find the centroid, find the centroid, find the moment of inertia with respect to the $z-$axis.