Signals & Systems (CNET - 221)
Chapter-4

Fourier Series

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Chapter Objective

Following are the objectives of Chapter-III

- Continuous and Discrete LTI Systems
- Representation of signal in terms of impulses
- Unit Impulse Signal response & Convolution
- LTI System Properties

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Examples : 3.2, 3.3, 3.4, 3.5

Course Description-Chapter-4

Fourier Series

4.1 Introduction Fourier Series Representation Of Continuous- Time Signals

4.2 Fourier Series Representation of Continuous -Time Periodic Signals

4.2.1 Linear Combination of Harmonically related complex Exponentials

4.2.2 Determination of the Fourier Series Representation of a Continuous-Time Periodic Signal

4.3 Convergence of the Fourier Series

4.4 Properties of Continuous-Time Fourier Series

Linearity, Time Shifting , Time Reversal , Time Scaling , Multiplication , Conjugation and Conjugate Symmetry, Parseval's Relation

4.5 Fourier Series Representation of Discrete-Time Periodic Signals

4.5.1 Linear Combination of harmonically related complex exponentials

4.5.2 Determination of the Fourier Series Representation of a Periodic Signal

Fourier Series

Fourier series is just a means to represent a periodic signal as an infinite sum of sine wave components.

$$\underbrace{\underbrace{f(t) = \frac{a_0}{2}}_{\text{DC Part}} + \underbrace{\sum_{n=1}^{\infty} a_n \cos \frac{2\pi nt}{T}}_{\text{Even Part}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin \frac{2\pi nt}{T}}_{\text{Odd Part}}}_{T \text{ is a period of all the above signals}}$$

Let $\omega_0 = 2\pi/T$.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

Fourier Series-Decomposition

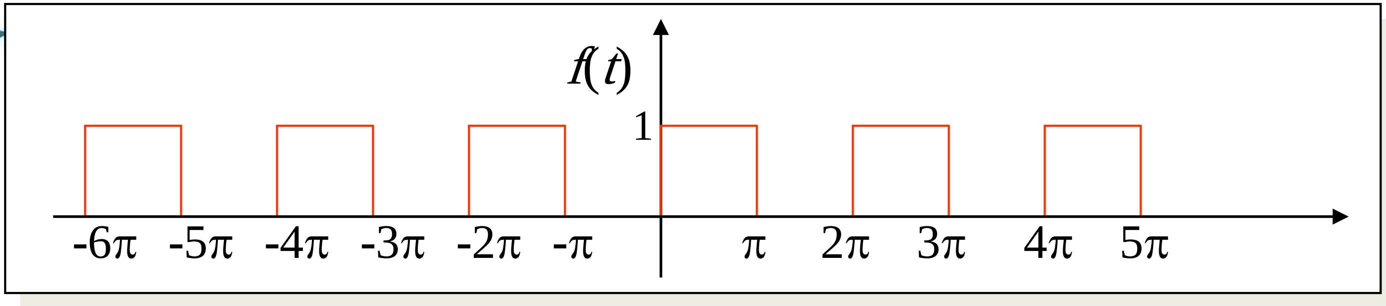
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

$$a_0 = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)dt$$

$$a_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos n\omega_0 t dt \qquad n = 1,2,\cdots$$

$$b_n = \frac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin n\omega_0 t dt \qquad n = 1,2,\cdots$$

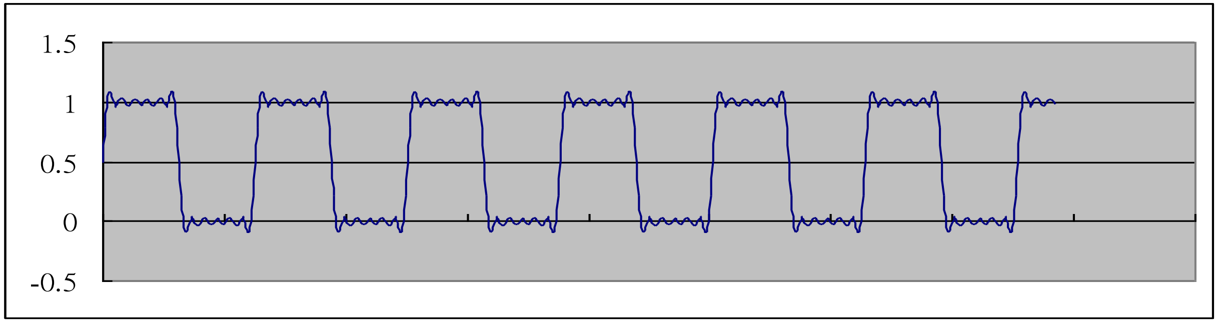
Example (Square Wave)



$$a_0 = \frac{2}{2\pi}\int_0^{\pi} 1dt = 1 \qquad a_n = \frac{2}{2\pi}\int_0^{\pi} \cos nt dt = \frac{1}{n\pi}\sin nt\Big|_0^{\pi} = 0 \qquad n = 1,2,\cdots$$

$$b_n = \frac{2}{2\pi}\int_0^{\pi} \sin nt dt = -\frac{1}{n\pi}\cos nt\Big|_0^{\pi} = -\frac{1}{n\pi}(\cos n\pi - 1) = \begin{cases} 2/n\pi & n = 1,3,5,\cdots \\ 0 & n = 2,4,6,\cdots \end{cases}$$

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t + \cdots\right)$$



Harmonics

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi nt}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi nt}{T}$$

$$f(t) = \underbrace{\frac{a_0}{2}}_{\text{DC Part}} + \underbrace{\sum_{n=1}^{\infty} a_n \cos(n\omega_0 t)}_{\text{Even Part}} + \underbrace{\sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)}_{\text{Odd Part}}$$

DC Part — Even Part — Odd Part

T is a period of all the above signals

Harmonics…….Continued

Define $\omega_0 = 2\pi f_0 = \dfrac{2\pi}{T}$, called the *fundamental angular frequency*.

Define $\omega_n = n\omega_0$, called the *n*-th *harmonic* of the periodic function.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$\Rightarrow \quad f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t$$

Harmonics…….Continued

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \omega_n t + \sum_{n=1}^{\infty} b_n \sin \omega_n t$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \omega_n t + b_n \sin \omega_n t)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left(\frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos \omega_n t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin \omega_n t \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} (\cos \theta_n \cos \omega_n t + \sin \theta_n \sin \omega_n t)$$

$$= C_0 + \sum_{n=1}^{\infty} C_n \cos(\omega_n t - \theta_n)$$

Complex Exponentials

$$e^{jn\omega_0 t} = \cos n\omega_0 t + j \sin n\omega_0 t$$

$$e^{-jn\omega_0 t} = \cos n\omega_0 t - j \sin n\omega_0 t$$

$$\cos n\omega_0 t = \frac{1}{2}\left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right)$$

$$\sin n\omega_0 t = \frac{1}{2j}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right) = -\frac{j}{2}\left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

Complex Form of the Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$= \frac{a_0}{2} + \frac{1}{2}\sum_{n=1}^{\infty} a_n \left(e^{jn\omega_0 t} + e^{-jn\omega_0 t}\right) - \frac{j}{2}\sum_{n=1}^{\infty} b_n \left(e^{jn\omega_0 t} - e^{-jn\omega_0 t}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\frac{1}{2}(a_n - jb_n)e^{jn\omega_0 t} + \frac{1}{2}(a_n + jb_n)e^{-jn\omega_0 t}\right]$$

$$= c_0 + \sum_{n=1}^{\infty} \left[c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t}\right]$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Complex Form of the Fourier Series

$$f(t) = c_0 + \sum_{n=1}^{\infty} \left[c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t} \right]$$

$$= c_0 + \sum_{n=1}^{\infty} c_n e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} c_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Complex Form of the Fourier Series

$$c_0 = \frac{a_0}{2} = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$= \frac{1}{T}\left[\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t dt - j\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t dt\right]$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)(\cos n\omega_0 t - j\sin n\omega_0 t)dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}dt$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n) = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{jn\omega_0 t}dt$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{1}{2}(a_n - jb_n)$$

$$c_{-n} = \frac{1}{2}(a_n + jb_n)$$

Complex Form of the Fourier Seri

$$f(t)=\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

$$c_n=\frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt$$

$$c_0=\frac{a_0}{2}$$

$$c_n=\frac{1}{2}(a_n-jb_n)$$

$$c_{-n}=\frac{1}{2}(a_n+jb_n)$$

If $f(t)$ is real,

$$\Rightarrow\quad c_{-n}=c_n^*$$

$$c_n=|c_n|e^{j\phi_n},\quad c_{-n}=c_n^*=|c_n|e^{-j\phi_n}$$

$$|c_n|=|c_{-n}|=\frac{1}{2}\sqrt{a_n^2+b_n^2}$$

$$n=\pm1,\pm2,\pm3,\cdots$$

$$\phi_n=\tan^{-1}\left(-\frac{b_n}{a_n}\right)$$

$$c_0=\frac{1}{2}a_0$$

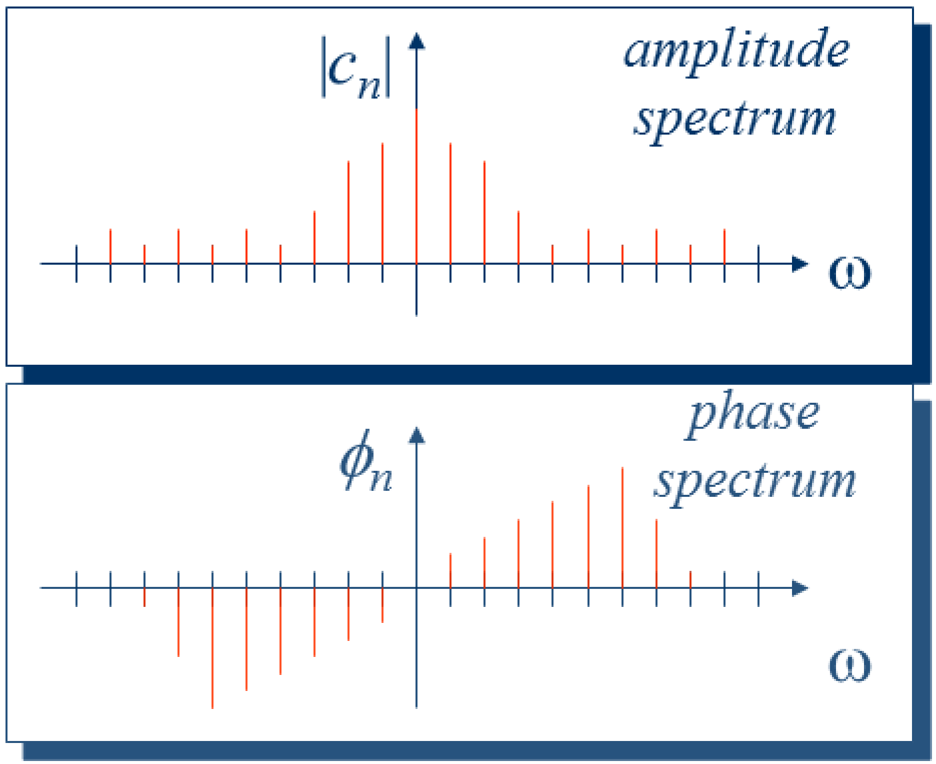
Complex Frequency Spectra

$$c_n = |c_n| e^{j\phi_n}, \quad c_{-n} = c_n^* = |c_n| e^{-j\phi_n}$$

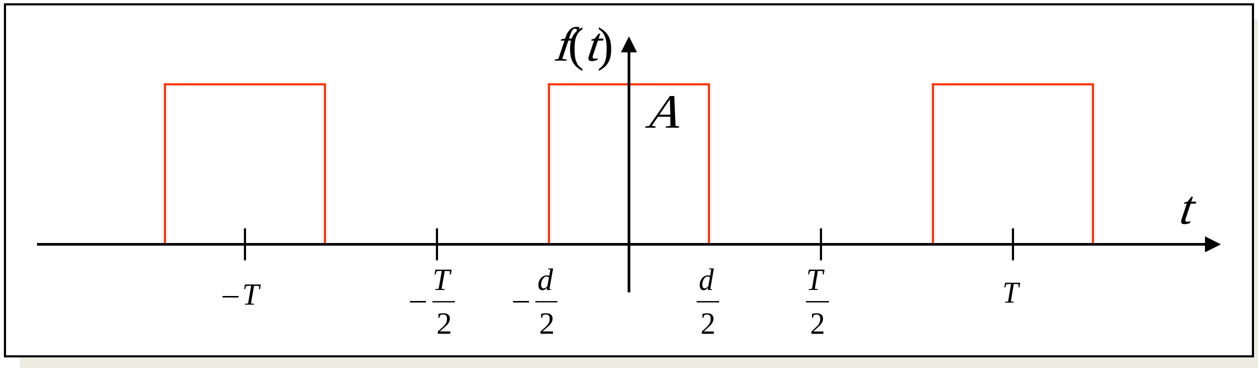
$$|c_n| = |c_{-n}| = \frac{1}{2}\sqrt{a_n^2 + b_n^2}$$

$$\phi_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right) \qquad n = \pm 1, \pm 2, \pm 3, \cdots$$

$$c_0 = \frac{1}{2} a_0$$



Example



$$c_n = \frac{A}{T}\int_{-d/2}^{d/2} e^{-jn\omega_0 t}\,dt$$

$$= \frac{A}{T}\frac{1}{-jn\omega_0}e^{-jn\omega_0 t}\Big|_{-d/2}^{d/2}$$

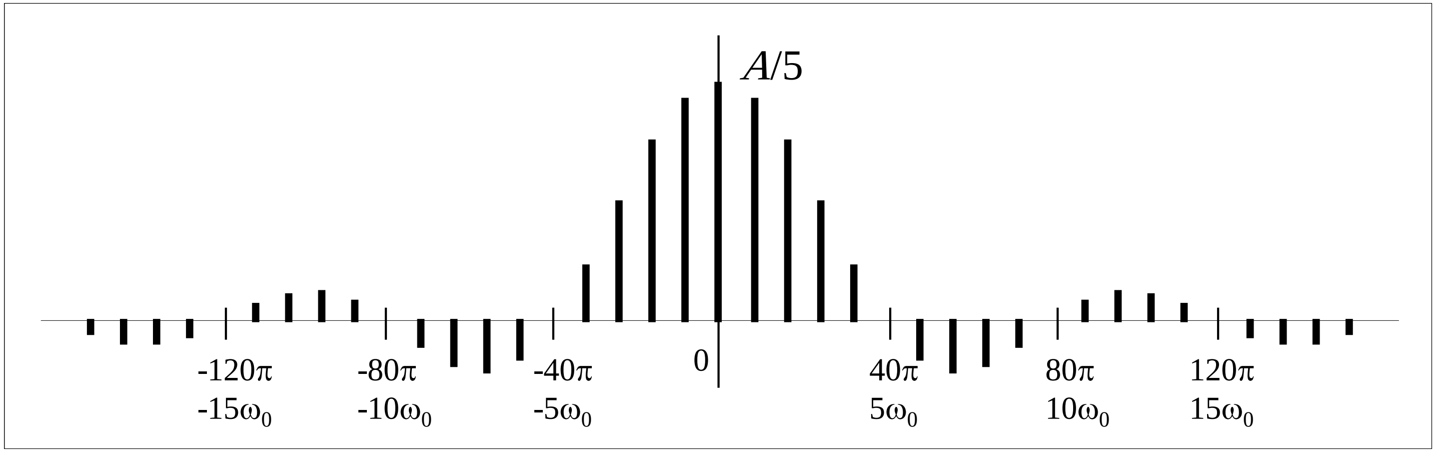
$$= \frac{A}{T}\left(\frac{1}{-jn\omega_0}e^{-jn\omega_0 d/2} - \frac{1}{-jn\omega_0}e^{jn\omega_0 d/2}\right)$$

$$= \frac{A}{T}\frac{1}{-jn\omega_0}(-2j\sin n\omega_0 d/2)$$

$$= \frac{A}{T}\frac{1}{\frac{1}{2}n\omega_0}\sin n\omega_0 d/2$$

$$= \frac{Ad}{T}\frac{\sin\left(\frac{n\pi d}{T}\right)}{\left(\frac{n\pi d}{T}\right)}$$

Example

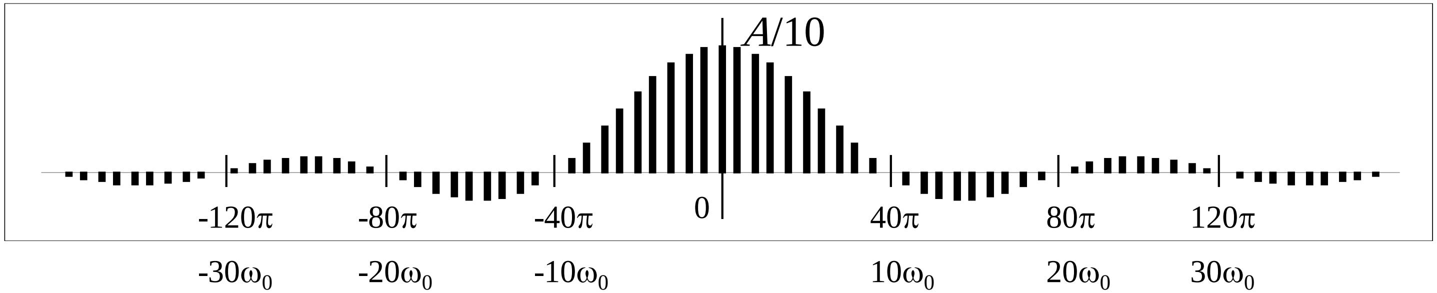


$$c_n = \frac{Ad}{T}\frac{\sin\left(\frac{n\pi d}{T}\right)}{\left(\frac{n\pi d}{T}\right)}$$

$$d = \frac{1}{20},\quad T = \frac{1}{4},\quad \frac{d}{\mathrm{T}} = \frac{1}{5}$$

$$\omega_0 = \frac{2\pi}{T} = 8\pi$$

Example



$$c_n = \frac{Ad}{T}\frac{\sin\left(\frac{n\pi d}{T}\right)}{\left(\frac{n\pi d}{T}\right)}$$

$$d = \frac{1}{20},\quad T = \frac{1}{2},\quad \frac{d}{\mathrm{T}} = \frac{1}{5}$$

$$\omega_0 = \frac{2\pi}{T} = 4\pi$$

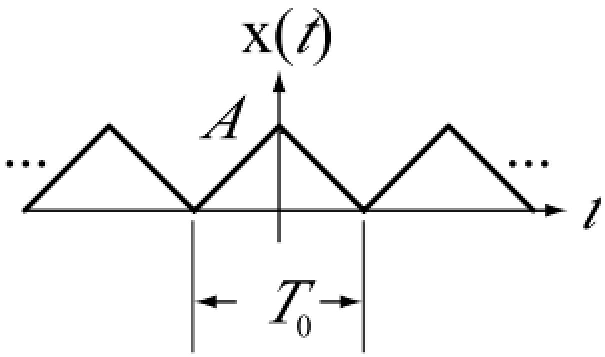
Convergence of the CTFS

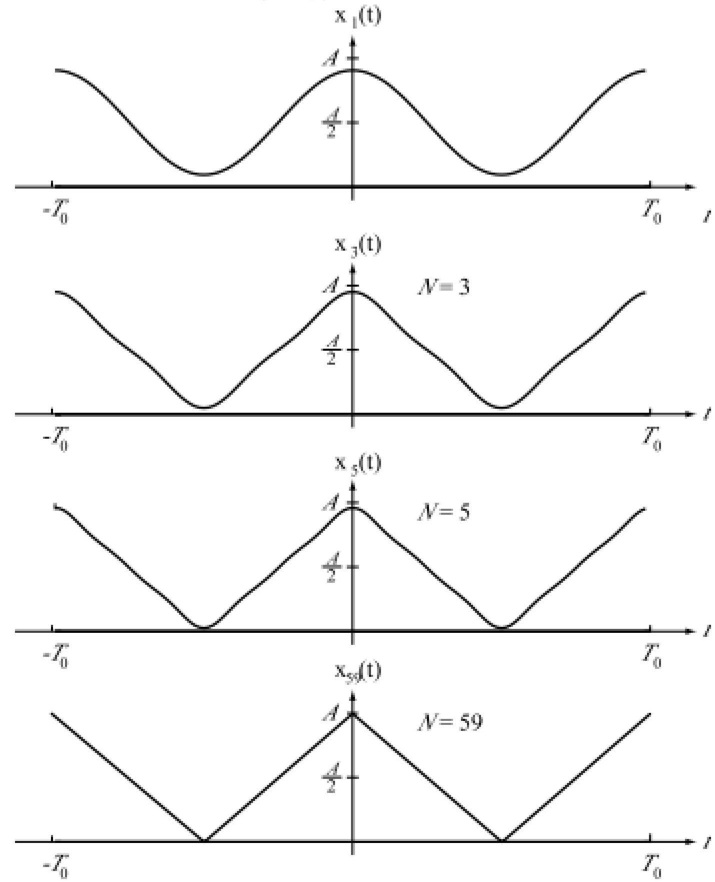
Partial CTFS Sums

$$x_N(t) = \sum_{k=-N}^{N} c_x[k] e^{j2\pi kt/T_0}$$

For continuous signals, **convergence** is exact at every point.

A Continuous Signal



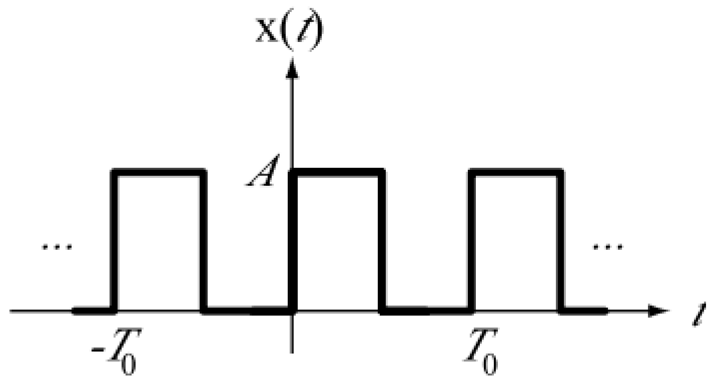


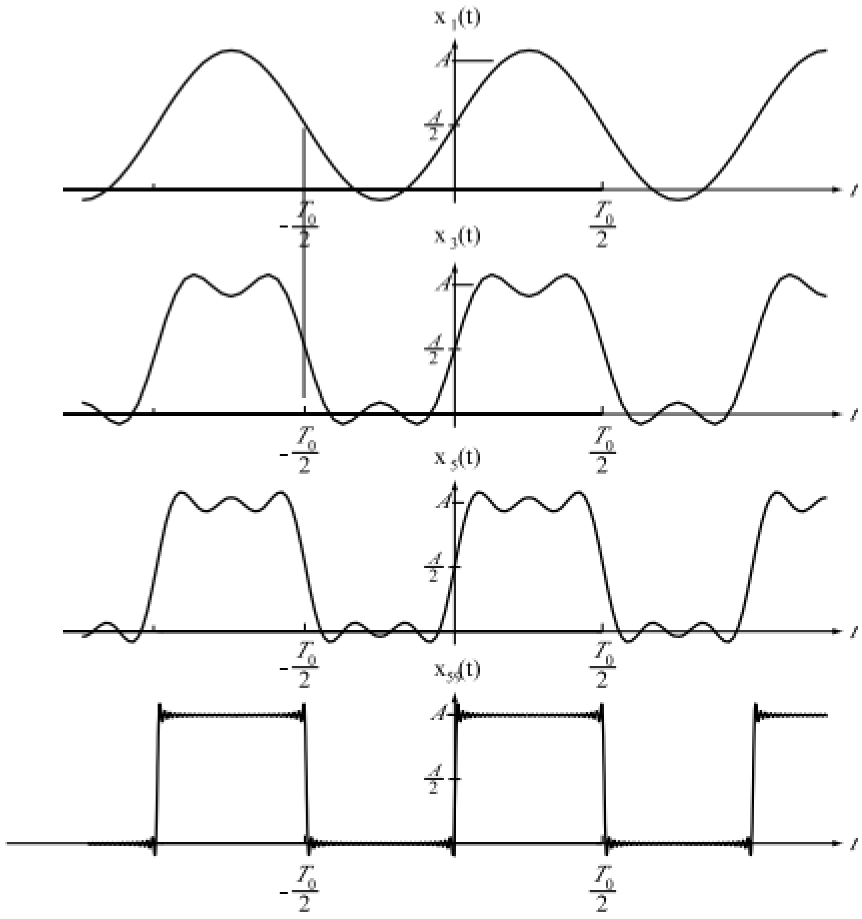
Convergence of the CTFS

Partial CTFS Sums

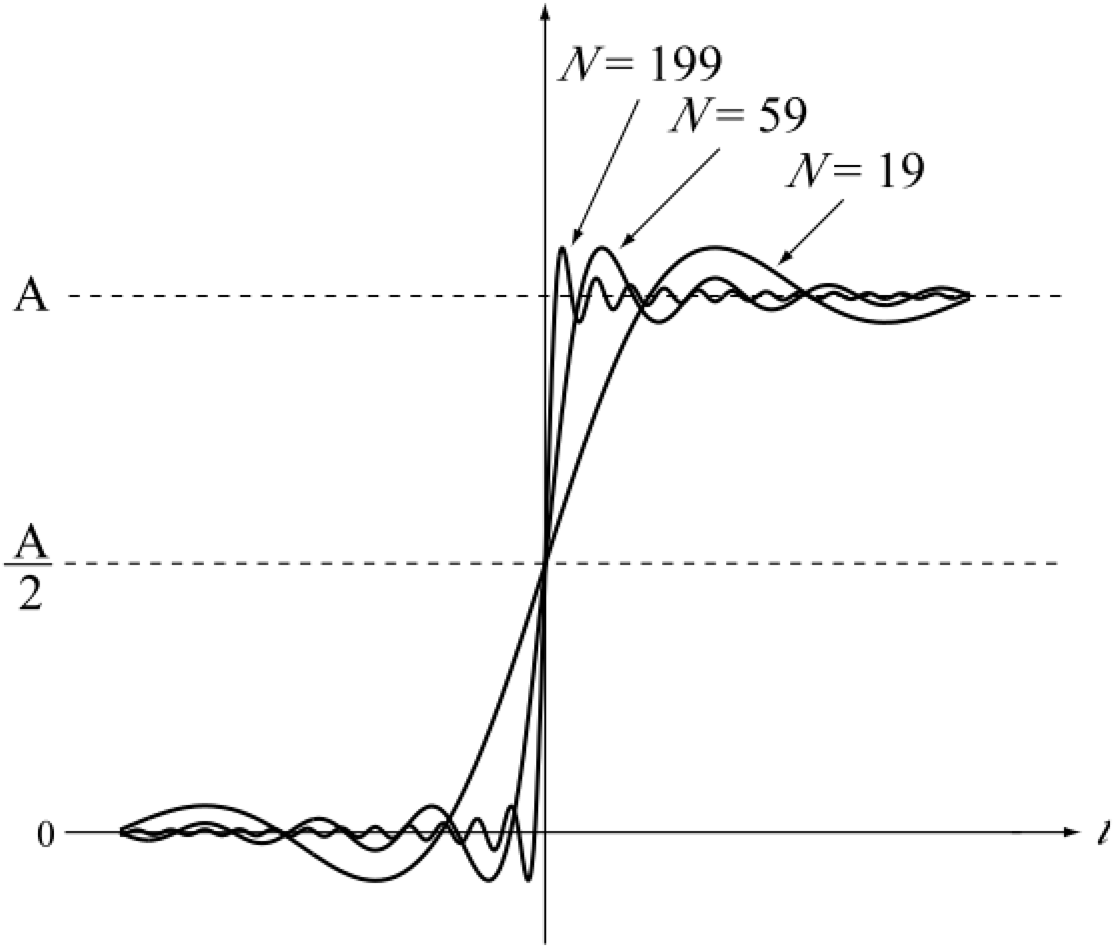
For discontinuous signals, convergence is exact at every point of continuity.

Discontinuous Signal





Convergence of the CTFS



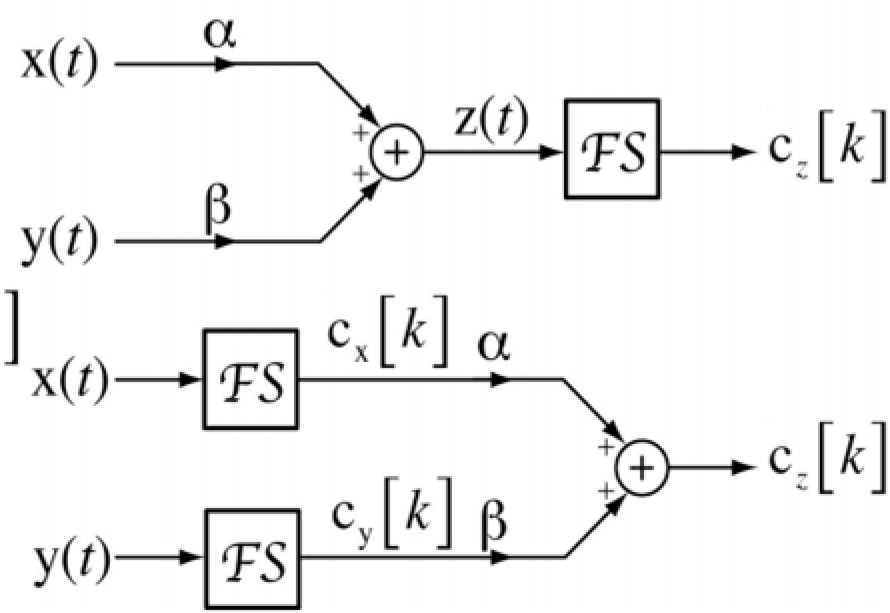
At points of discontinuity the Fourier series representation converges to the mid-point of the discontinuity.

CTFS Properties

Let a signal $x(t)$ have a fundamental period T_{0x} and let a signal $y(t)$ have a fundamental period T_{0y}. Let the CTFS harmonic functions, each using a common period T as the representation time, be $c_x[k]$ and $c_y[k]$. Then the following properties apply.

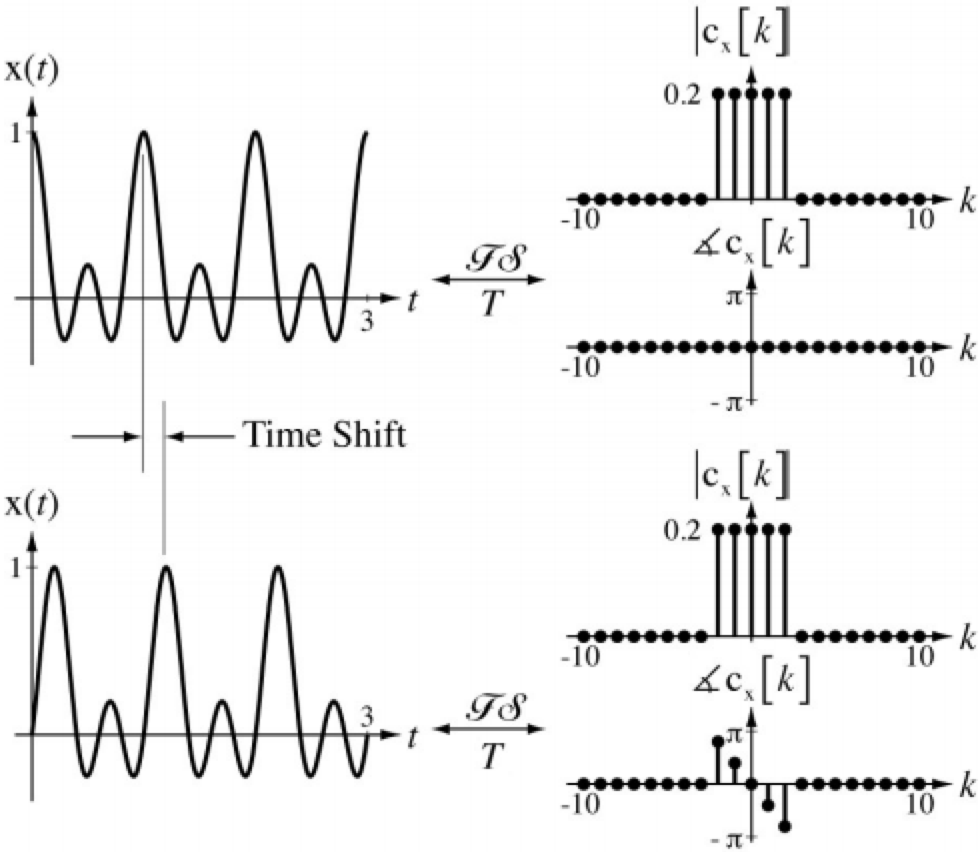
Linearity

$$\alpha x(t)+\beta y(t) \xleftrightarrow[T]{\mathcal{FS}} \alpha c_x[k]+\beta c_y[k]$$



CTFS Properties……..Continued

Time Shifting $\mathrm{x}(t-t_0) \xleftrightarrow[T]{\mathcal{FS}} e^{-j2\pi k t_0/T}\,\mathrm{c_x}[k]$



CTFS Properties........Continued

Frequency Shifting (Harmonic Number Shifting)

$$e^{j2\pi k_0 t/T}\,\mathrm{x}(t) \xleftrightarrow[T]{\mathcal{FS}} \mathrm{c}_{\mathrm{x}}[k-k_0]$$

A shift in frequency (harmonic number) corresponds to multiplication of the time function by a complex exponential.

Time Reversal

$$\mathrm{x}(-t) \xleftrightarrow[T]{\mathcal{FS}} \mathrm{c}_{\mathrm{x}}[-k]$$

CTFS Properties……..Continued

Time Scaling

$$\text{Let } z(t) = x(at),\ a > 0$$

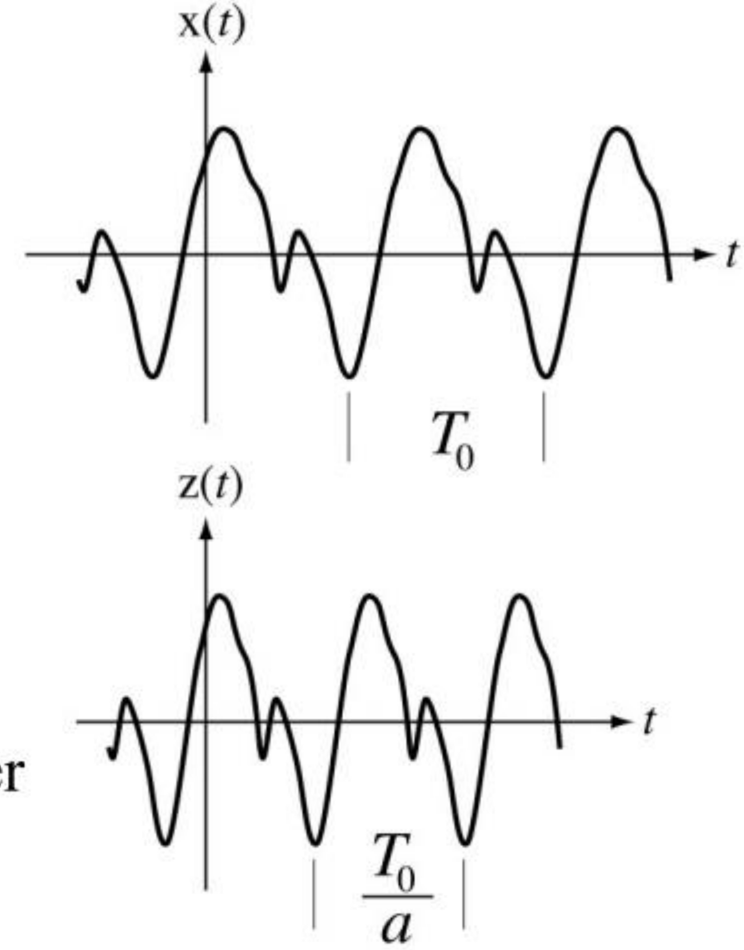
Case 1. $T = T_{0x} / a = T_{0z}$ for $z(t)$

$$c_z[k] = c_x[k]$$

Case 2. $T = T_{0x}$ for $z(t)$

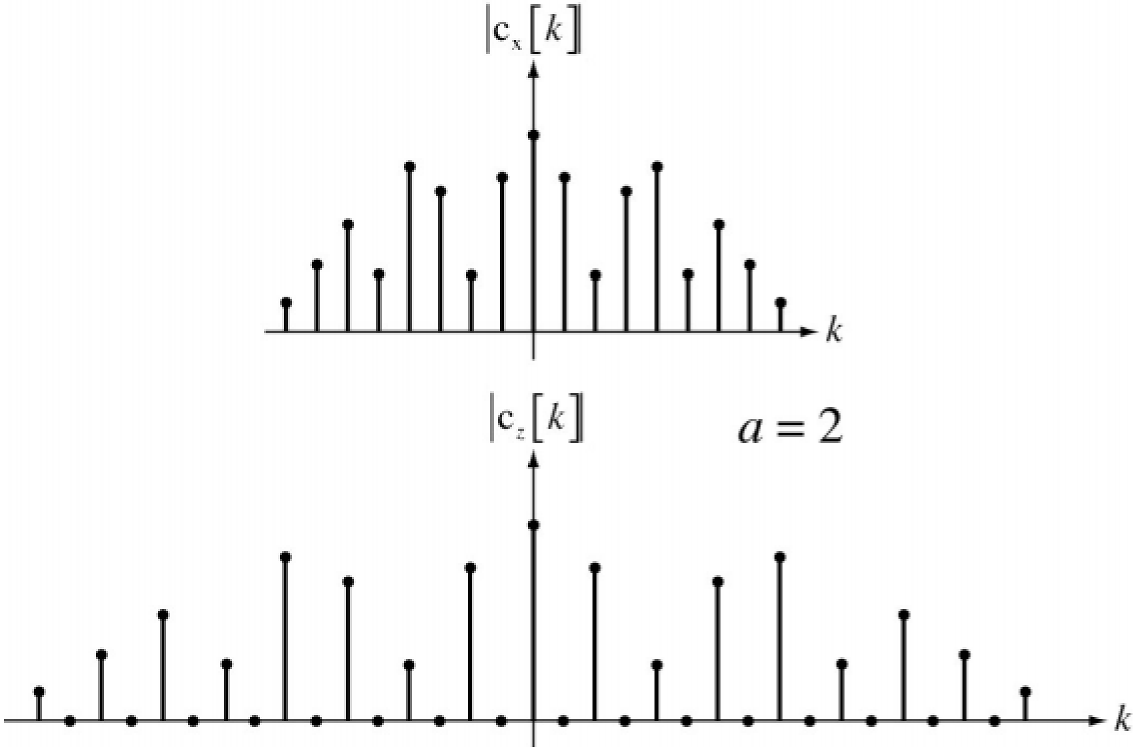
If a is an integer,

$$c_z[k] = \begin{cases} c_x[k/a] & , \ k/a \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$$



CTFS Properties........Continued

Time Scaling (continued)



CTFS Properties........Continued

Change of Representation Time

With $T = T_{0x}$, $\mathrm{x}(t) \xleftrightarrow[T]{\mathcal{FS}} \mathrm{c_x}[k]$

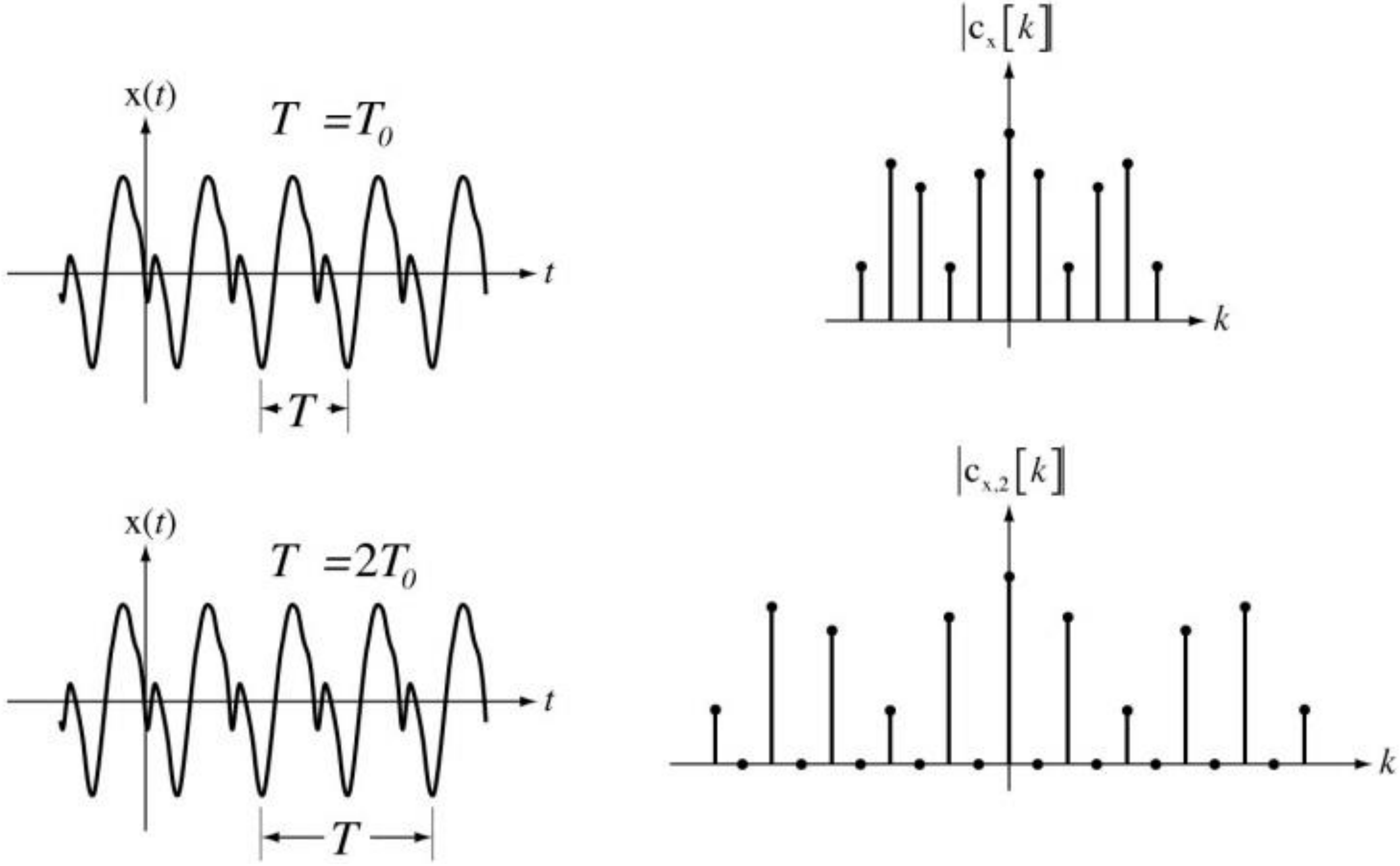
With $T = mT_{0x}$, $\mathrm{x}(t) \xleftrightarrow[T]{\mathcal{FS}} \mathrm{c}_{\mathrm{x},m}[k]$

$$\mathrm{c}_{\mathrm{x},m}[k] = \begin{cases} \mathrm{c_x}[k/m] & , \ k/m \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$$

(m is any positive integer)

CTFS Properties……..Continued

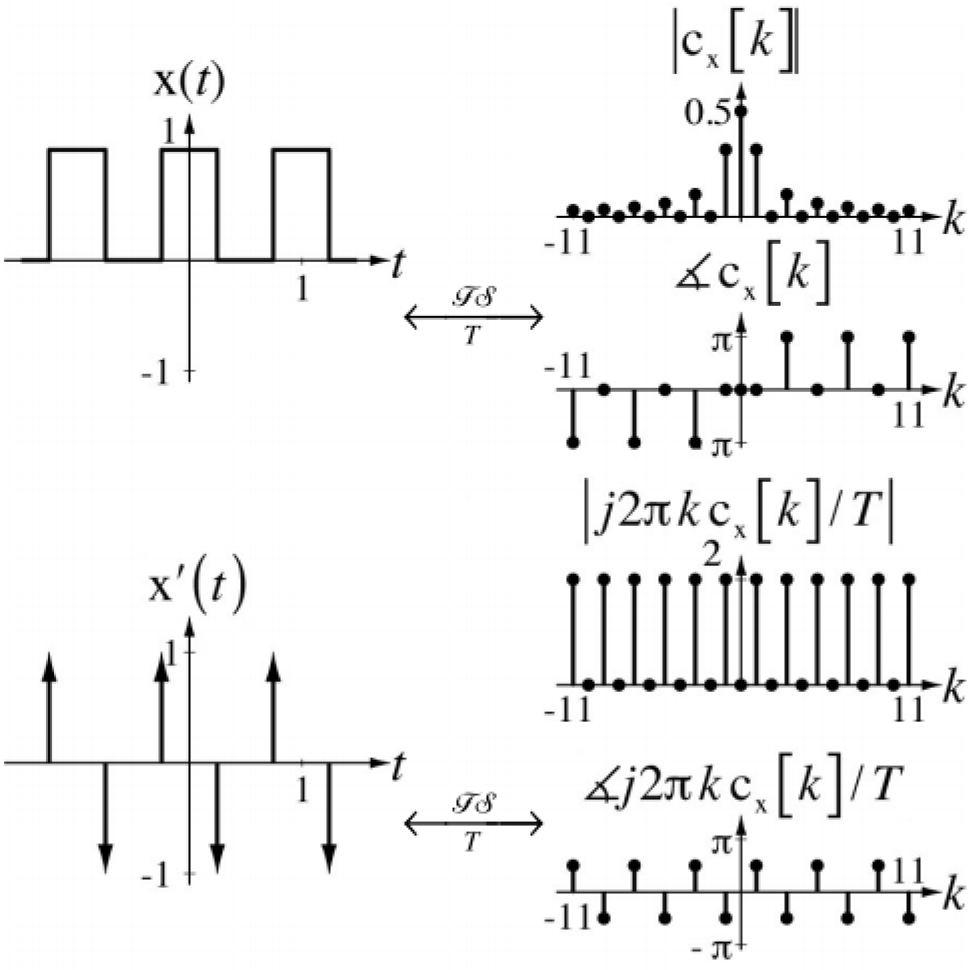
Change of Representation Time



CTFS Properties……..Continued

Time Differentiation

$$\frac{d}{dt}\left(\mathrm{x}(t)\right) \xleftrightarrow[T]{\mathcal{FS}} j2\pi k\,\mathrm{c_x}[k]/T$$



CTFS Properties……..Continued

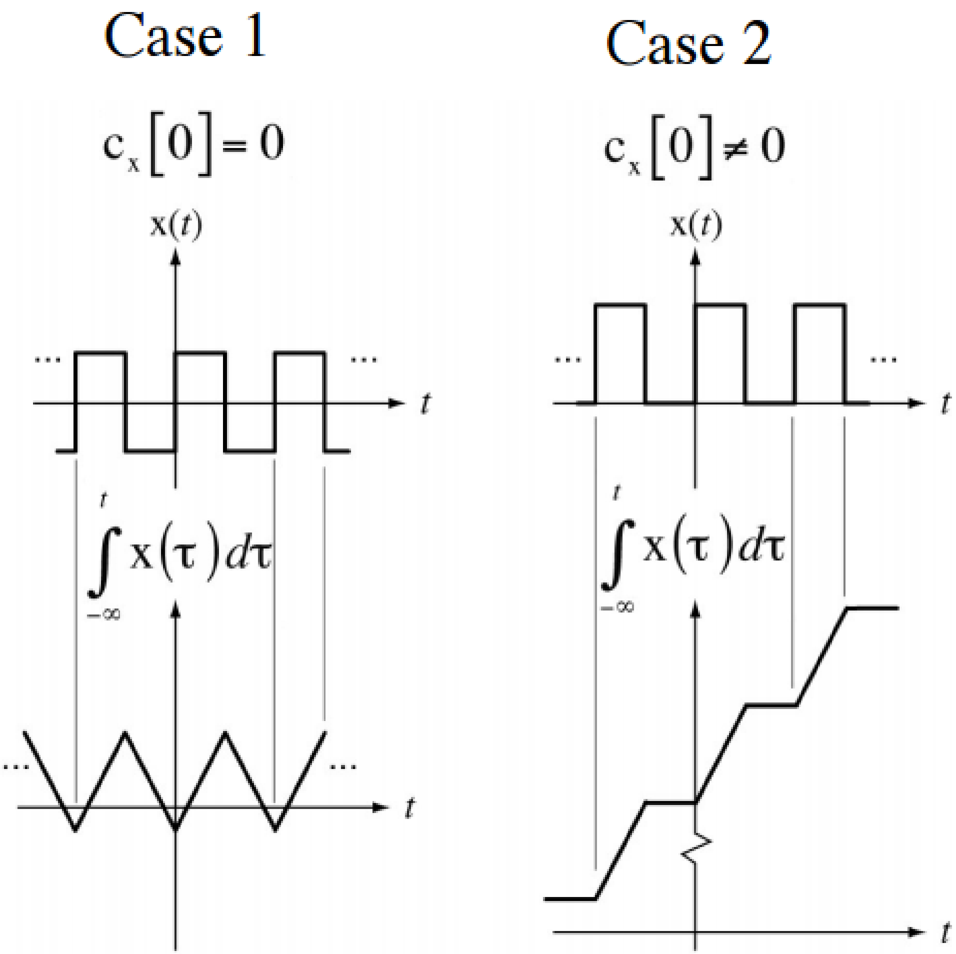
Time Integration

Case 1. $c_x[0]=0$

$$\int_{-\infty}^{t} x(\lambda)\,d\lambda \xleftrightarrow[T]{\mathcal{FS}} \frac{c_x[k]}{j2\pi k/T}\ ,k\neq 0$$

Case 2. $c_x[0]\neq 0$

$\int_{-\infty}^{t} x(\lambda)\,d\lambda$ is not periodic



CTFS Properties........Continued

Multiplication - Convolution Duality

$$x(t)y(t) \xleftrightarrow[T]{\mathcal{FS}} c_x[k] * c_y[k]$$

(The harmonic functions $c_x[k]$ and $c_y[k]$ must be based on the same representation time T.)

$$x(t) \circledast y(t) \xleftrightarrow[T]{\mathcal{FS}} T\, c_x[k] c_y[k]$$

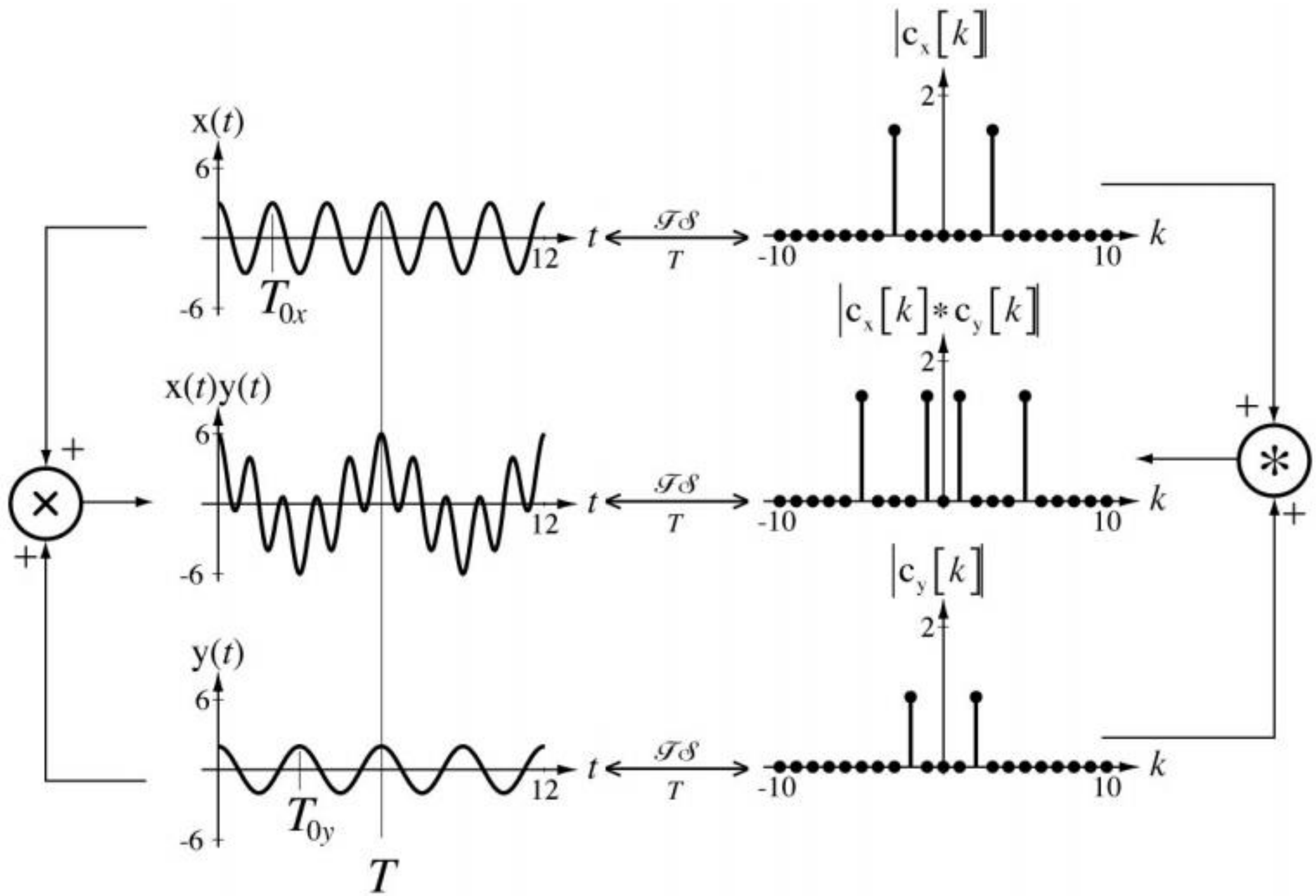
The symbol $\circledast$ indicates **periodic convolution**.

Periodic convolution is defined mathematically by

$$x(t) \circledast y(t) = \int_T x(\tau) y(t-\tau) d\tau$$

$x(t) \circledast y(t) = x_{ap}(t) * y(t)$ where $x_{ap}(t)$ is any single period of $x(t)$

CTFS Properties……..Continued



CTFS Properties……..Continued

Conjugation

$$x^*(t) \xleftrightarrow[T]{\mathcal{FS}} c_x^*[-k]$$

Parseval's Theorem

$$\frac{1}{T}\int_T |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |c_x[k]|^2$$

The **average power** of a periodic signal is the sum of the average powers in its harmonic components.

Some Common CTFS Pairs

$$1 \xleftrightarrow[T]{\mathcal{FS}} \delta[k] \;,\; T \text{ arbitrary}$$

$$\delta_{T_0}(t) \xleftrightarrow[mT_0]{\mathcal{FS}} \begin{cases} (1/T_0) & , \; k/m \text{ an integer} \\ 0 & , \text{otherwise} \end{cases}$$

$$e^{j2\pi qt/T_0} \xleftrightarrow[mT_0]{\mathcal{FS}} \delta[k-mq]$$

$$\sin(2\pi qt/T_0) \xleftrightarrow[mT_0]{\mathcal{FS}} (j/2)\left(\delta[k+mq]-\delta[k-mq]\right)$$

$$\cos(2\pi qt/T_0) \xleftrightarrow[mT_0]{\mathcal{FS}} (1/2)\left(\delta[k-mq]+\delta[k+mq]\right)$$

$$\text{rect}(t/w) * \delta_{T_0}(t) \xleftrightarrow[mT_0]{\mathcal{FS}} (w/T_0)\,\text{sinc}(wk/mT_0)\,\delta_m[k]$$

$$\text{tri}(t/w) * \delta_{T_0}(t) \xleftrightarrow[mT_0]{\mathcal{FS}} (w/T_0)\,\text{sinc}^2(wk/mT_0)\,\delta_m[k]$$

$$(m \text{ an integer})$$

Parseval's Theorem

- Let $x(t)$ be a periodic signal with period T
- The *average power* P of the signal is defined as

$$P = \frac{1}{T}\int_{-T/2}^{T/2} x^2(t)\,dt$$

- Expressing the signal as $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}, \quad t \in \mathbb{R}$

  it is also

$$P = \sum_{k=-\infty}^{\infty} |c_k|^2$$

Videos

1. https://www.youtube.com/watch?v=7Z3LE5uM-6Y&list=PLbMVogVj5nJQQZbah2uRZIRZ_9kfoqZyx

2. Signals & Systems Tutorial

https://www.youtube.com/watch?v=yLezP5ziz0U&list=PL56ED47DCECCD69B2