Module 1 : Signals in Natural Domain
Lecture : Properties of LTI Systems Objectives

In this lecture you will learn the following

We shall look into the properties of convolution (as shown below) in both continuous and discrete domain

- Associative
- Commutative
- Distributive properties
- As a LTI system is completely specified by its impulse response, we look into the conditions on the impulse response for the LTI system to obey properties like memory, stability, invertibility, and causality.

Properties of LTI System

In the preceding chapters,we have already derived expressions for discrete as well as continuous time convolution operations.

Discrete :

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

Continuous :

$$y(t) = \int_{-\infty}^{+\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

We shall now discuss the important properties of convolution for LTI systems.

1) Commutative property **:**

By the commutative property,the following equations hold true :

a) Discrete time:

$$x[n] * h[n] = h[n] * x[n]$$

Proof : We know that

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

Hence we make the following substitution (n - k = l)

$\therefore$ The above expression can be written as

$$y[n] = \sum_{l=-\infty}^{+\infty} x[n-l]\, h[l] = h[n] * x[n]$$

So it is clear from the derived expression that

$$x[n] * h[n] = h[n] * x[n]$$

Note :

1. 'n' remains constant during the convolution operation so 'n' remains constant in the substitution “n-k = l” even as 'k' and 'l' change.

2. “l” goes from $-\infty$ to $+\infty$, this would not have been so had 'k' been bounded.(e.g :- 0 < k < 11 would make n < l < n – 11)

b) Continuous Time:

$$x(t) * h(t) = h(t) * x(t)$$

Proof:

We know that

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

Making the substitution $t - \lambda = \phi$ Limits:

$d\phi = -d\lambda$

λ	$-\infty$	∞
ϕ	∞	$-\infty$

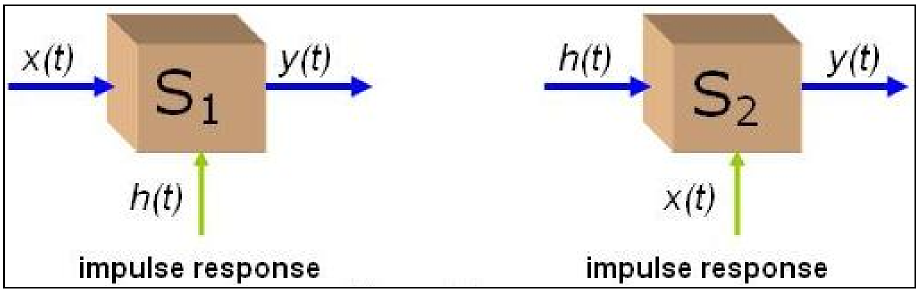
$$\therefore\ y(t) = \int_{-\infty}^{\infty} x(t-\phi)\, h(\phi)\, d\phi$$

$$= h(t) * x(t)$$

$$\therefore\ x(t) * h(t) = h(t) * x(t)$$

Thus we proved that convolution is commutative in both discrete and continuous variables.

Thus the following two systems : One with input signal x(t)and impulse response h(t) and the other with input signal *h(t)* and impulse response x(t) both give the same output y(t).



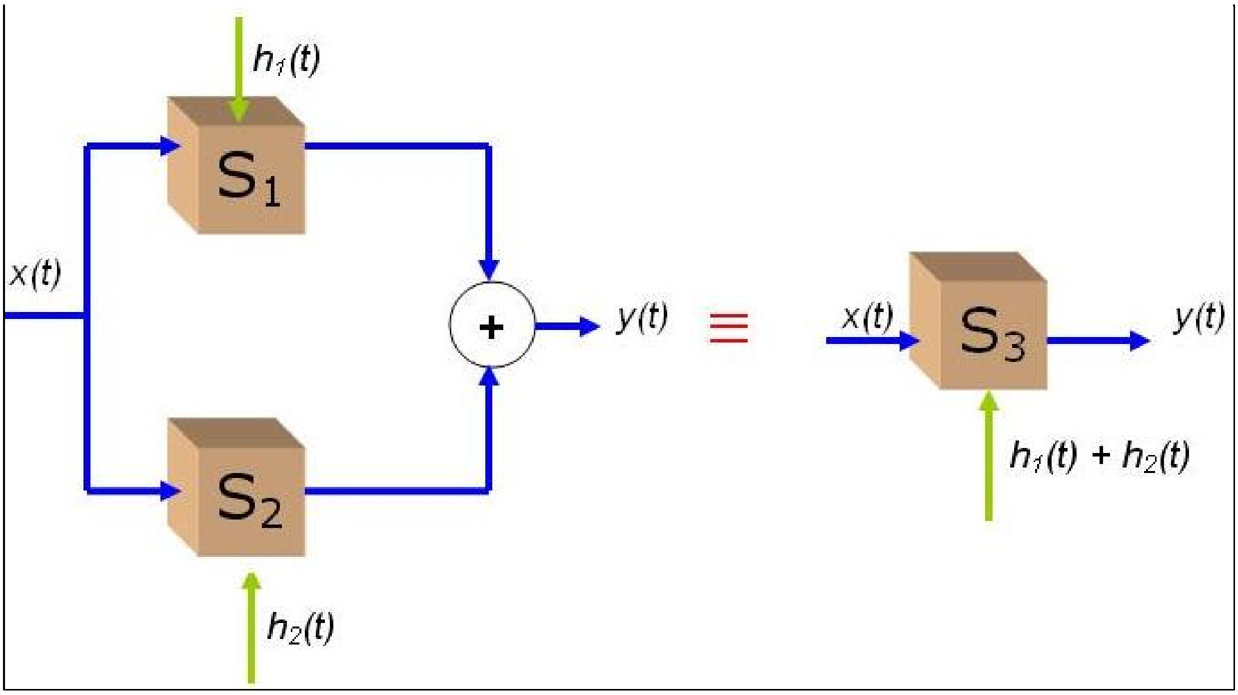
2) Distributive Property :

By this property we mean that convolution isdistributive over addition.

a) Discrete : $x[n] * \{ \alpha h_1[n] + \beta h_2[n] \} = \alpha \{ x[n] * h_1[n] \} + \beta \{ x[n] * h_2[n] \}$ α, β are constants

b) Continuous : $x(t) * \{ \alpha h_1(t) + \beta h_2(t) \} = \alpha \{ x(t) * h_1(t) \} + \beta \{ x(t) * h_2(t) \}$ α, β are constants

A parallel combination of LTI systems can be replaced by an equivalent LTI system which is described by the sum of the individual impulse responses in the parallel combination.



3) Associative property

a) Discrete time :

$$y[n] = x[n]*h[n]*g[n]$$

Proof : We know that

$$(x[n]*h_1[n])*h_2[n] = \sum_{l=-\infty}^{+\infty} (x*h_1)[l]h_2[n-l]$$

$$= \sum_{l=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} x[k]h_1[l-k]h_2[n-l] \longrightarrow (1)$$

$$x[n]*(h_1[n]*h_2[n]) = \sum_{p=-\infty}^{+\infty} x[p](h_1*h_2)[n-p]$$

$$= \sum_{p=-\infty}^{+\infty} x[p] \sum_{q=-\infty}^{+\infty} h_1[q]h_2[n-p-q]$$

$$= \sum_{p=-\infty}^{+\infty} \sum_{q=-\infty}^{+\infty} x[p]h_1[q]h_2[n-p-q] \longrightarrow (2)$$

Making the substitutions: p = k ; q = (l - k) and comparing the two equations makes our proof complete.

Note: As k and l go from $-\infty$ to $+\infty$ independently of each other, so do p and q, however p depends on k, and q depends on l and k.

b) Continuous time :

$$\{x(t)*h_1(t)\}*h_2(t)=x(t)*\{h_1(t)*h_2(t)\}$$

$$\{x(t)*h_1(t)\}*h_2(t)=\int_{-\infty}^{\infty}(x*h_1)(\lambda_1)h_2(t-\lambda_1)d\lambda_1$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(\lambda_2)*h_1(\lambda_1-\lambda_2)h_2(t-\lambda_1)d\lambda_1 d\lambda_2 \quad \rightarrow (1)$$

$$x(t)*\{h_1(t)*h_2(t)\}=\int_{-\infty}^{\infty}x(\lambda_3)(h_1*h_2)(t-\lambda_3)d\lambda_3$$

$$=\int_{-\infty}^{\infty}x(\lambda_3)\int_{-\infty}^{\infty}h_1(\lambda_4)h_2(t-\lambda_3-\lambda_4)d\lambda_3 d\lambda_4$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x(\lambda_3)h_1(\lambda_4)h_2(t-\lambda_3-\lambda_4)d\lambda_3 d\lambda_4 \quad \rightarrow (2)$$

Lets substitute

$$\lambda_3 = \lambda_2$$

$$\lambda_4 = \lambda_1 - \lambda_2$$

The Jacobian for the above transformation is

$$J = \frac{\partial(\lambda_3, \lambda_4)}{\partial(\lambda_1, \lambda_2)} = 1$$

Doing some further algebra helps us see equation **(2)** transforming into equation **(1)** ,i.e. essentially they are the same. The limits are also the same. Thus the proof is complete.

Implications

This property (Associativity) makes the representation y[n] = x[n]*h[n] *g[n] unambiguous.

From this property, we can conclude that the effective impulse response of acascaded LTI system is given by the convolution of their individual impulse responses.

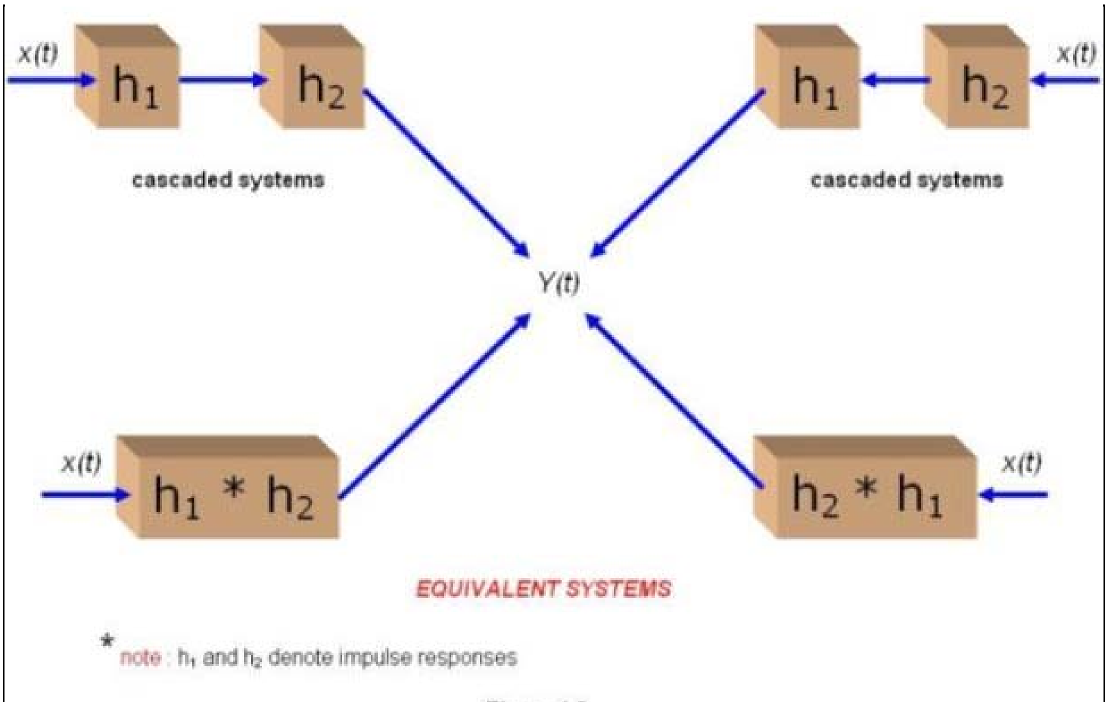


Figure 1.3

Consequently the unit impulse response of a cascaded LTI system is independent of the order in which the individual LTI systems are connected.

Note :All the above three properties are certainly obeyed by LTI systemsbuthold for non-LTI systems in, as seen from the following example:

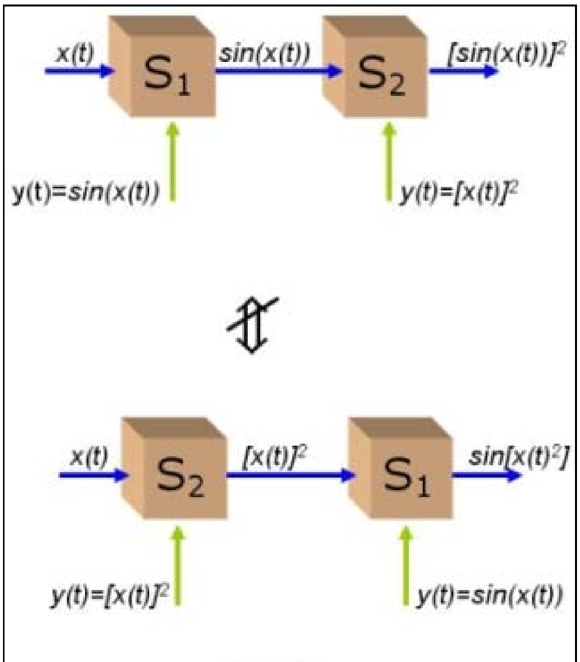


Figure 1.4

4) LTI systems and Memory

Recall that a system is memoryless if its output depends on the current input only. From the expression :

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k]$$

It is easily seen that **y[n]** depends **only** on **x[n]**if and only if $\mathbf{h[n] = 0 \text{ for } n \neq 0}$

i.e. $\mathbf{h[n] = k\, \delta[n]}$ (k is a constant)

Hence

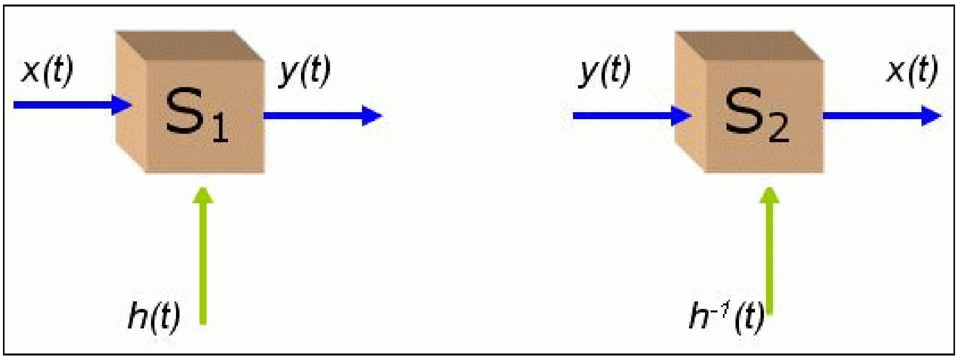
$$
\begin{aligned}
y[n] &= x[n] * h[n] \\
&= x[n] * k\delta[n] \\
&= k(x * \delta)[n] \\
&= kx[n]
\end{aligned}
$$

5) Invertibility :

A system is said to be invertible if there exists an inverse system which when connected in series with the original system produces an output identical to the input.

We know that

$$
\begin{aligned}
&(x*\delta)[n] = x[n] \\
&(x*h*h^{-1})[n] = x[n] \\
&\quad (h*h^{-1})[n] = \delta[n]
\end{aligned}
$$



6) Causality :

a) Discrete time :

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]\, h[n-k] = \sum_{k=-\infty}^{+\infty} h[k]\, x[n-k]$$ { By Commutative Property }

In order for a discrete LTI system to be causal, **y[n]** must not depend on x[k] for k > n. For this to be true h[n-k]'s corresponding to the x[k]'s for k > n must be zero. This then requires the impulse response of a causal discrete time LTI system satisfy the following conditions :

$$h[n] = 0 \qquad \forall n < 0$$

Essentially the system output depends only on the past and the present values of the input.

Proof : (By contradiction)

Let in particular h[k] is not equal to 0, for some k<0

$$y[0] = \sum_{k=0}^{\infty} h[k]\, x[-k] + \sum_{k<0} h[k]\, x[n-k] = \sum_{k<0} h[k]\, x[n-k]$$ {Refer the eqn. above}

So we need to prove that for all x[n] = 0, n < 0, y[0] = 0

$$
\begin{aligned}
h[k] &= 2 - j \;, \quad k = -3 \\
&= 1 + j \;, \quad k = -2
\end{aligned}
$$

Now we take a signal defined as

$$
\begin{aligned}
x[n-2] &= 1 - j \;, \quad n = 2 \\
x[n-3] &= 2 + j \;, \quad n = 3
\end{aligned}
$$

This signal is zero elsewhere. Therefore we get the following result :

$$y[0] = \sum_{k<0} h[k]\, x[n-k]$$

$$= \sum_{k=-2,-3} (1-j)(1+j) \;+\; (2+j)(2-j)$$

$$= 2+5 \;=\; 7 \;\neq\; 0$$

We have come to the result that y[0] $\neq$ 0, for the above assumption. $\therefore$ our assumption stands void. So we conclude that y[n] cannot be independent of x[k] unless h[k] = 0 for k < 0

Note : Here we ensured a non-zero summation by choosing x[n-k]'s as conjugate of h[k]'s.

b) Continuous time :

$$y(t) = \int_{-\infty}^{+\infty} x(v)\, h(t-v)\, dv$$

In order for a continuous LTI system to be causal, y(t) must not depend on x(v) for v > t . For this to be true h(t-v)'s corresponding to the x(v)'s for v > t must be zero.

This then requires the impulse response of a causal continuous time LTI system satisfy the following conditions :

$$h(t) = 0 \qquad \forall\ t < 0$$

As stated before in the discrete time analysis,the system output depends only on the past and the present values of the input.

Proof : (By contradiction)

Suppose, there exists α > 0, such that h(-α) $\neq$ 0.

Now consider $\boxed{x(t) = \delta(t-\alpha)}$

Since,

$$y(t) = \int_{-\infty}^{+\infty} x(v)\, h(t-v)\, dv$$

$$y(0) = \int_{-\infty}^{+\infty} x(v)\, h(-v)\, dv$$

$$\therefore\ y(0) = h(-\alpha) \;\neq\; 0$$

$\Longrightarrow$ System is not causal, a contradiction. Hence,

$$h(t) = 0 \quad \forall\ t < 0$$

7) Stability :

A system is said to be stable if its impulse response satisfies the following criterion :

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

Theorem:

Stability $\Longleftrightarrow$ $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$, in the Discrete domain, OR

Stability $\Leftrightarrow \int_{-\infty}^{\infty} |h(t)|dt < \infty$, in the Continuous domain.

Proof of sufficiency:

Suppose $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$,

We have $y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$

If x[n] is bounded i.e. $0 \leq |x[n]| \leq M_x < \infty \;\; \forall\, n$, then:

$$|y[n]| = \left| \sum_k h[k].x[n-k] \right|$$
$$\leq \sum_k | h[k]x[n-k] |$$
$$\leq \sum_k M_x | h[k] | = M_x \left(\sum_k | h[k] | \right)$$

But as $\sum | h[k] | < \infty \Rightarrow |y[n]| < \infty$

Proof of Necessity:

Take any n.

$$x[n-k] = \frac{\overline{h[k]}}{|h[k]|} \quad \text{if } |h[k]| \neq 0;$$
$$= 0 \quad \text{if } |h[k]| = 0;$$

If | h[k] | = 0, then x[n-k] is bounded with bound 0 $\Rightarrow \sum |h[k]| < \infty$

Then, $y[n] = \sum_k | h[k] . \frac{\overline{h[k]}}{h[k]|} \quad h[k] \neq 0$

Hence $| y[n] | = \sum | h[k] |$. But since the system is stable $\Rightarrow | y[n] | < \infty$, which in turn implies that $\sum |h[k]| < \infty$

Hence if y[n] is bounded then the condition $\sum |h[k]| < \infty$ must hold.

Hence Proved

A similar proof follows in continuous time when you replace $\sum |h[k]|$ by integral $\int_{-\infty}^{\infty} | h(t) |dt$.

Conclusion:

In this lecture you have learnt:

- **Convolution** obeys **commutative**, **distributive** (over addition) and **associative** properties in both continuous and discrete domains.
- Commutativity implies the system with input signal x(t) and impulse response h(t) and the other with input signal *h(t)* and impulse response x(t) both give the same output y(t).
- Distributivity implies a parallel combination of LTI systems can be replaced by an equivalent LTI system which is described by the sum of the individual impulse responses in the parallel combination.
- Associativity implies the unit impulse response of a cascaded LTI system is independent of the order in which the individual LTI systems are connected.
- A system is **memoryless** if and only if **h[n] = 0 for all non-zero n** .
- LTI system is **invertible** if the the convolution of the **impulse response and its inverse** results in **unit impulse**
- For a **causal** discrete time LTI system, **h[n] = 0 for all n<0**. (similarly for continuous time)
- For a **stable** system ,the **impulse response** must be **absolutely integrable**.

Congratulations, you have finished Lecture 10.