Signals & Systems (CNET - 221)
Chapter-5
Fourier Transform

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Outline of Chapter-5

Fourier Transform

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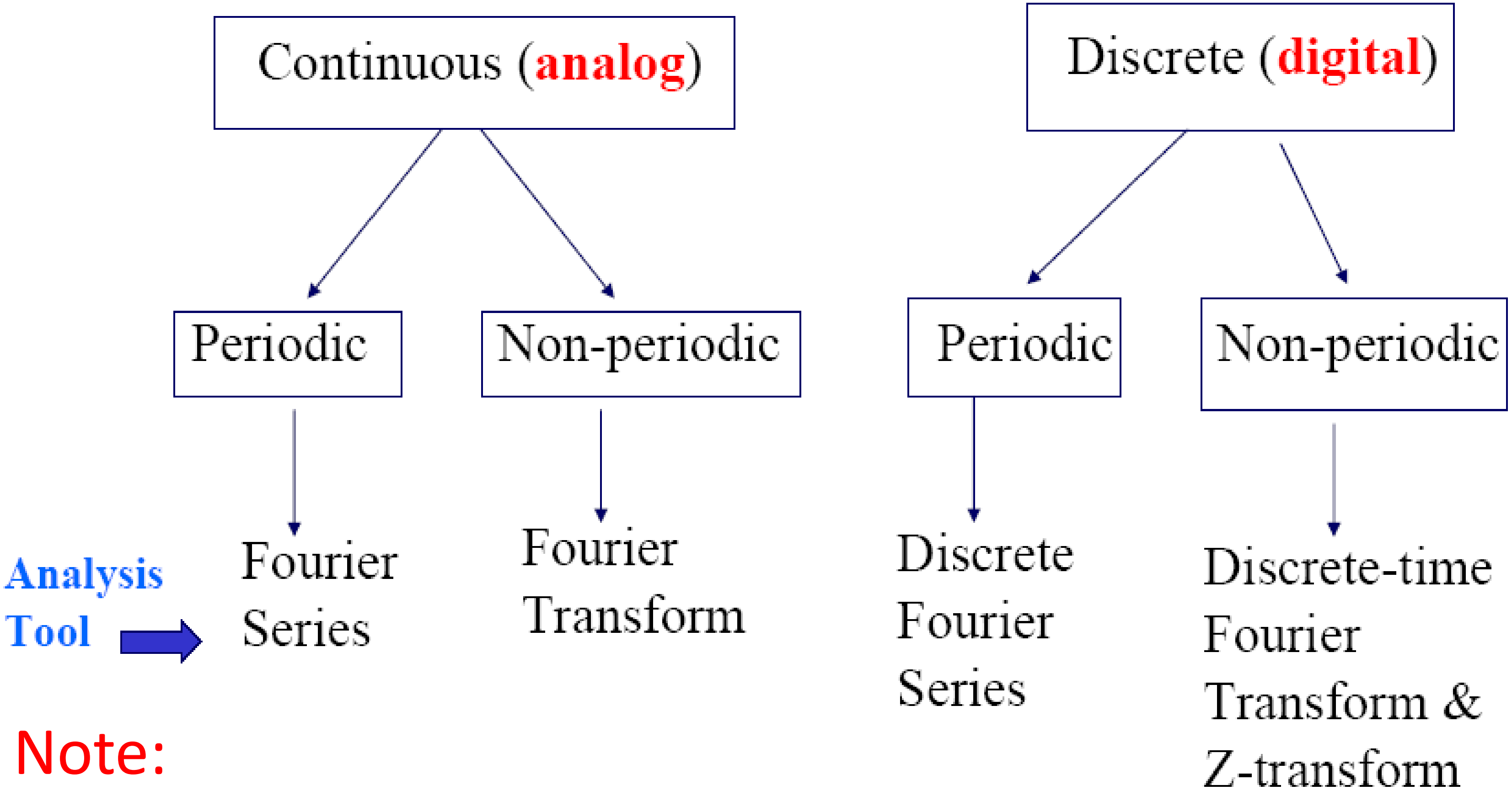
Objective of Chapter-5

- To generalize the Fourier series to include aperiodic signals by defining the Fourier transform
- To establish which type of signals can or cannot be described by a Fourier transform
- To derive and demonstrate the properties of the Fourier transform

Fourier Representations



Note:

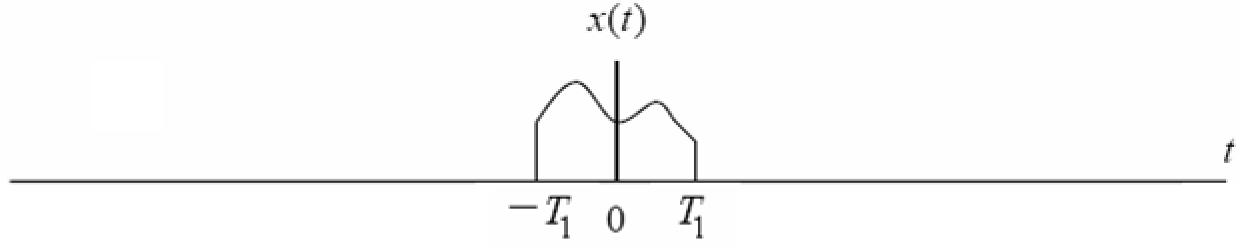
- **A Fourier representation is unique, i.e., no two same signals in time domain give the same function in frequency domain**

Representation of Aperiodic Signals : Continuous Time Fourier Transform

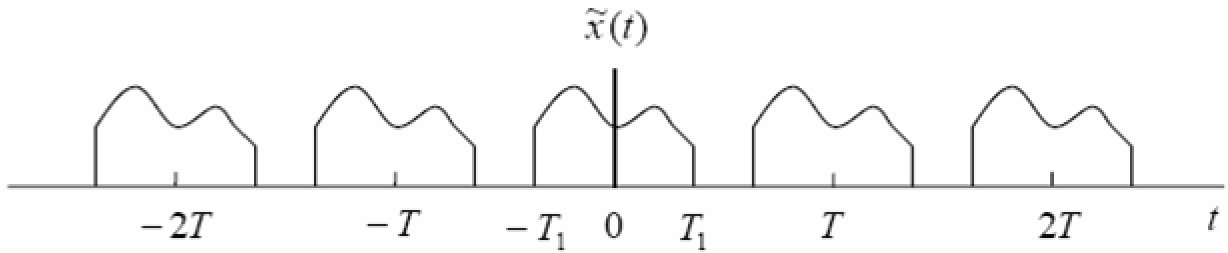
- **Development of the FT : Representation of an Aperiodic Signal**
- **Fourier Series (FS):** A discrete representation of a **periodic** signal as a linear combination of complex exponentials
- The CT Fourier Series cannot represent an aperiodic signal for all time
- **Fourier Transform (FT):** A continuous representation of a **not periodic** signal as a linear combination of complex exponentials
- The CT Fourier transform represents an aperiodic signal for all time.
- A not-periodic signal can be viewed as a periodic signal with an **infinite** period.

Fourier Transform – CT Aperiodic signals

- Consider the CT aperiodic signal given below:



One can construct a periodic signal equal to $x(t)$ over the interval $-T_1 \leq t \leq T_1$ as follows:



Equations of Fourier Transform

Fourier Transform:

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

Fourier Inverse Transform:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$$

Convergence of Fourier Transform

Dirichlet's sufficient conditions for the convergence of Fourier transform are similar to the conditions for the CT-FS:

1. x(t) must be absolutely integrable

$$\int_{-\infty}^{+\infty} |x(t)| dt < \infty$$

2. x(t) must have a finite number of maxima and minima within any finite interval.

3. x(t) must have a finite number of discontinuities, all of finite size, within any finite interval.

Examples of Continuous Time-FT

- Example 4.1 : P.no: 290

Find the Fourier transform of the following signal:

$$x(t) = e^{-at}u(t), \quad \mathrm{Re}\{a\} > 0$$

Solution: $x(t)$ meets all three Dirichlet conditions. Using $X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ we will have:

$$X(j\omega) = \int_{0}^{+\infty} e^{-at}e^{-j\omega t}dt$$

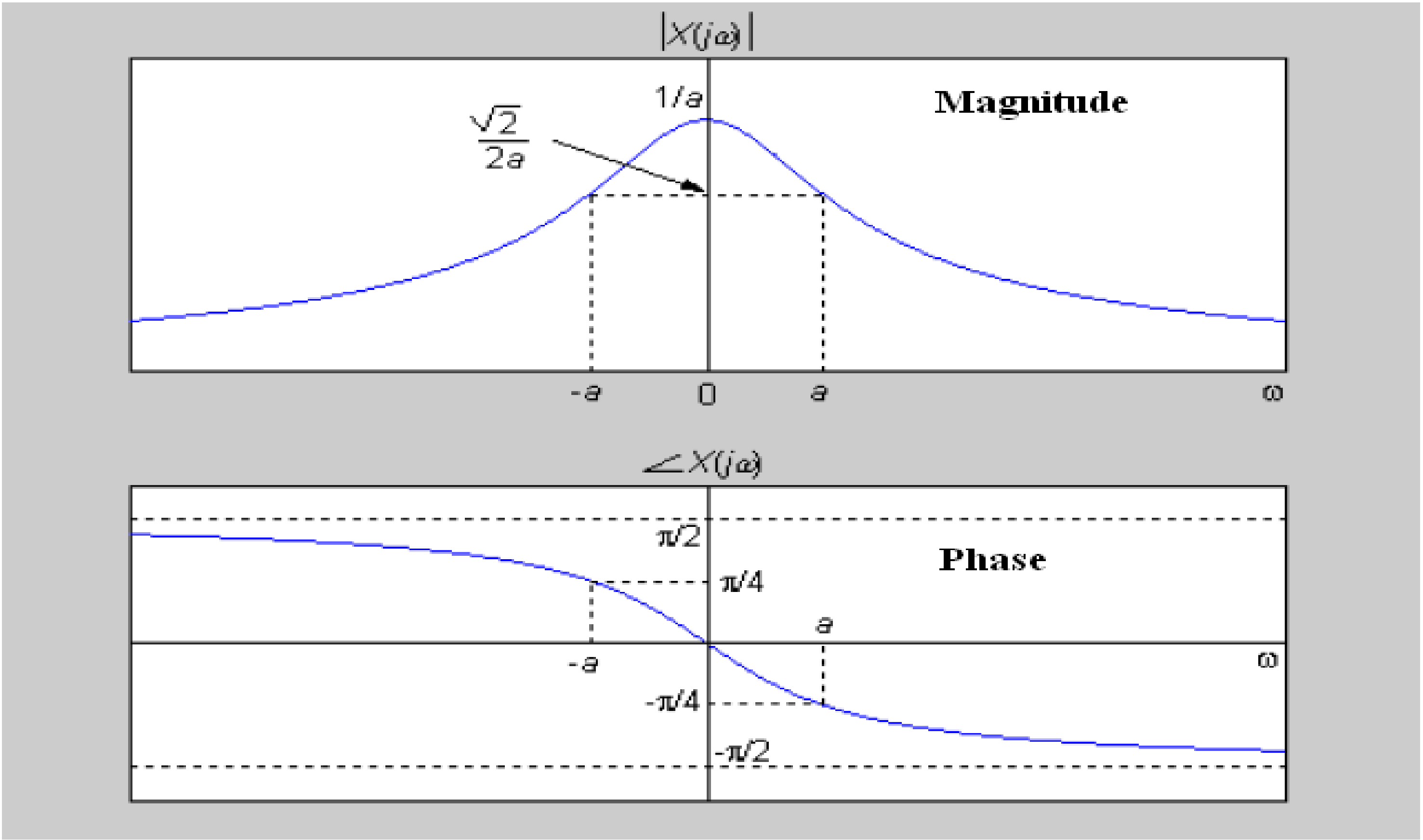
$$= -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty}$$

$$= \frac{1}{a+j\omega}, \quad \mathrm{Re}\{a\} > 0$$

Like the Fourier series coefficients, the Fourier transform is a complex function of frequency in general and must be represented in two separate figures

Example – 4.1 Continued……

The magnitude and phase of $X(j\omega)$ for a real value of a for this example are



Example 4.2 : P.no: 291

Find the Fourier transform of the following signal:

$$x(t) = e^{-a|t|}u(t), \quad \mathrm{Re}\{a\} > 0$$

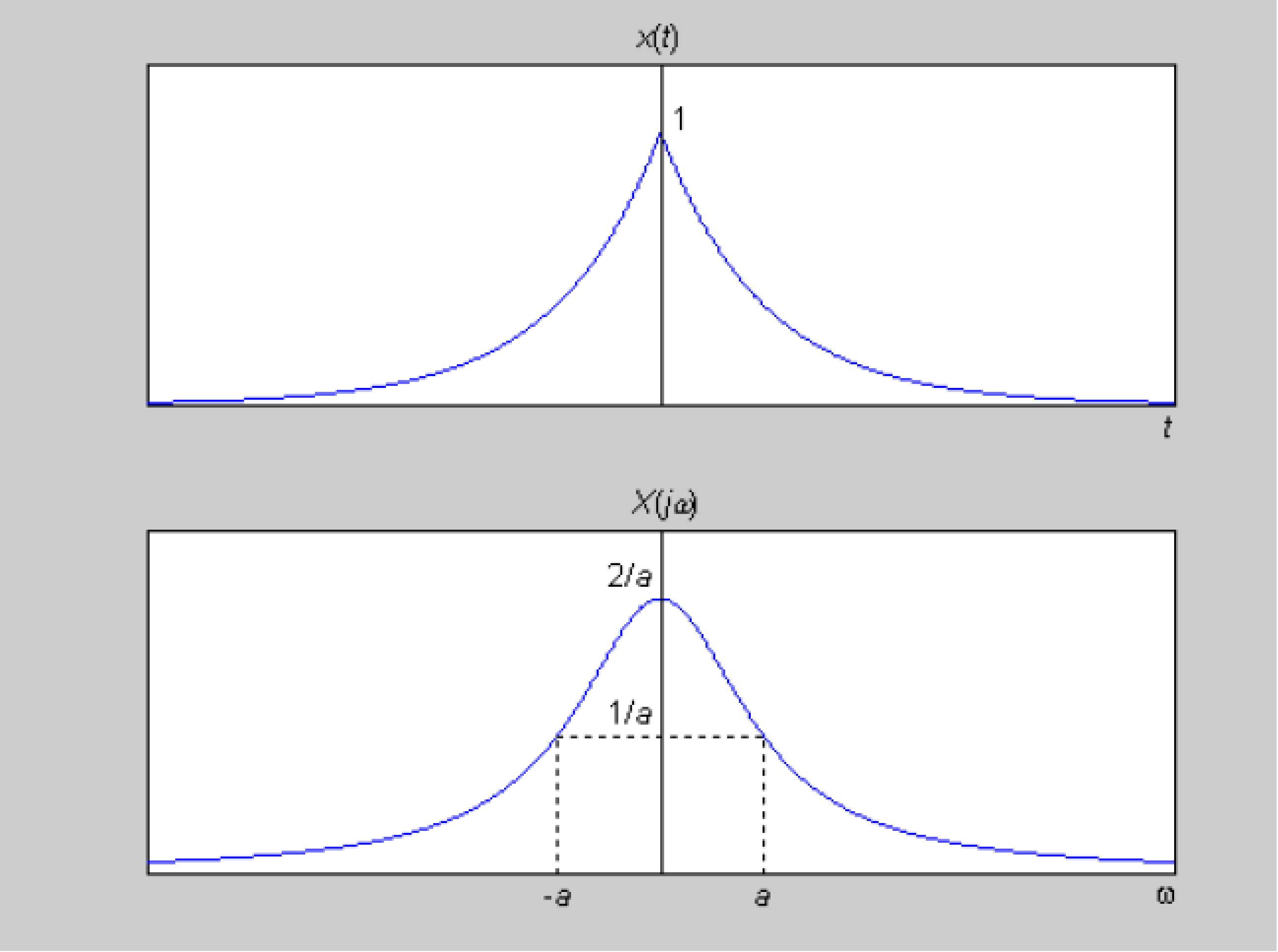
Solution: $x(t)$ meets all three Dirichlet conditions. We know that $|t| := \begin{cases} -t & t < 0 \\ t & t > 0 \end{cases}$

So, using $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$ we will have:

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{+\infty} e^{-a|t|} e^{-j\omega t} dt \\ &= \int_{-\infty}^{0} e^{at} e^{-j\omega t} dt + \int_{0}^{+\infty} e^{-at} e^{-j\omega t} dt \\ &= \frac{1}{a - j\omega} + \frac{1}{a + j\omega} \\ &= \frac{2a}{a^2 + \omega^2}, \quad \mathrm{Re}\{a\} > 0 \end{aligned}$$

Example 4.2 Continued..

- The Fourier transform for this example is real at all frequencies
- The time signal and its Fourier transform are



Example 4.2 : P.no: 292

- Properties of unit impulse:

$$\delta(t) = \frac{du(t)}{dt}. \qquad \int_{-\infty}^{\infty} x(t)\delta(t - t_0)dt = x(t_0)$$

- Example:

Find the Fourier transform of the unit impulse signal

Solution: Using $X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ and the sifting property of the impulse we will have:

$$X(j\omega) = \int_{-\infty}^{+\infty} \delta(t)e^{-j\omega t}dt = 1$$

- Note: Unit Impulse has a Fourier Transform consisting of equal contributions at all frequencies.

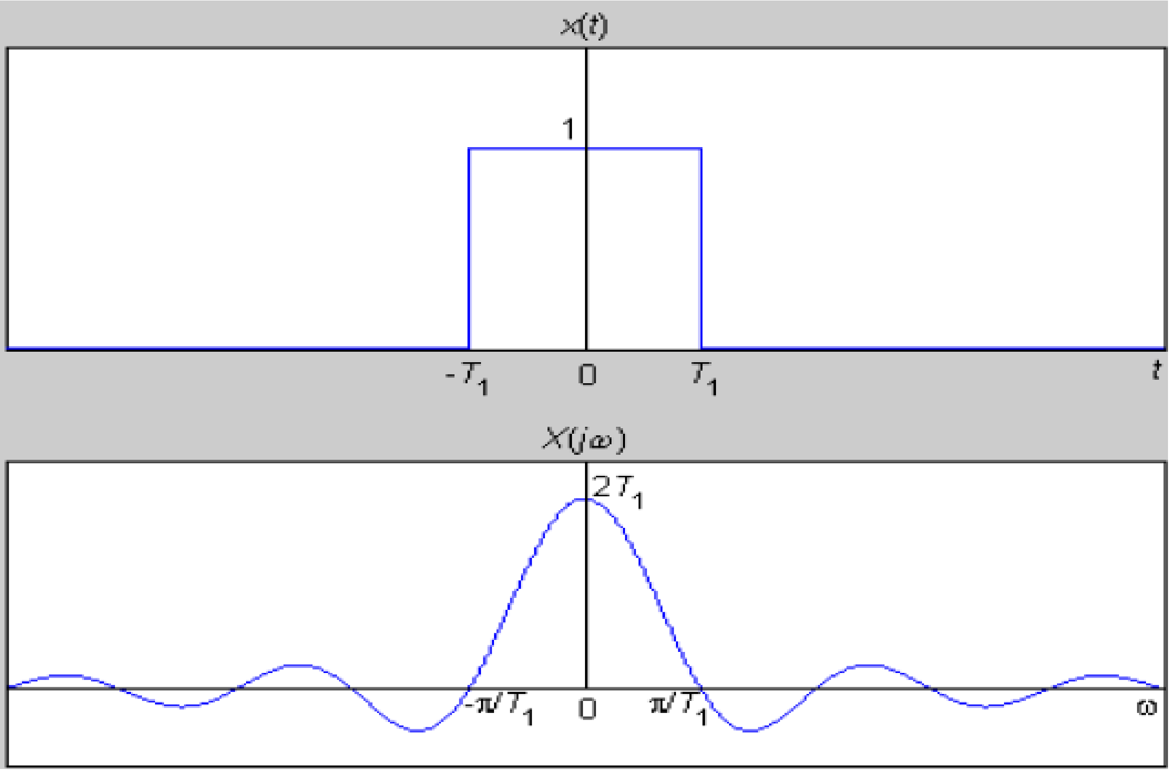
Example 4.4 P.no:293

Find the Fourier transform of the following rectangular pulse signal

$$x(t) = \begin{cases} 1 & |t| < T_1 \\ 0 & |t| > T_1 \end{cases} .$$

Solution: Using $X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$ we will have:

$$X(j\omega) = \int_{-T_1}^{+T_1} e^{-j\omega t} dt = 2\frac{\sin(\omega T_1)}{\omega} .$$



Example 4.5 P.no:294

Find a time signal $x(t)$ whose Fourier transform is given by:

$$X(j\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & |\omega| > W \end{cases}$$

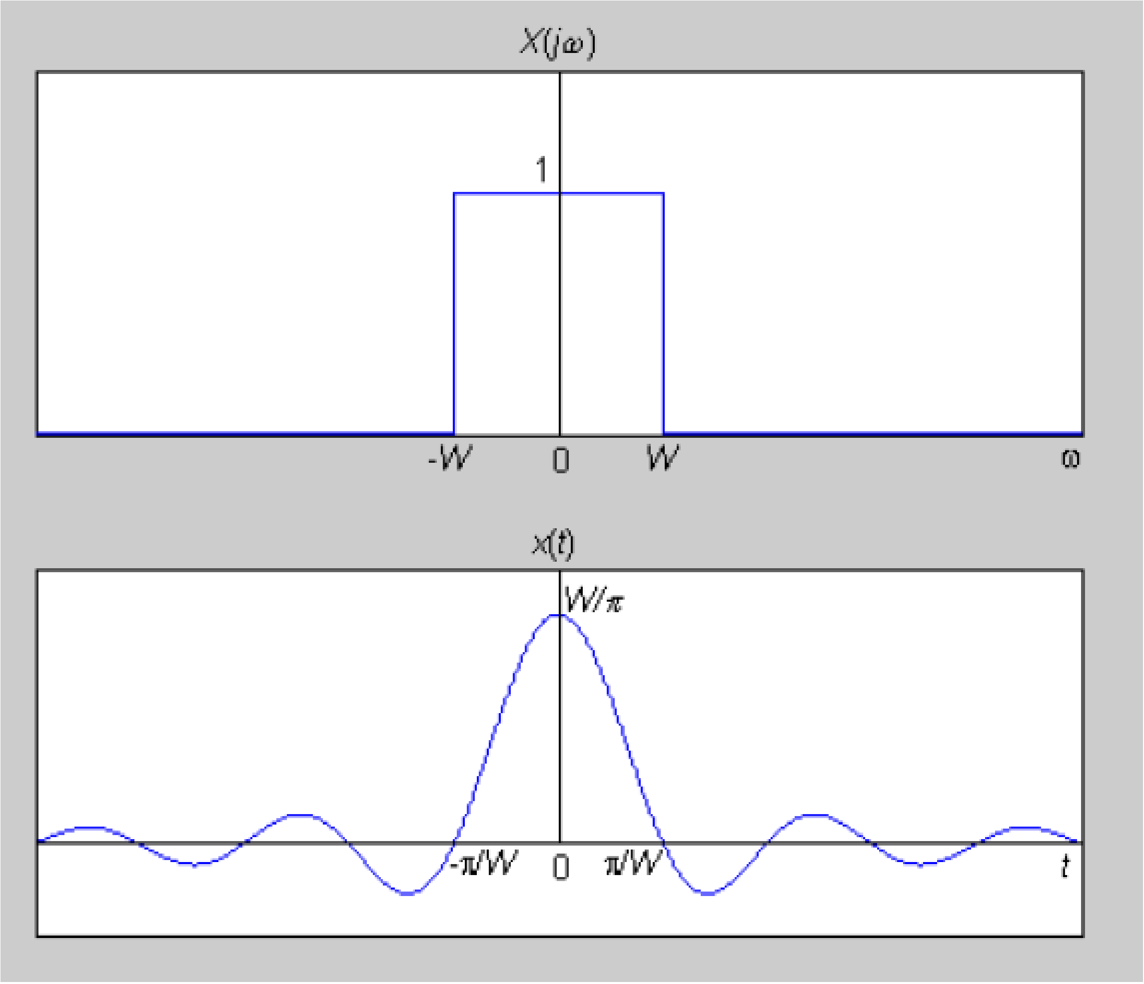
Solution: Using $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ we will have:

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}d\omega = \frac{\sin(Wt)}{\pi t}$$

The Fourier transform $X(j\omega)$ and the corresponding time signal $x(t)$ are shown in the next slide

Example 4.5 Continued…

- FT



Properties of Fourier Transform

The equations of Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

1. Linearity
2. Time Shifting
3. Conjugation and Conjugate Symmetry
4. Differentiation and Integration
5. Time and Frequency Scaling
6. Duality
7. Parseval’s Relation
8. Convolution and Multiplication

1. Linearity

If

$$x(t) \xleftrightarrow{FT} X(j\omega), \quad y(t) \xleftrightarrow{FT} Y(j\omega)$$

then

$$ax(t) + by(t) \xleftrightarrow{FT} aX(j\omega) + bY(j\omega)$$

2. Time Shifting

If

$$x(t) \stackrel{FT}{\longleftrightarrow} X(j\omega)$$

Then

$$x(t - t_0) \stackrel{FT}{\longleftrightarrow} X(j\omega)e^{-j\omega t_0}$$

3. Conjugation and Conjugate Symmetry

The Conjugate property states that if

$$x(t) \stackrel{FT}{\longleftrightarrow} X(j\omega)$$

$$x^*(t) \stackrel{FT}{\longleftrightarrow} X(-j\omega)$$

- Conjugate symmetry property :

$$X(-j\omega) = X^*(j\omega)$$

4. Differentiation and Integration

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

$$\frac{d}{dt}(x(t)) \xleftrightarrow{F} j2\pi f\, X(f)$$

$$\frac{d}{dt}(x(t)) \xleftrightarrow{F} j\omega\, X(j\omega)$$

$$\Rightarrow \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

The term $\pi X(0)\delta(\omega)$ in the integration property reflects the DC or average value of the signal

5. Time and Frequency Scaling

If

$$x(t) \xleftrightarrow{\text{FT}} X(j\omega)$$

then

$$x(at) \xleftrightarrow{\text{FT}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

$$x(-t) \xleftrightarrow{\text{FT}} X(-j\omega)$$

6. Duality

$$X(t) \overset{F}{\longleftrightarrow} x(-f) \quad \text{and} \quad X(-t) \overset{F}{\longleftrightarrow} x(f)$$

$$X(jt) \overset{F}{\longleftrightarrow} 2\pi\, x(-\omega) \quad \text{and} \quad X(-jt) \overset{F}{\longleftrightarrow} 2\pi\, x(\omega)$$

7. Parseval's Relation

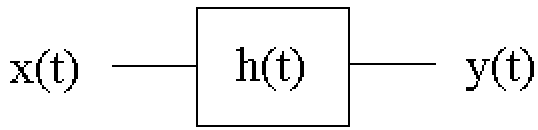
If x(t) and X(jω) are a Fourier transform pair, then

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

8(i). Convolution Property

Consider LTI System

h(t) is the impulse response of the LTI system.



Convolution Property:

$$y(t) = h(t) * x(t) \Rightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

$H(j\omega)$ is called the frequency response of the system

8(ii). Multiplication

- The Convolution property states that convolution in the time domain corresponds to multiplication in the frequency domain.

$$x(t) * y(t) \xleftrightarrow{F} X(f)Y(f)$$

$$x(t) * y(t) \xleftrightarrow{F} X(j\omega)Y(j\omega)$$

$$x(t)y(t) \xleftrightarrow{F} X(f) * Y(f)$$

$$x(t)y(t) \xleftrightarrow{F} \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

Summary of properties of CTFT

Linearity	$\alpha x(t) + \beta y(t) \xleftrightarrow{\mathcal{F}} \alpha X(f) + \beta Y(f)$
	$\alpha x(t) + \beta y(t) \xleftrightarrow{\mathcal{F}} \alpha X(j\omega) + \beta Y(j\omega)$
Time shifting	$x(t - t_0) \xleftrightarrow{\mathcal{F}} X(f)e^{-j2\pi f t_0}$
	$x(t - t_0) \xleftrightarrow{\mathcal{F}} X(j\omega)e^{-j\omega t_0}$
Frequency shifting	$x(t)e^{+j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} X(f - f_0)$
	$x(t)e^{+j\omega_0 t} \xleftrightarrow{\mathcal{F}} X[j(\omega - \omega_0)]$
Time scaling	$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{\lvert a\rvert} X\left(\frac{f}{a}\right)$
	$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{\lvert a\rvert} X\left(j\frac{\omega}{a}\right)$
Frequency scaling	$\frac{1}{\lvert a\rvert} x\left(\frac{t}{a}\right) \xleftrightarrow{\mathcal{F}} X(af)$
	$\frac{1}{\lvert a\rvert} x\left(\frac{t}{a}\right) \xleftrightarrow{\mathcal{F}} X(ja\omega)$
Transform of a conjugate	$x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-f)$

Summary of properties of CTFT

Multiplication–convolution duality

$$\mathrm{x}(t) * \mathrm{y}(t) \xleftrightarrow{\mathcal{F}} \mathrm{X}(f)\mathrm{Y}(f)$$

$$\mathrm{x}(t) * \mathrm{y}(t) \xleftrightarrow{\mathcal{F}} \mathrm{X}(j\omega)\mathrm{Y}(j\omega)$$

$$\mathrm{x}(t)\mathrm{y}(t) \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) * \mathrm{Y}(f)$$

$$\mathrm{x}(t)\mathrm{y}(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}\mathrm{X}(j\omega) * \mathrm{Y}(j\omega)$$

Differentiation

$$\frac{d}{dt}(\mathrm{x}(t)) \xleftrightarrow{\mathcal{F}} j2\pi f\mathrm{X}(f)$$

$$\frac{d}{dt}(\mathrm{x}(t)) \xleftrightarrow{\mathcal{F}} j\omega\mathrm{X}(j\omega)$$

Modulation

$$\mathrm{x}(t)\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}[\mathrm{X}(f - f_0) + \mathrm{X}(f + f_0)]$$

$$\mathrm{x}(t)\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}[\mathrm{X}(j(\omega - \omega_0)) + \mathrm{X}(j(\omega + \omega_0))]$$

Transforms of periodic signals

$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k]e^{-j2\pi(kf_F)t} \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k]\delta(f - kf_0)$$

$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k]e^{-j(k\omega_F)t} \xleftrightarrow{\mathcal{F}} \mathrm{X}(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} \mathrm{X}[k]\delta(\omega - k\omega_0)$$