# Signals & Systems (CNET - 221)

# Linear Time Invariant System

Mr. HANEEF KHAN
Department of Computer Networks
Faculty of CS&IS
Jazan University

# Chapter Objective

**Following are the objectives of Chapter-III**

- Continuous LTI Systems
- Representation of signal in terms of impulses
- Unit Impulse Signal response & Convolution
- LTI System Properties

# Outline of Chapter-3

3.0 Introduction

3.1 Continuous-Time LTI Systems

- 3.1.1 Representation of Continuous -Time Signals In Terms of Impulses
- 3.1.2 Continuous- Time Unit Impulse Response And The Convolution-Sum representation of LTI System

3.2 Properties Of Linear Time Invariant System

- 3.2.1 Commutative Property
- 3.2.2 Distributive Property
- 3.2.3 Associative Property
- 3.2.4 LTI Systems With And Without Memory
- 3.2.5 Invertibility of LTI System
- 3.2.6 Causality For LTI System
- 3.2.7 Stability For LTI Systems
- 3.2.8 Unit Step Response of An LTI System

# Linear Time Invariant (LTI) Systems

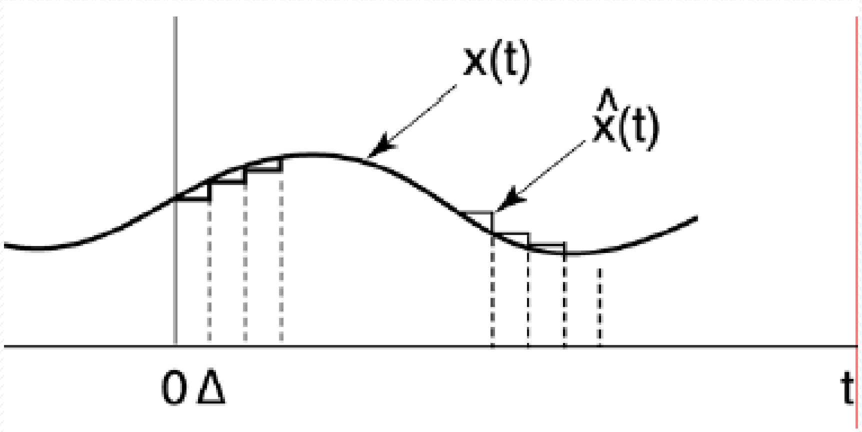
- A system satisfying both the **linearity** and the **time-invariance** property is called LTI system.

  **Linearity** – Linear system is a system that possesses the property of *superposition*.

  **Time Invariance** – A system is time invariant if the behavior and characteristics of the system are *fixed over time*.
- Highly useful signal processing algorithms have been developed utilizing this class of systems over the last several decades.
- Any linear time-invariant system (LTI) system, continuous-time or discrete-time, can be uniquely characterized by its
  - **Impulse response:** response of system to an impulse
  - **Frequency response**: response of system to a complex exponential $e^{j 2 p f}$ for all possible frequencies f.
  - **Transfer function**: Laplace transform of impulse response
- Given one of the three, we can find other two provided that they exist

# Representation of Continuous-Time Signals in Terms of Impulses

Apply any signal x(t) as sum of shifted, scaled impulses.



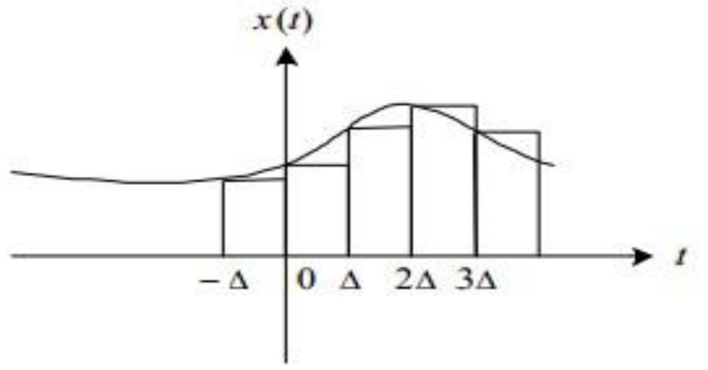

$$\hat{x}(t) = x(k\Delta)\ ,\ k\Delta < t < (k+1)\Delta$$

# Representation of Continuous-Time Signals in Terms of Impulses

- A continuous-time signal can be viewed as a linear combination of continuous impulses:
- Any input can be expressed using the unit impulse function

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

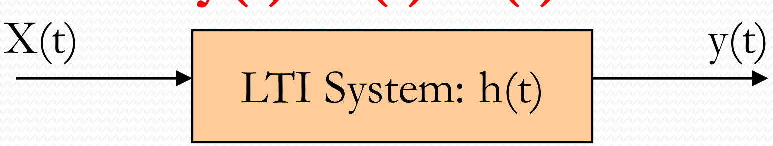
- The result is obtained by chopping up the signal x(t) in sections of width D, and taking sum



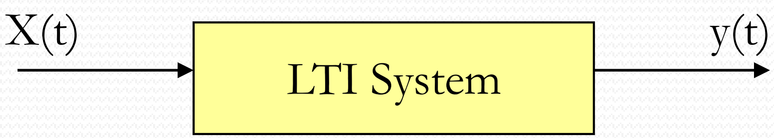
# Continuous-Time LTI systems: The Convolution Integral

- The response of a continuous-time LTI system can be computed by convolution of the impulse response of the system with the input signal, using a convolution integral, rather than a sum.
- That is: A system can be characterized using its impulse response:

**y(t)=x(t)*h(t)**




Do not confuse convolution with multiplication!




$$y(t) = T\{x(t)\}$$

$$h(t) = T\{\delta(t)\}$$

$$h(t-\tau) = T\{\delta(t-\tau)\}$$

$$y(t) = T\left[\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau\right]$$

$$Linearity: y(t) = \int_{-\infty}^{\infty} x(\tau)T[\delta(t-\tau)]d\tau$$

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

By definition

y(t)=x(t)*h(t)

# Continuous-Time LTI systems: The Convolution Integral

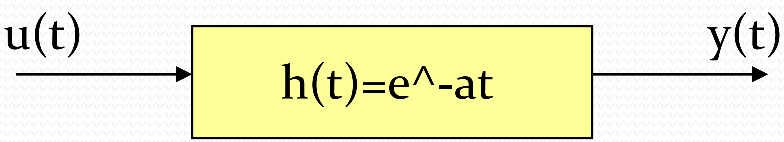
$$y(t) \quad = \quad x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, \mathrm{d}\tau$$

- Key point: The system output at $t$ is in response to all past and present values of the input signal, not just at time $t$
- Conceptually, you can think of convolution as weighted averaging
- The signal $x(t)$ is being averaged
- The impulse response $h(t)$ determines how it is weighted
- Lowpass filters tend to have smooth, slowly varying weights
- Highpass and bandpass filters tend to be oscillatory

# Example 1:

- Consider a CT-LTI system. Assume the impulse response of the system is h(t)=e^(-at) for all a>o and t>o and input x(t)=u(t). Find the output.

$$y(t) = h(t) * x(t) = h(t) * u(t)$$




$$y(t) = \int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} \left(e^{-a\tau} \cdot u(\tau)\right) u(t-\tau) d\tau$$

Because t>o

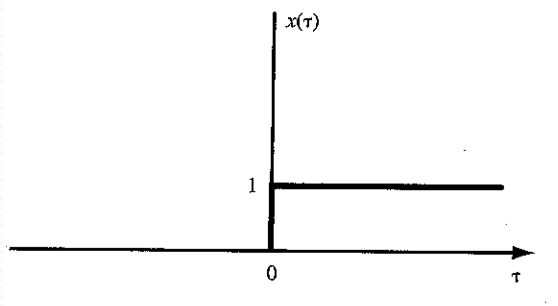
$$\int_{0}^{t} \left(e^{-a\tau}\right) d\tau = \frac{1}{-a}(e^{-at} - 1)$$

Draw x(τ), h(τ), h(t-τ),etc. → next slide

$$= \frac{1}{a}(1 - e^{-at}) u(t)$$

The fact that a>o is not an issue!

# Example – Cont.

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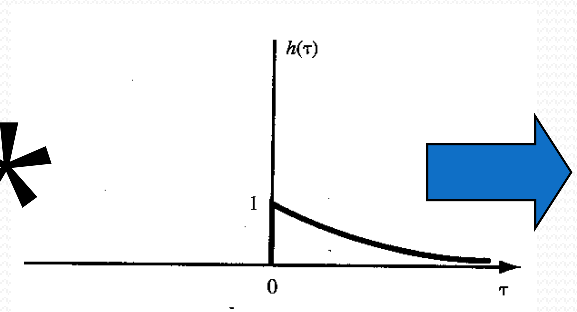
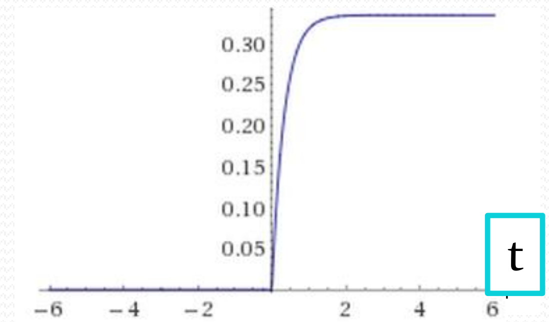
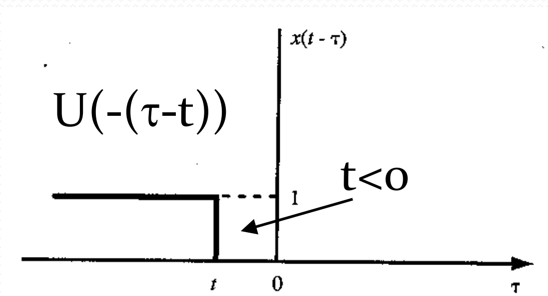
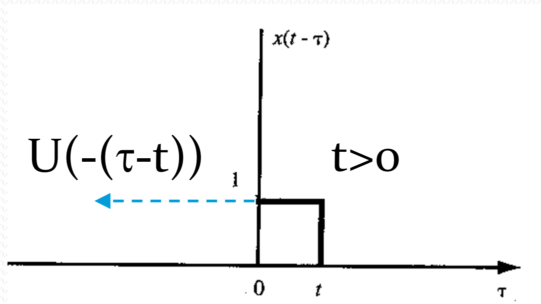



$$y(t) = h(t) * x(t) = h(t) * u(t)$$

y(t); for a=3



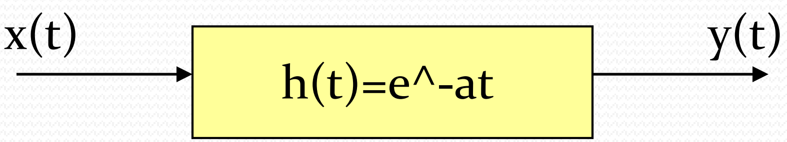




$$= \frac{1}{a}(1 - e^{-at})u(t)$$

Remember we are plotting it over τ and t is the variable

# Example: 2

- Consider a CT-LTI system. Assume the impulse response of the system is h(t)=e^-at for all a>o and t>o and input **x(t)= e^at u(-t).** Find the output.




$$y(t) = h(t) * x(t) = h(t) * u(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

$$= \int_{-\infty}^{\infty} \left(e^{a\tau} \cdot u(-\tau)\right)\left(e^{-a(t-\tau)} \cdot u(t-\tau)\right) d\tau$$

$$\forall t < 0 \rightarrow \left(e^{-at}\right) \int_{-\infty}^{t} \left(e^{2a\tau}\right) d\tau = \left(e^{-at}\right) \frac{1}{2a} e^{2at} = \frac{1}{2a} e^{at}$$

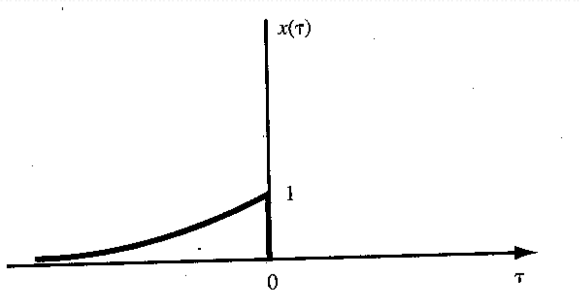
$$\forall t > 0 \rightarrow \left(e^{-at}\right) \int_{-\infty}^{0} \left(e^{2a\tau}\right) d\tau = \left(e^{-at}\right) \frac{1}{2a} [1-0] = \frac{1}{2a} e^{-at}$$

$$\Rightarrow y(t) = \frac{1}{2a} e^{-a|t|} \forall a > 0$$

Note that for t>o; x(t) =o; so the integration can only be valid up to t=o

Draw x(τ), h(τ), h(t-τ),etc. → next slide

# Example – Cont.




x(t)= e^at u(-t)

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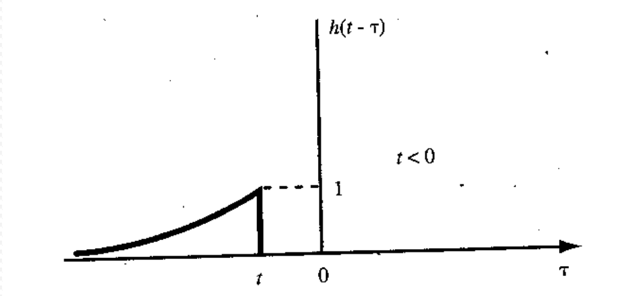
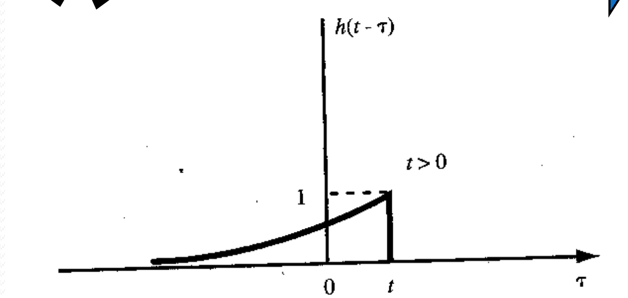
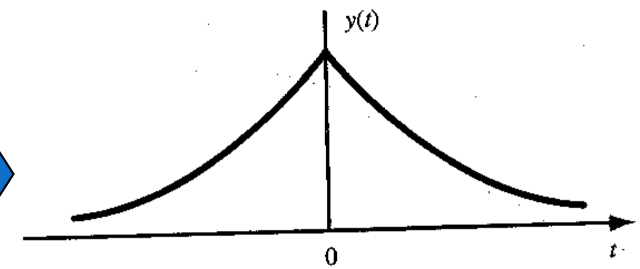



h(t)=e^-at u(t)




$$y(t) = h(t) * x(t) = h(t) * u(t)$$

$$\Rightarrow y(t) = \frac{1}{2a} e^{-a|t|} \forall a > 0$$

$$y(t) = h(t) * x(t) = h(t) * u(t)$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \quad ?$$

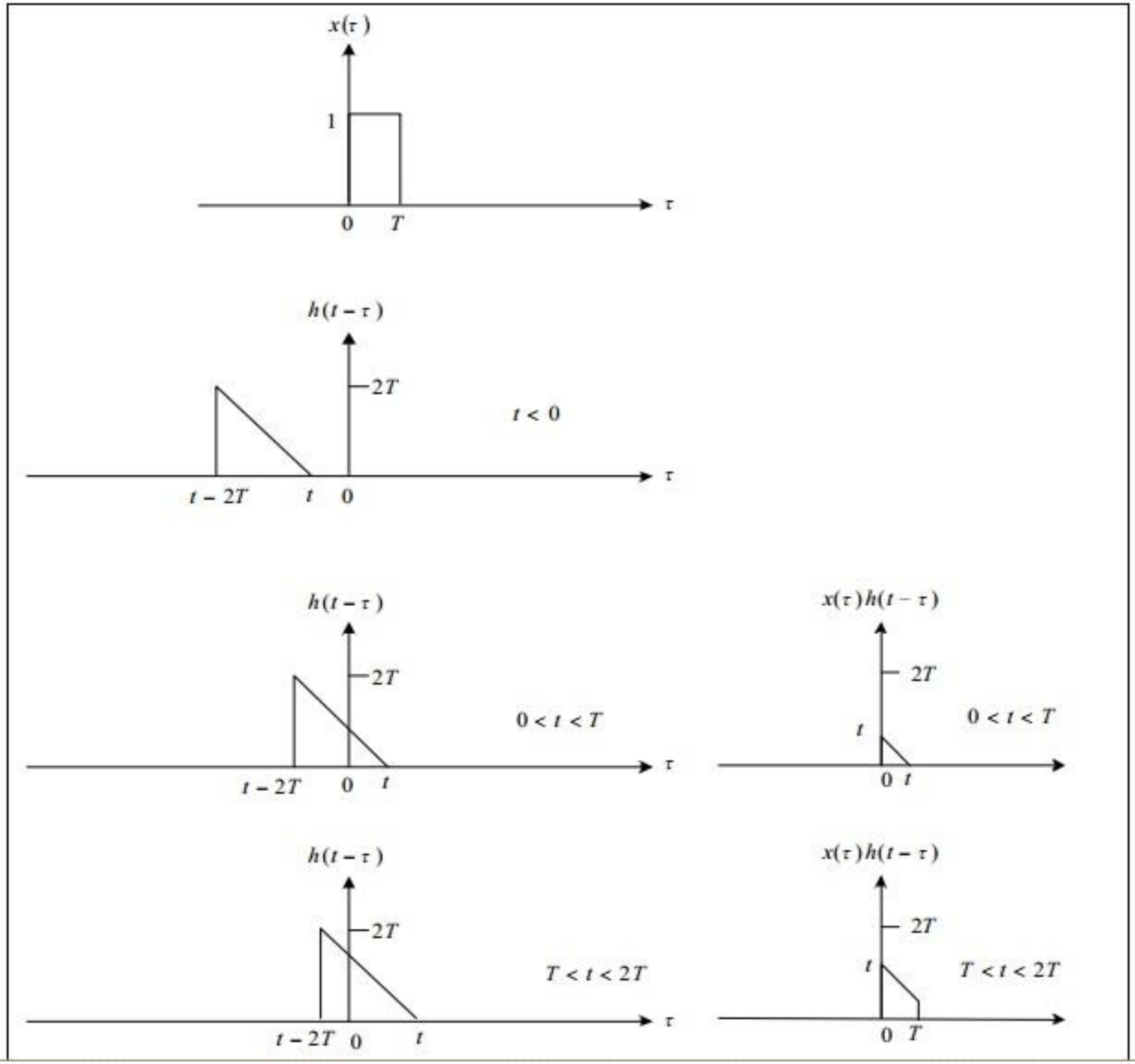
**Example**: Compute the convolution of the two signals below:

$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & otherwise \end{cases} \text{ and } h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & otherwise \end{cases}$$

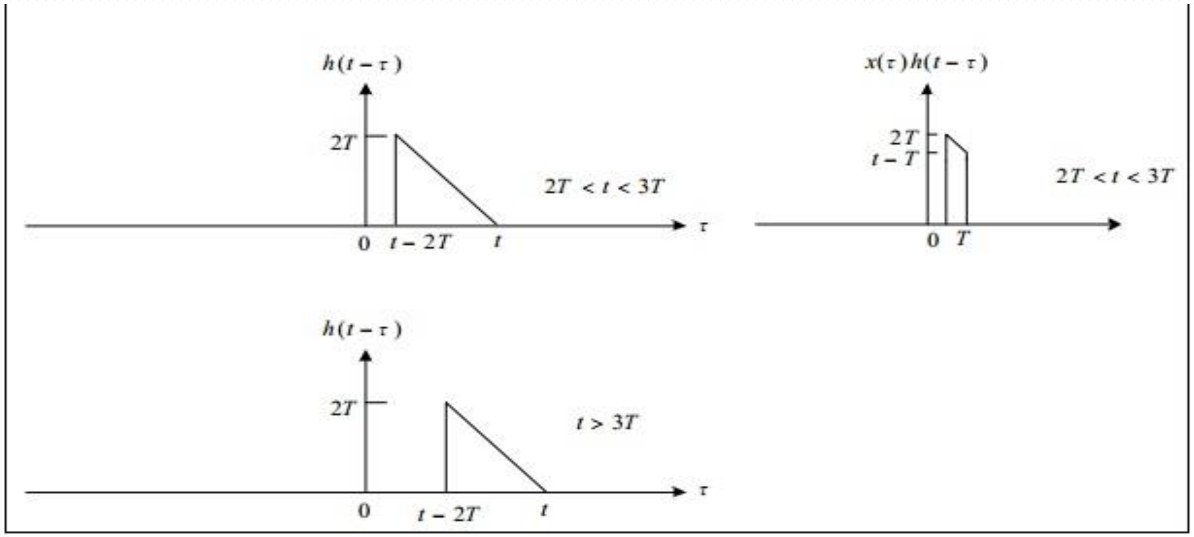
For this example, it is convenient to calculate the convolution in separate intervals. $x(\tau)$ is sketched and $h(t-\tau)$ is sketched in each of the intervals:

For $t < 0$, and $t > 3T$, $x(\tau)h(t-\tau) = 0$ for all the values of $\tau$, and consequently $y(t)$=0.

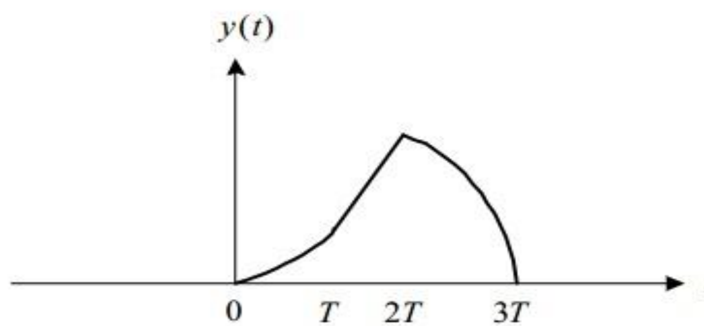
For other intervals, the product $x(\tau)h(t-\tau)$ can be found in the figure on the next page. Thus for these three intervals, the integration can be calculated with the result shown below:



x(τ)
1
0
T
τ
h(t − τ)
2T
t < 0
t − 2T
t
0
τ
h(t − τ)
2T
0 < t < T
t − 2T
0
t
τ
x(τ)h(t − τ)
2T
t
0 < t < T
0
t
h(t − τ)
2T
T < t < 2T
t − 2T
0
t
τ
x(τ)h(t − τ)
2T
t
T < t < 2T
0
T




$$y(t)=\begin{cases} 0, & t<0 \\ \frac{1}{2}t^2, & 0<t<T \\ Tt-\frac{1}{2}T^2, & T<t<2T \\ -\frac{1}{2}t^2+Tt+\frac{3}{2}T^2, & 2T<t<3T \\ 0, & t>3T \end{cases}$$



# Properties of Convolution

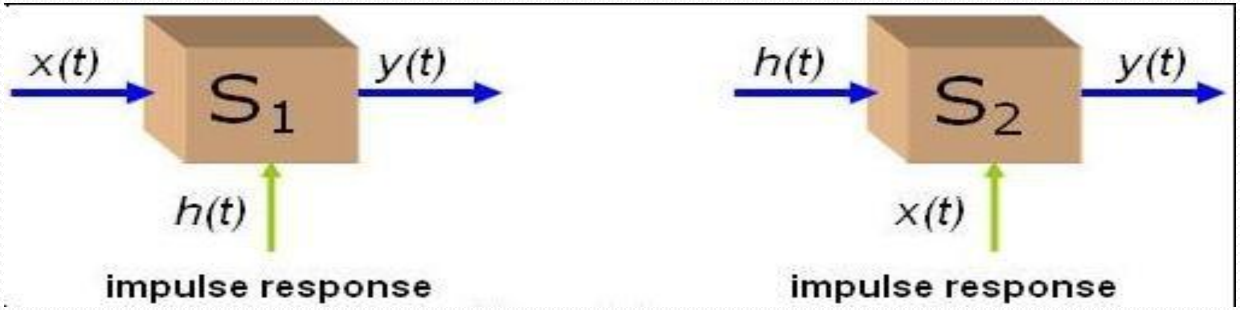
1. Commutative Property
2. Distributive Property
3. Associative Property

# 1. Commutative Property

Convolution in both continuous time and discrete time is that it is a *commutative* operation.

***x(t) * h(t) = h(t) * x(t)……Continuous Time***

***x[n] * h[n] = h[n] * x[n]……Discrete Time***



## 2. Distributive Property

Another basic property is the distributive property, convolution distributes over addition.

***x(t)*[h1(t)+h2(t)]=x(t)*h1(t)+x(t)*h2(t)……..***

*Continuous Time*

***x[n]*(h1[n]+h2[t])=x[t]*h1[n]+x[n]*h2[n]……..***

*Discrete Time*

# 3. Associative Property

According to associative property, the series interconnection of two systems is equivalent to the single system.

***x(t)*[h1(t) *h2(t)] = [x(t)*h1(t)]*h2(t)……..***

*Continuous Time*

***x[n]*[h1[n] *h2[t]]=(x[t]*h1[n])*h2[n]……..***

*Discrete Time*

# Properties of LTI system

1. LTI systems with and without memory
2. Invertibility of LTI systems
3. Causality of LTI systems
4. Stability of LTI systems
5. The Unit Step response of LTI systems

# 1. LTI Systems with and without Memory

A system is memory less if its output at any time depends only on the value of its input at the same time.

**<u>Continuous-time system</u>**

*System is memoryless if* ***h(t) = 0 for t ≠ 0,*** *and such a memoryless LTI system has the form*

$$y(t) = K\,x(t)$$

*For some constant K and has the impulse response*

$$h(t) = K\,x(t)$$

***Note:-*** *If K* = 1 the systems become ***identity systems***, with output equal to the input.

# Continue…..

## Discrete-time System

System is memoryless, if $h[n] = 0$ for $n \neq 0$, in this case, the impulse response has the form.

$$h[n] = K\,\delta[n]$$

Where $K = h[0]$ is a constant and the convolution sum reduces to the relation

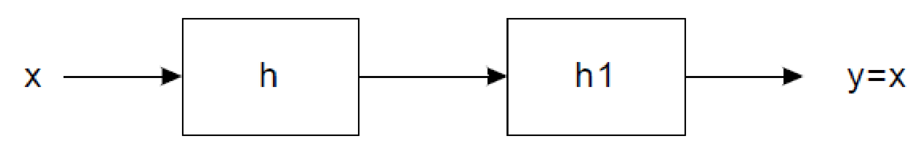
$$y[n] = K\,x[n]$$

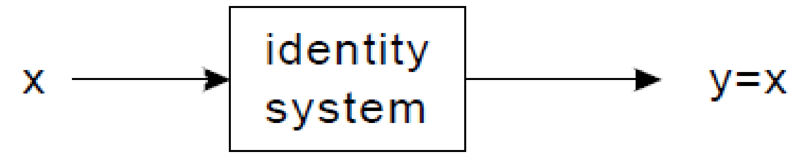
**Note:-** *If* $K = 1$, the systems become ***identity systems***, with output equal to the input.

***Otherwise the LTI system has memory.***

# 2. Invertibility of LTI systems

We have seen that a system S is invertible if and only if there exists an inverse system $S^{-1}$ such that $S^{-1}S$ produce an output equal to the input to the first system.




Since the overall impulse response in the first figure is ***h(t)* h1(t)***, ***h1*** must satisfy for it to be ***the impulse response of the inverse system***, namely

$$h(t) * h1(t) = \partial(t) \ldots\ldots \textit{Continuos LTI system}$$

$$h[n] * h1[n] = \delta[n] \ldots\ldots \textit{Discrete LTI system}$$

## 3. Causality

A system is causal if the output depends only on present and past, but not future inputs. A necessary and sufficient condition for causality is

$$h(t) = 0 \qquad \forall\, t < 0$$

where ***h(t)*** is the impulse response.

**Note:-** Causality is a necessity if the independent variable is time, but not all systems have time as an independent variable.

# 4. Stability

A system stable if every bounded input produces a bounded output.

A LTI system is said to be stable if its impulse response satisfies the following criterion :

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

$$\int_{-\infty}^{\infty} |h(t)|dt < \infty$$

# 5. Unit Step Response of an LTI System

Unit step response is define as s(t) or s[n] with corresponding to the output when x(t)=u(t) or x[n]=u[n].

The convolution-sum representation, the step response of a LTI system is the convolution of the unit step with the impulse response, that is,

$$s(t) = \int h(\tau)\, d\tau; \text{ where } \tau = -\infty \text{ to } t \quad \text{[Continuos Time LTI system]}$$

$$s[n]=u[n]*h[n] \quad \text{[ Discrete Time LTI system]}$$

- ***s(t) or s[n]*** is the step response
- ***h(t) or h[n]*** is the impulse response
- ***u(t) or u[n]*** is the unit step.

**THANKS**