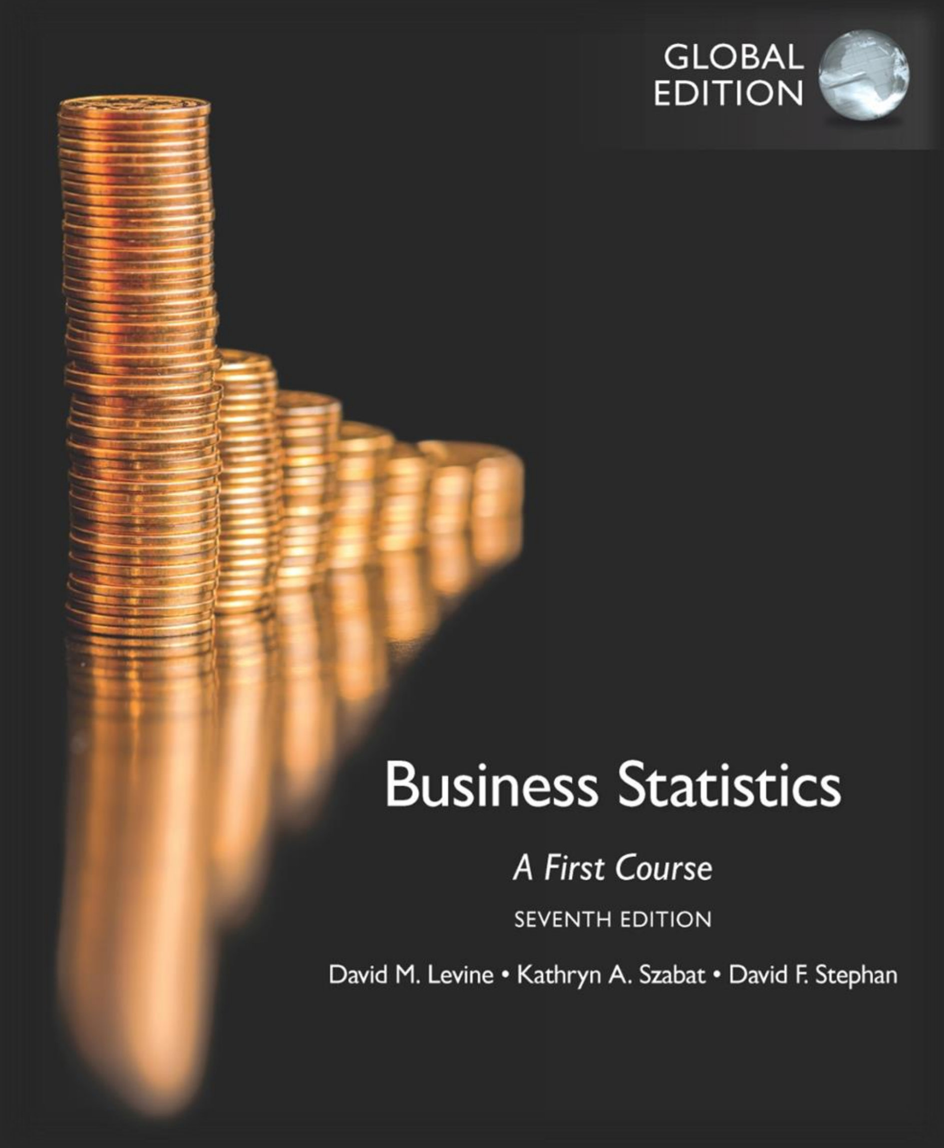


Two-Sample T-Tests

Objectives

In this chapter, you learn:

- How to use hypothesis testing for comparing the difference between
  - The means of two independent populations
  - The means of two related populations
  - The proportions of two independent populations
  - The variances of two independent populations
  - The means of more than two populations

Two-Sample Tests

DCOVA

Two-Sample Tests

- Population Means, Independent Samples
- Population Means, Related Samples
- Population Proportions
- Population Variances

Examples:

- Group 1 vs. Group 2
- Same group before vs. after treatment
- Proportion 1 vs. Proportion 2
- Variance 1 vs. Variance 2

Difference Between Two Means

DCOVA

Population means, independent samples *

- σ_1 and σ_2 unknown, assumed equal
- σ_1 and σ_2 unknown, not assumed equal

Goal: Test hypothesis or form a confidence interval for the difference between two population means, $\mu_1 - \mu_2$

The point estimate for the difference is

$$\bar{X}_1 - \bar{X}_2$$

Difference Between Two Means: Independent Samples

DCOVA

- Different data sources
  - Unrelated
  - Independent
    - Sample selected from one population has no effect on the sample selected from the other population

Population means, independent samples *

- σ_1 and σ_2 unknown, assumed equal → Use S_p to estimate unknown σ. Use a **Pooled-Variance t test.**
- σ_1 and σ_2 unknown, not assumed equal → Use S_1 and S_2 to estimate unknown σ_1 and σ_2. Use a **Separate-variance t test**

Hypothesis Tests for Two Population Means

DCOVA

Two Population Means, Independent Samples

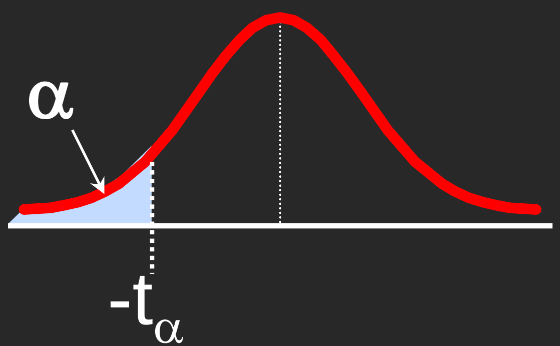
Lower-tail test:	Upper-tail test:	Two-tail test:
$H_0: \mu_1 \geq \mu_2$ $H_1: \mu_1 < \mu_2$	$H_0: \mu_1 \leq \mu_2$ $H_1: \mu_1 > \mu_2$	$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 \neq \mu_2$
i.e.,	i.e.,	i.e.,
$H_0: \mu_1 - \mu_2 \geq 0$ $H_1: \mu_1 - \mu_2 < 0$	$H_0: \mu_1 - \mu_2 \leq 0$ $H_1: \mu_1 - \mu_2 > 0$	$H_0: \mu_1 - \mu_2 = 0$ $H_1: \mu_1 - \mu_2 \neq 0$

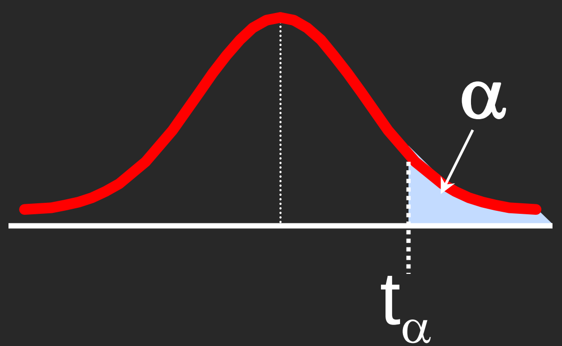
Hypothesis tests for $\mu_1 - \mu_2$

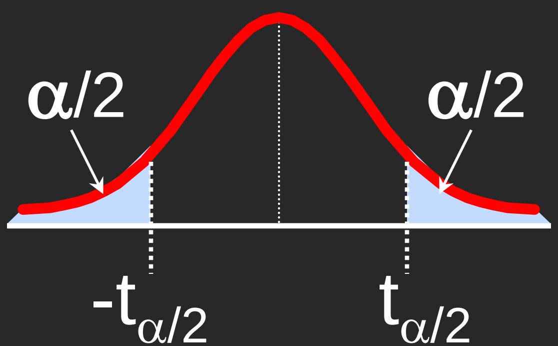
DCOVA

Two Population Means, Independent Samples

Lower-tail test:	Upper-tail test:	Two-tail test:
$H_0: \mu_1 - \mu_2 \geq 0$ $H_1: \mu_1 - \mu_2 < 0$	$H_0: \mu_1 - \mu_2 \leq 0$ $H_1: \mu_1 - \mu_2 > 0$	$H_0: \mu_1 - \mu_2 = 0$ $H_1: \mu_1 - \mu_2 \neq 0$







Reject H_0 if $t_{STAT} < -t_{\alpha}$

Reject H_0 if $t_{STAT} > t_{\alpha}$

Reject H_0 if $t_{STAT} < -t_{\alpha/2}$ or $t_{STAT} > t_{\alpha/2}$

Hypothesis tests for $\mu_1 - \mu_2$ with σ_1 and σ_2 unknown and assumed equal

DCOVA

Population means, independent samples

- σ_1 and σ_2 unknown, assumed equal *
- σ_1 and σ_2 unknown, not assumed equal

Assumptions:

- Samples are randomly and independently drawn
- Populations are normally distributed or both sample sizes are at least 30
- Population variances are unknown but assumed equal

Hypothesis tests for $\mu_1 - \mu_2$ with σ_1 and σ_2 unknown and assumed equal

(continued)

DCOVA

Population means, independent samples

- σ_1 and σ_2 unknown, assumed equal *
- σ_1 and σ_2 unknown, not assumed equal

- The pooled variance is:

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{(n_1 - 1) + (n_2 - 1)}$$

- The test statistic is:

$$t_{STAT} = \frac{\left(\bar{X}_1 - \bar{X}_2\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{S_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

- Where t_{STAT} has d.f. = $(n_1 + n_2 - 2)$

Confidence interval for μ_1 - μ_2 with σ_1 and σ_2 unknown and assumed equal

DCOVA

Population means, independent samples

σ_1 and σ_2 unknown, assumed equal *

σ_1 and σ_2 unknown, not assumed equal

The confidence interval for $\mu_1 - \mu_2$ is:

$$\left(\bar{X}_1 - \bar{X}_2\right) \pm t_{\alpha/2} \sqrt{S_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

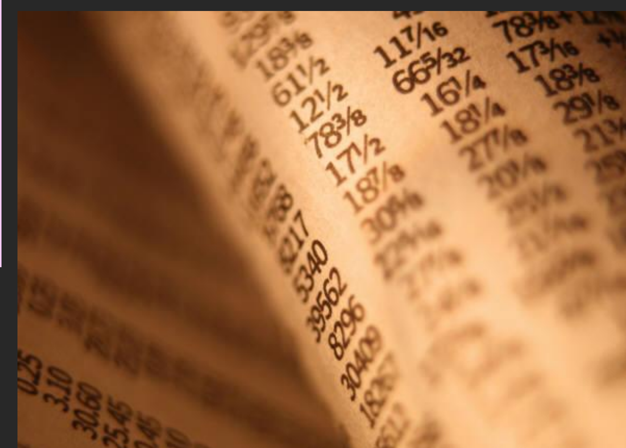
Where $t_{\alpha/2}$ has d.f. = $n_1 + n_2 - 2$

Pooled-Variance t Test Example

DCOVA

You are a financial analyst for a brokerage firm. Is there a difference in dividend yield between stocks listed on the NYSE & NASDAQ? You collect the following data:

	NYSE	NASDAQ
Number	**21**	**25**
Sample mean	**3.27**	**2.53**
Sample std dev	**1.30**	**1.16**



Assuming both populations are approximately normal with equal variances, is there a difference in mean yield (α = 0.05)?

Pooled-Variance t Test Example: Calculating the Test Statistic

(continued)

DCOVA

H0: $\mu_1 - \mu_2 = 0$ i.e. ($\mu_1 = \mu_2$)
H1: $\mu_1 - \mu_2 \neq 0$ i.e. ($\mu_1 \neq \mu_2$)

The test statistic is:

$$t = \frac{\left(\bar{X}_1 - \bar{X}_2\right) - \left(\mu_1 - \mu_2\right)}{\sqrt{S_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \frac{\left(3.27 - 2.53\right) - 0}{\sqrt{1.5021\left(\frac{1}{21} + \frac{1}{25}\right)}} = 2.040$$

$$S_p^2 = \frac{(n_1 - 1)S_1^{\,2} + (n_2 - 1)S_2^{\,2}}{(n_1 - 1) + (n_2 - 1)} = \frac{(21 - 1)1.30^2 + (25 - 1)1.16^2}{(21 - 1) + (25 - 1)} = 1.5021$$

Pooled-Variance t Test Example: Hypothesis Test Solution

DCOVA

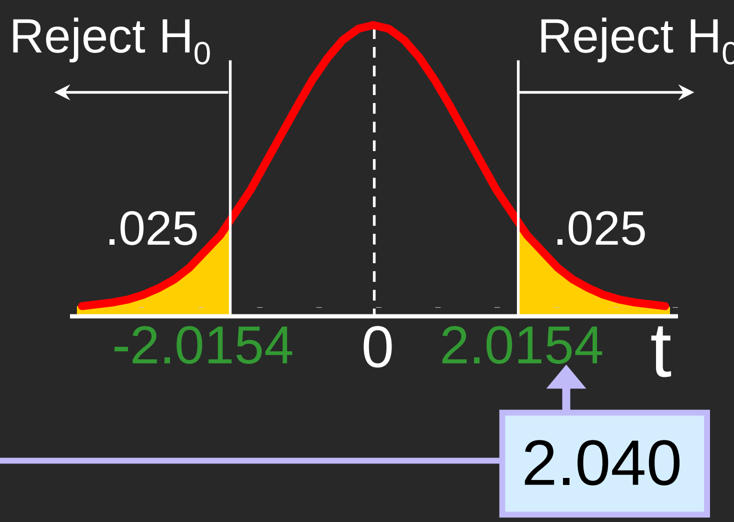
$H_0: \mu_1 - \mu_2 = 0$ i.e. $(\mu_1 = \mu_2)$

$H_1: \mu_1 - \mu_2 \neq 0$ i.e. $(\mu_1 \neq \mu_2)$

$\alpha = 0.05$

df = 21 + 25 - 2 = 44

Critical Values: t = ± 2.0154



Test Statistic:

$$t = \frac{3.27 - 2.53}{\sqrt{1.5021\left(\frac{1}{21} + \frac{1}{25}\right)}} = 2.040$$

Decision:

Reject H_0 at $\alpha = 0.05$

Conclusion:

There is evidence of a difference in means.

Pooled-Variance t Test Example: Confidence Interval for $\mu_1 - \mu_2$

DCOVA

Since we rejected H_0 can we be 95% confident that $\mu_{NYSE} > \mu_{NASDAQ}$?

95% Confidence Interval for $\mu_{NYSE} - \mu_{NASDAQ}$

$$\left(\bar{X}_1 - \bar{X}_2\right) \pm t_{\alpha/2} \sqrt{S_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = \mathbf{0.74 \pm 2.0154 \times 0.3628 = (0.009,\ 1.471)}$$

Since 0 is less than the entire interval, we can be 95% confident that $\mu_{NYSE} > \mu_{NASDAQ}$

Hypothesis tests for $\mu_1 - \mu_2$ with σ_1 and σ_2 unknown, not assumed equal

DCOVA

Population means, independent samples

- σ_1 and σ_2 unknown, assumed equal
- σ_1 and σ_2 unknown, not assumed equal *

Assumptions:

- Samples are randomly and independently drawn
- Populations are normally distributed or both sample sizes are at least 30
- Population variances are unknown and cannot be assumed to be equal

Hypothesis tests for $\mu_1 - \mu_2$ with σ_1 and σ_2 unknown and not assumed equal

(continued)

DCOVA

Population means, independent samples

σ_1 and σ_2 unknown, assumed equal

σ_1 and σ_2 unknown, not assumed equal *

The formulae for this test are not covered in this book.

See reference 8 from this chapter for more details.

This test utilizes two separate sample variances to estimate the degrees of freedom for the t test

Separate-Variance t Test Example

DCOVA

You are a financial analyst for a brokerage firm. Is there a difference in dividend yield between stocks listed on the NYSE & NASDAQ? You collect the following data:

	NYSE	NASDAQ
Number	21	25
Sample mean	3.27	2.53
Sample std dev	1.30	1.16

Assuming both populations are approximately normal with unequal variances, is there a difference in mean yield (α = 0.05)?

Separate-Variance t Test Example: Calculating the Test Statistic

(continued)

DCOVA

$H0: \mu_1 - \mu_2 = 0$ i.e. $(\mu_1 = \mu_2)$

$H1: \mu_1 - \mu_2 \neq 0$ i.e. $(\mu_1 \neq \mu_2)$

	A	B	C
1	**Separate-Variances t Test**		
2	(assumes unequal population variances)		
3	**Data**		
4	Hypothesized Difference	0	
5	Level of Significance	0.05	
6	**Population 1 Sample**		
7	Sample Size	21	=COUNT(DATACOPY!$A:$A)
8	Sample Mean	3.27	=AVERAGE(DATACOPY!$A:$A)
9	Sample Standard Deviation	1.3	=STDEV.S(DATACOPY!$A:$A)
10	**Population 2 Sample**		
11	Sample Size	25	=COUNT(DATACOPY!$B:$B)
12	Sample Mean	2.53	=AVERAGE(DATACOPY!$B:$B)
13	Sample Standard Deviation	1.16	=STDEV.S(DATACOPY!$C:$C)
14			
15	Intermediate Calculations		
16	Pop. 1 Sample Variance	1.69	=B9^2
17	Pop. 2 Sample Variance	1.3456	=B13^2
18	Pop. 1 Sample Variance / Sample Siz	0.0804762	=B16/B7
19	Pop. 2 Sample Variance / Sample Si:	0.053824	=B17/B11
20	Numerator of Degrees of Freedom	0.0180365	=(B18+B19)^2
21	Denomiator of Degrees of Freedom	0.0004445	=(B18^2)/(B7-1) + (B19^2)/(B11-1)
22	Total Degrees of Freedom	40.574393	=B20/B21
23	Degrees of Freedom	40	=INT(B22)
24	Separate Variance Denominator	0.3665	=SQRT(B18+B19)
25	Difference in Sample Means	0.74	=B8 - B12
26	**t Test Statistic**	2.019	=(B25 - B4)/B24
27			
28	**Two-Tail Test**		
29	**Lower Critical Value**	-2.021	=-T.INV.2T(B5,B23)
30	**Upper Critical Value**	2.021	=T.INV.2T(B5,B23)
31	**p-value**	0.050	=T.DIST.2T(ABS(B26),B23)-B4
32	**Do not reject the null hypothesis**		=IF(B31<B5,"Reject the null hypothesis","Do not reject the null hypothesis")

Separate-Variance t Test Example: Hypothesis Test Solution

DCOVA

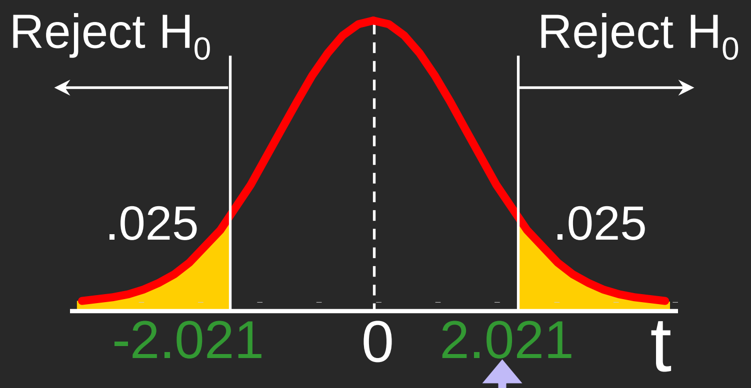
H_0: $\mu_1 - \mu_2 = 0$ i.e. $(\mu_1 = \mu_2)$

H_1: $\mu_1 - \mu_2 \neq 0$ i.e. $(\mu_1 \neq \mu_2)$

$\alpha = 0.05$

df = 40

Critical Values: t = ± 2.021



2.019

Test Statistic:

T = 2.019

Decision:

Fail To Reject H_0 at α = 0.05

Conclusion:

There is insufficient evidence of a difference in means.

Related Populations
The Paired Difference Test

DCOVA

Related samples

Tests Means of 2 Related Populations

- Paired or matched samples
- Repeated measures (before/after)
- Use difference between paired values:

$$D_i = X_{1i} - X_{2i}$$

- Eliminates Variation Among Subjects
- Assumptions:
  - Differences are normally distributed
  - Or, if not Normal, use large samples

Related Populations The Paired Difference Test

(continued)

DCOVA

Related samples

The i^{th} paired difference is D_i , where

$$D_i = X_{1i} - X_{2i}$$

The point estimate for the paired difference population mean μ_D is $\overline{D}$:

$$\overline{D} = \frac{\sum_{i=1}^{n} D_i}{n}$$

The sample standard deviation is S_D

$$S_D = \sqrt{\frac{\sum_{i=1}^{n} (D_i - \overline{D})^2}{n-1}}$$

n is the number of pairs in the paired sample

The Paired Difference Test: Finding t_{STAT}

DCOVA

Paired samples

- The test statistic for μ_D is:

$$t_{STAT} = \frac{\bar{D} - \mu_D}{\frac{S_D}{\sqrt{n}}}$$

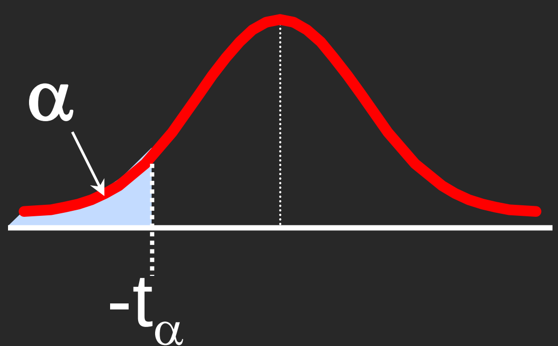
- Where t_{STAT} has $n - 1$ d.f.

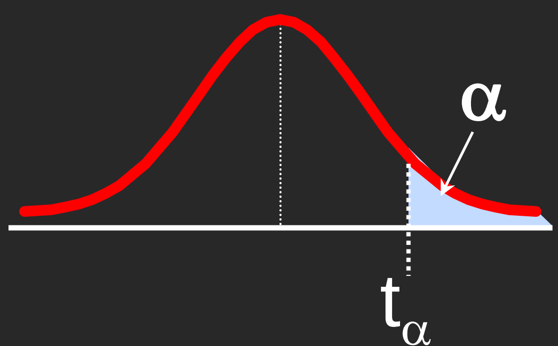
The Paired Difference Test: Possible Hypotheses

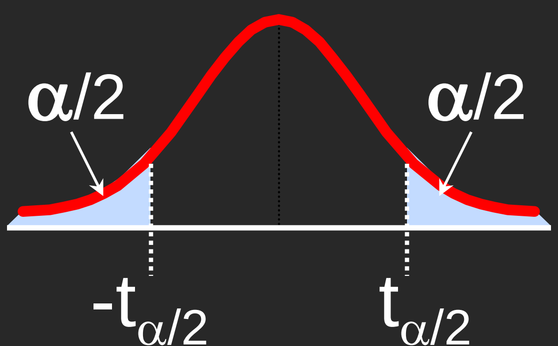
DCOVA

Paired Samples

Lower-tail test:	Upper-tail test:	Two-tail test:
$H_0: \mu_D \geq 0$ $H_1: \mu_D < 0$	$H_0: \mu_D \leq 0$ $H_1: \mu_D > 0$	$H_0: \mu_D = 0$ $H_1: \mu_D \neq 0$







Reject H_0 if $t_{STAT} < -t_{\alpha}$

Reject H_0 if $t_{STAT} > t_{\alpha}$

Reject H_0 if $t_{STAT} < -t_{\alpha/2}$ or $t_{STAT} > t_{\alpha/2}$

Where t_{STAT} has n - 1 d.f.

The Paired Difference Confidence Interval

DCOVA

Paired samples

The confidence interval for μ_D is

$$\bar{D} \pm t_{\alpha/2} \frac{S_D}{\sqrt{n}}$$

where $$S_D = \sqrt{\frac{\sum_{i=1}^{n}(D_i - \bar{D})^2}{n-1}}$$

Paired Difference Test: Example

DCOVA

- Assume you send your salespeople to a "customer service" training workshop. Has the training made a difference in the number of complaints? You collect the following data:

Salesperson	Number of Complaints: Before (1)	After (2)	(2) - (1) Difference, D_i
C.B.	6	4	- 2
T.F.	20	6	-14
M.H.	3	2	- 1
R.K.	0	0	0
M.O.	4	0	- 4
			-21

$$\bar{D} = \frac{\sum D_i}{n}$$

$$= -4.2$$

$$S_D = \sqrt{\frac{\sum (D_i - \bar{D})^2}{n-1}}$$

$$= 5.67$$

Paired Difference Test: Solution

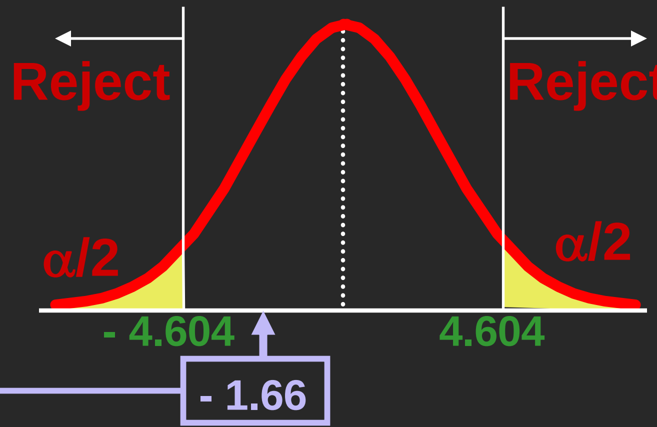
DCOVA

- Has the training made a difference in the number of complaints (at the 0.01 level)?

H_0: $\mu_D = 0$
H_1: $\mu_D \neq 0$

$\alpha = .01 \quad \bar{D} = -4.2$

$t_{0.005} = \pm 4.604$
d.f. = n - 1 = 4



Test Statistic:

$$t_{STAT} = \frac{\bar{D} - \mu_D}{S_D/\sqrt{n}} = \frac{-4.2 - 0}{5.67/\sqrt{5}} = -1.66$$

Decision: Do not reject H_0
(t_{stat} is not in the rejection region)

Conclusion: There is insufficient of a change in the number of complaints.

The Paired Difference Confidence Interval -- Example

DCOVA

The confidence interval for μ_D is:

$$\bar{D} \pm t_{\alpha/2} \frac{S_D}{\sqrt{n}}$$

$\bar{D} = -4.2$, $S_D = 5.67$

$$99\% \text{ CI for } \mu_D : -4.2 \pm 4.604 \frac{5.67}{\sqrt{5}}$$

$$= (-15.87, 7.47)$$

Since this interval contains 0 you are 99% confident that $\mu_D = 0$

Two Population Proportions

DCOVA

Population proportions

Goal: test a hypothesis or form a confidence interval for the difference between two population proportions,

$$\pi_1 - \pi_2$$

Assumptions:

$n_1 \pi_1 \geq 5$, $n_1(1- \pi_1) \geq 5$

$n_2 \pi_2 \geq 5$, $n_2(1- \pi_2) \geq 5$

The point estimate for the difference is $p_1 - p_2$

Two Population Proportions

DCOVA

Population proportions

In the null hypothesis we assume the null hypothesis is true, so we assume $\pi_1 = \pi_2$ and pool the two sample estimates

The pooled estimate for the overall proportion is:

$$\bar{p} = \frac{X_1 + X_2}{n_1 + n_2}$$

where X_1 and X_2 are the number of items of interest in samples 1 and 2

Two Population Proportions

(continued)

DCOVA

Population proportions

The test statistic for $\pi_1 - \pi_2$ is a Z statistic:

$$Z_{STAT} = \frac{(p_1 - p_2) - (\pi_1 - \pi_2)}{\sqrt{\bar{p}(1-\bar{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

where

$$\bar{p} = \frac{X_1 + X_2}{n_1 + n_2}, \quad p_1 = \frac{X_1}{n_1}, \quad p_2 = \frac{X_2}{n_2}$$

Hypothesis Tests for Two Population Proportions

DCOVA

Population proportions

Lower-tail test:

$H_0: \pi_1 \geq \pi_2$
$H_1: \pi_1 < \pi_2$

i.e.,

$H_0: \pi_1 - \pi_2 \geq 0$
$H_1: \pi_1 - \pi_2 < 0$

Upper-tail test:

$H_0: \pi_1 \leq \pi_2$
$H_1: \pi_1 > \pi_2$

i.e.,

$H_0: \pi_1 - \pi_2 \leq 0$
$H_1: \pi_1 - \pi_2 > 0$

Two-tail test:

$H_0: \pi_1 = \pi_2$
$H_1: \pi_1 \neq \pi_2$

i.e.,

$H_0: \pi_1 - \pi_2 = 0$
$H_1: \pi_1 - \pi_2 \neq 0$

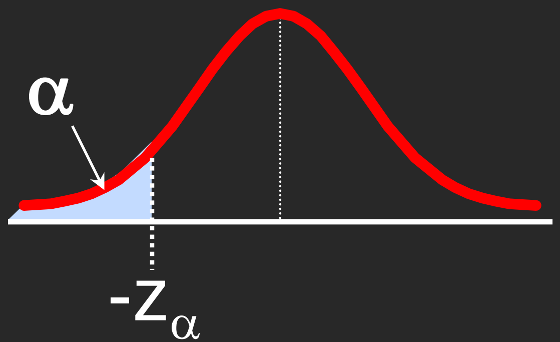
Hypothesis Tests for Two Population Proportions

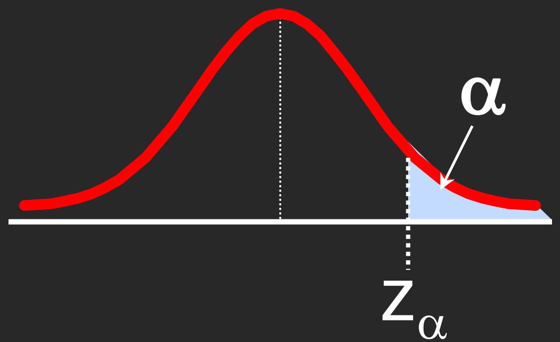
(continued)

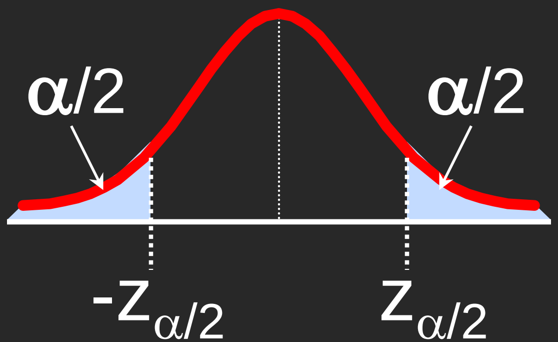
DCOVA

Population proportions

Lower-tail test:	Upper-tail test:	Two-tail test:
$H_0: \pi_1 - \pi_2 \geq 0$ $H_1: \pi_1 - \pi_2 < 0$	$H_0: \pi_1 - \pi_2 \leq 0$ $H_1: \pi_1 - \pi_2 > 0$	$H_0: \pi_1 - \pi_2 = 0$ $H_1: \pi_1 - \pi_2 \neq 0$







Reject H_0 if $Z_{STAT} < -Z_\alpha$

Reject H_0 if $Z_{STAT} > Z_\alpha$

Reject H_0 if $Z_{STAT} < -Z_{\alpha/2}$ or $Z_{STAT} > Z_{\alpha/2}$

Hypothesis Test Example: Two population Proportions

DCOVA

Is there a significant difference between the proportion of men and the proportion of women who will vote Yes on Proposition A?

- In a random sample, 36 of 72 men and 35 of 50 women indicated they would vote Yes
- Test at the .05 level of significance

Hypothesis Test Example: Two population Proportions

(continued)

DCOVA

- The hypothesis test is:

$H_0: \pi_1 - \pi_2 = 0$ (the two proportions are equal)
$H_1: \pi_1 - \pi_2 \neq 0$ (there is a significant difference between proportions)

- The sample proportions are:
  - Men: $p_1 = 36/72 = 0.50$
  - Women: $p_2 = 35/50 = 0.70$

- The pooled estimate for the overall proportion is:

$$\bar{p} = \frac{X_1 + X_2}{n_1 + n_2} = \frac{36 + 35}{72 + 50} = \frac{71}{122} = .582$$

Hypothesis Test Example: Two Population Proportions

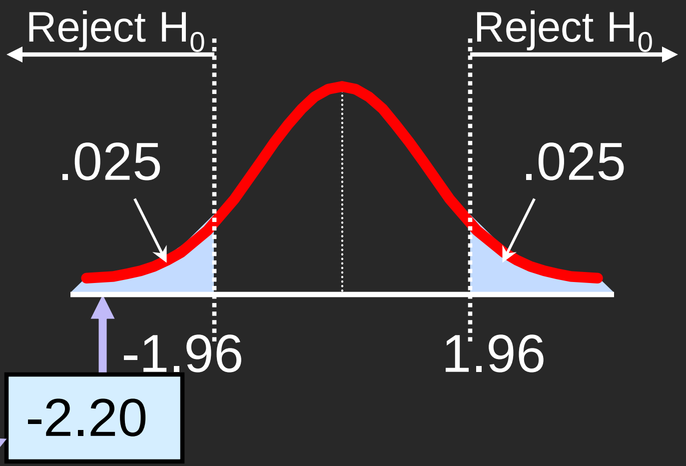
(continued)

DCOVA

The test statistic for $\pi_1 - \pi_2$ is:

$$z_{STAT} = \frac{(p_1 - p_2) - (\pi_1 - \pi_2)}{\sqrt{\bar{p}(1-\bar{p})\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}$$

$$= \frac{(.50 - .70) - (0)}{\sqrt{.582(1 - .582)\left(\frac{1}{72} + \frac{1}{50}\right)}} = -2.20$$



Critical Values = ±1.96
For α = .05

Decision: Reject H_0

Conclusion: There is evidence of a significant difference in the proportion of men and women who will vote yes.

Confidence Interval for Two Population Proportions

DCOVA

Population proportions

The confidence interval for $\pi_1 - \pi_2$ is:

$$\left(p_1 - p_2\right) \pm Z_{\alpha/2} \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

Confidence Interval for Two Population Proportions -- Example

DCOVA

The 95% confidence interval for $\pi_1 - \pi_2$ is:

$$\left(0.50 - 0.70\right) \pm 1.96\sqrt{\frac{0.50(0.50)}{72} + \frac{0.70(0.30)}{50}}$$

$$= (-0.37, -0.03)$$

Since this interval does not contain 0 can be 95% confident the two proportions are different.

Testing for the Ratio Of Two Population Variances

DCOVA

Tests for Two Population Variances *

F test statistic

Hypotheses	F_{STAT}
$H_0: \sigma_1^2 = \sigma_2^2$ $H_1: \sigma_1^2 \neq \sigma_2^2$	
$H_0: \sigma_1^2 \leq \sigma_2^2$ $H_1: \sigma_1^2 > \sigma_2^2$	S_1^2 / S_2^2

Where:

S_1^2 = Variance of sample 1 (the larger sample variance)

n_1 = sample size of sample 1

S_2^2 = Variance of sample 2 (the smaller sample variance)

n_2 = sample size of sample 2

$n_1 - 1$ = numerator degrees of freedom

$n_2 - 1$ = denominator degrees of freedom

The F Distribution

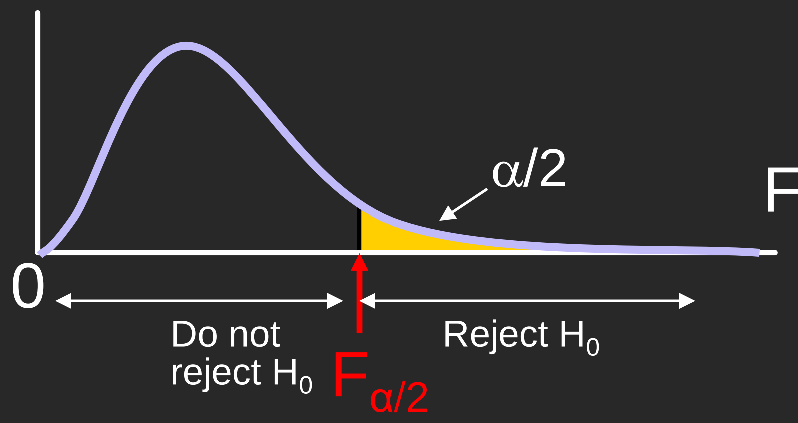
DCOVA

- The F critical value is found from the F table
- There are two degrees of freedom required: numerator and denominator
- The larger sample variance is always the numerator
- When $$F_{STAT} = \frac{S_1^2}{S_2^2}$$ $df_1 = n_1 – 1$; $df_2 = n_2 – 1$
- In the F table,
  - numerator degrees of freedom determine the column
  - denominator degrees of freedom determine the row

Finding the Rejection Region

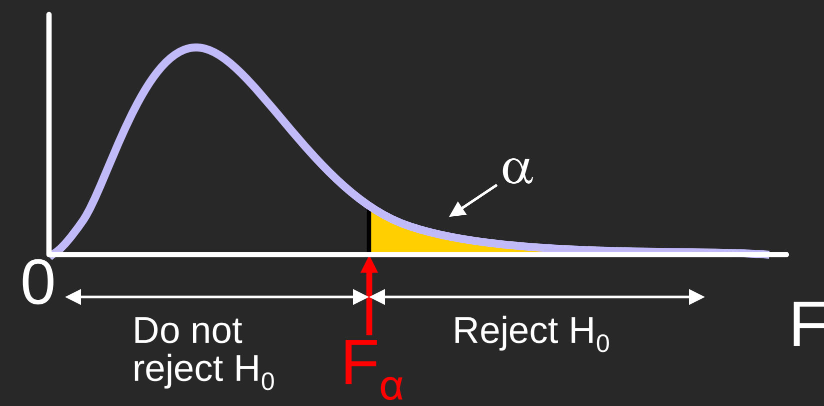
DCOVA

$H_0: \sigma_1^2 = \sigma_2^2$
$H_1: \sigma_1^2 \neq \sigma_2^2$



Reject H_0 if $F_{STAT} > F_{\alpha/2}$

$H_0: \sigma_1^2 \leq \sigma_2^2$
$H_1: \sigma_1^2 > \sigma_2^2$



Reject H_0 if $F_{STAT} > F_{\alpha}$

F Test: An Example

DCOVA

You are a financial analyst for a brokerage firm. You want to compare dividend yields between stocks listed on the NYSE & NASDAQ. You collect the following data:

	NYSE	NASDAQ
Number	**21**	**25**
Mean	**3.27**	**2.53**
Std dev	**1.30**	**1.16**

Is there a difference in the variances between the NYSE NASDAQ at the α = 0.05 level?

F Test: Example Solution

DCOVA

- Form the hypothesis test:

$H_0: \sigma^2_1 = \sigma^2_2$ (there is no difference between variances)

$H_1: \sigma^2_1 \neq \sigma^2_2$ (there is a difference between variances)

- Find the F critical value for $\alpha = 0.05$:

- Numerator d.f. = $n_1 - 1 = 21 - 1 = 20$

- Denominator d.f. = $n_2 - 1 = 25 - 1 = 24$

- $F_{\alpha/2} = F_{.025,\ 20,\ 24} = 2.33$

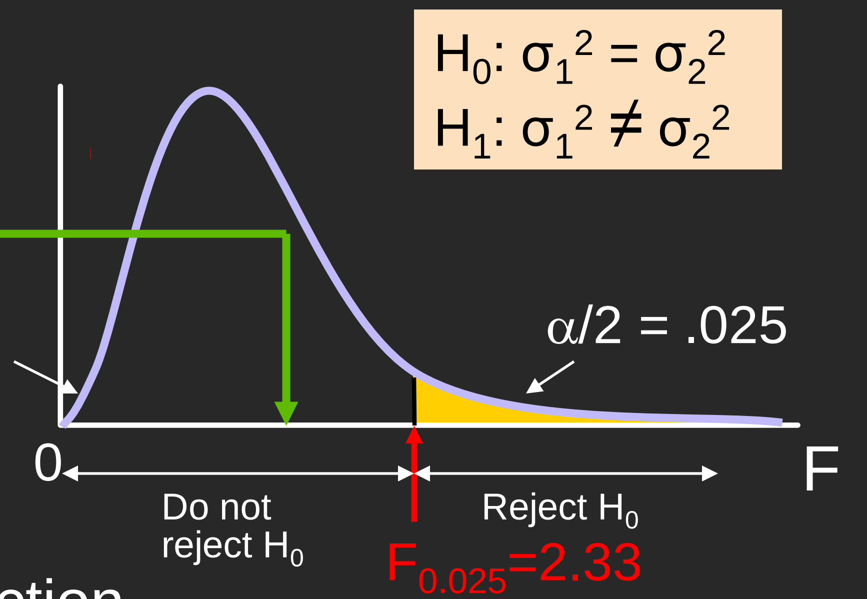
F Test: Example Solution

DCOVA

(continued)

- The test statistic is:

$$F_{STAT} = \frac{S_1^2}{S_2^2} = \frac{1.30^2}{1.16^2} = 1.256$$



- $F_{STAT} = 1.256$ is not in the rejection region, so we do not reject H_0

- Conclusion: There is not sufficient evidence of a difference in variances at $\alpha = .05$

General ANOVA Setting

DCOVA

- Investigator controls one or more factors of interest
  - Each factor contains two or more levels
  - Levels can be numerical or categorical
  - Different levels produce different groups
  - Think of each group as a sample from a different population
- Observe effects on the dependent variable
  - Are the groups the same?
- Experimental design: the plan used to collect the data

Completely Randomized Design

DCOVA

- Experimental units (subjects) are assigned randomly to groups
  - Subjects are assumed homogeneous
- Only one factor or independent variable
  - With two or more levels
- Analyzed by one-factor analysis of variance (ANOVA)

One-Way Analysis of Variance

DCOVA

- Evaluate the difference among the means of three or more groups

Examples:	Number of accidents for 1st, 2nd, and 3rd shift
	Expected mileage for five brands of tires

- Assumptions
  - Populations are normally distributed
  - Populations have equal variances
  - Samples are randomly and independently drawn

Hypotheses of One-Way ANOVA

DCOVA

- $H_0 : \mu_1 = \mu_2 = \mu_3 = \cdots = \mu_c$
  - All population means are equal
  - i.e., no factor effect (no variation in means among groups)
- $H_1 : \text{Not all of the population means are equal}$
  - At least one population mean is different
  - i.e., there is a factor effect
  - Does not mean that all population means are different (some pairs may be the same)

One-Way ANOVA

DCOVA

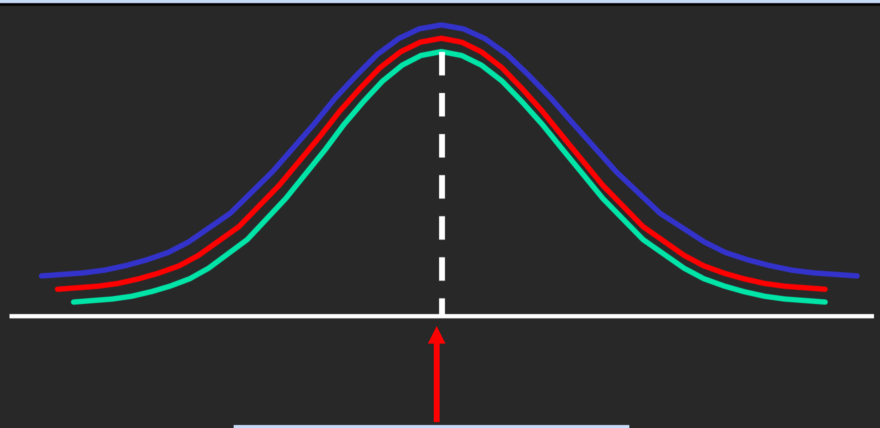
$$H_0 : \mu_1 = \mu_2 = \mu_3 = \cdots = \mu_c$$

$$H_1 : \text{Not all } \mu_j \text{ are equal}$$

When The Null Hypothesis is True

All Means are the same:

(No Factor Effect)



$$\mu_1 = \mu_2 = \mu_3$$

One-Way ANOVA

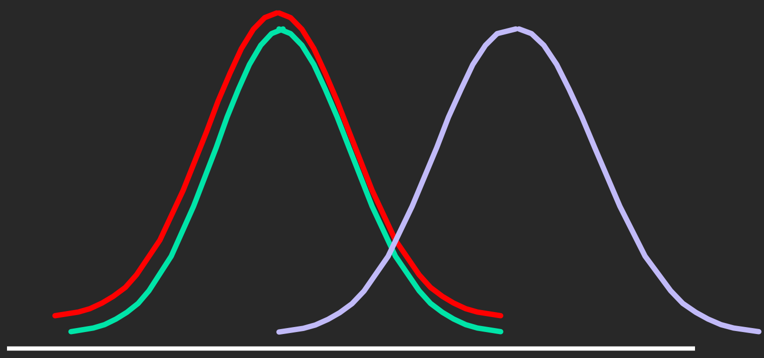
(continued)

DCOVA

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \cdots = \mu_c$$

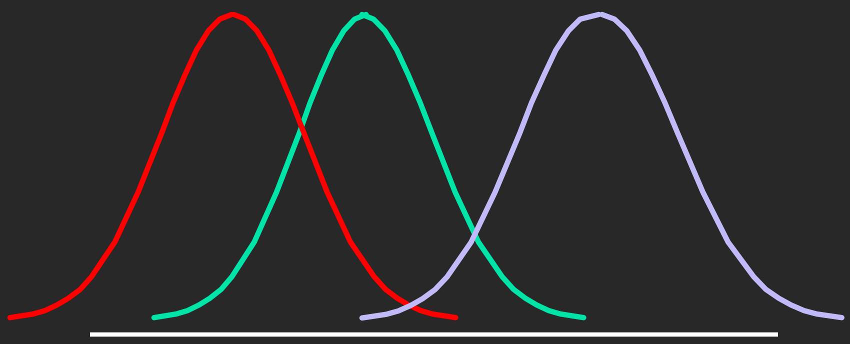
$$H_1 : \text{Not all } \mu_j \text{ are equal}$$

When The Null Hypothesis is NOT true
At least one of the means is different
(Factor Effect is present)



$$\mu_1 = \mu_2 \neq \mu_3$$

or



$$\mu_1 \neq \mu_2 \neq \mu_3$$

Partitioning the Variation

DCOVA

- Total variation can be split into two parts:

$$SST = SSA + SSW$$

SST = Total Sum of Squares
(Total variation)
SSA = Sum of Squares Among Groups
(Among-group variation)
SSW = Sum of Squares Within Groups
(Within-group variation)

Partitioning the Variation

(continued)

DCOVA

$$SST = SSA + SSW$$

Total Variation = the aggregate variation of the individual data values across the various factor levels (SST)

Among-Group Variation = variation among the factor sample means (SSA)

Within-Group Variation = variation that exists among the data values within a particular factor level (SSW)

Partition of Total Variation

DCOVA

Total Variation (SST)

= **Variation Due to Factor (SSA)** + **Variation Due to Random Error (SSW)**

Total Sum of Squares

DCOVA

$$SST = SSA + SSW$$

$$SST = \sum_{j=1}^{c} \sum_{i=1}^{n_j} (X_{ij} - \bar{\bar{X}})^2$$

Where:

SST = Total sum of squares

c = number of groups or levels

n_j = number of values in group j

X_{ij} = i^{th} observation from group j

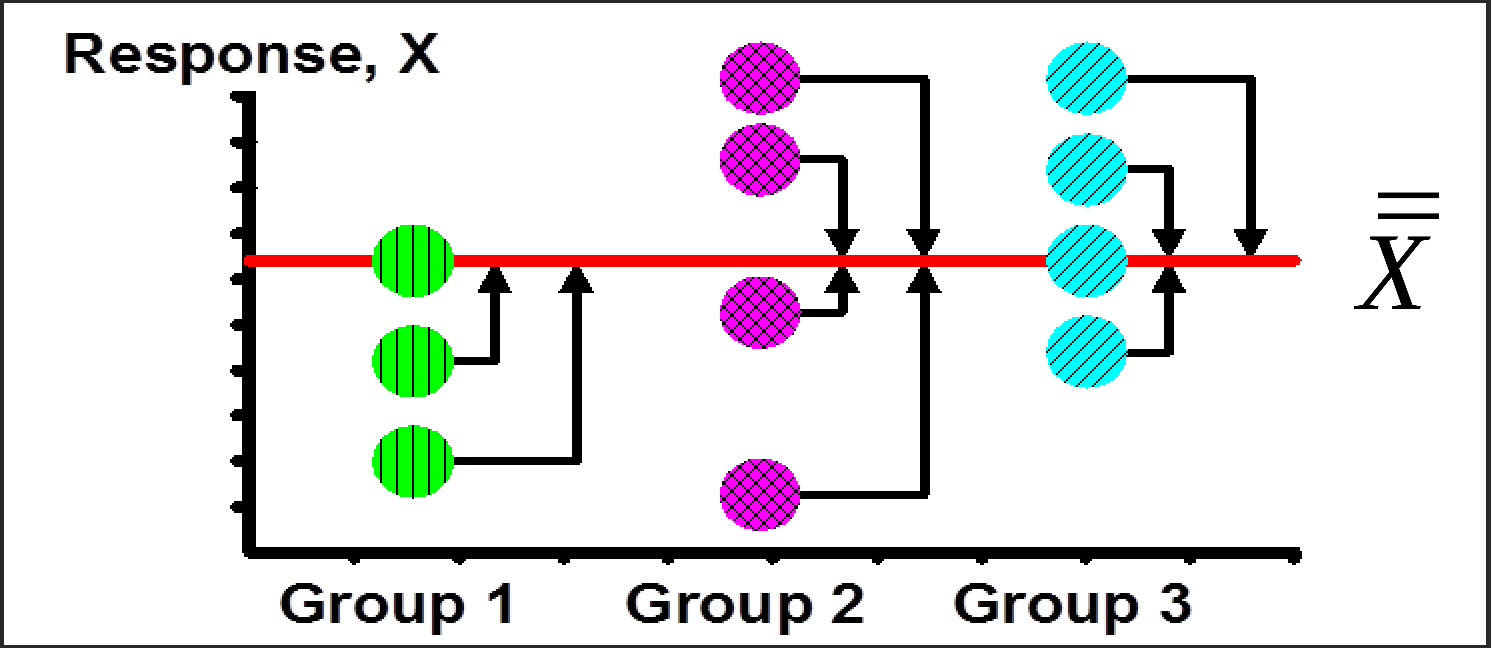
$\bar{\bar{X}}$ = grand mean (mean of all data values)

Total Variation

DCOVA

(continued)

$$SST = (X_{11} - \bar{\bar{X}})^2 + (X_{12} - \bar{\bar{X}})^2 + \cdots + (X_{cn_c} - \bar{\bar{X}})^2$$



Among-Group Variation

DCOVA

$$SST = SSA + SSW$$

$$SSA = \sum_{j=1}^{c} n_j (\bar{X}_j - \bar{\bar{X}})^2$$

Where:

SSA = Sum of squares among groups

c = number of groups

n_j = sample size from group j

$\bar{X}_j$ = sample mean from group j

$\bar{\bar{X}}$ = grand mean (mean of all data values)

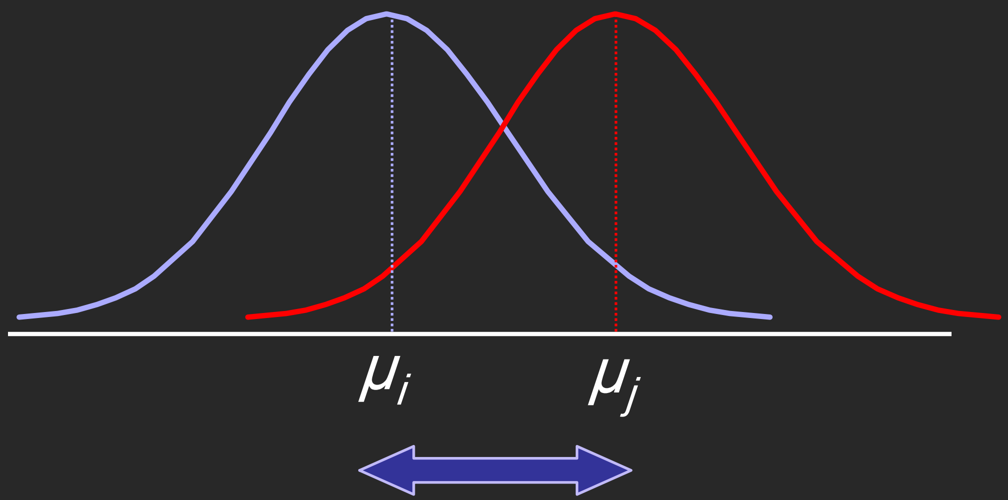
Among-Group Variation

(continued)

DCOVA

$$SSA = \sum_{j=1}^{c} n_j (\bar{X}_j - \bar{\bar{X}})^2$$

Variation Due to
Differences Among Groups



$$MSA = \frac{SSA}{c-1}$$

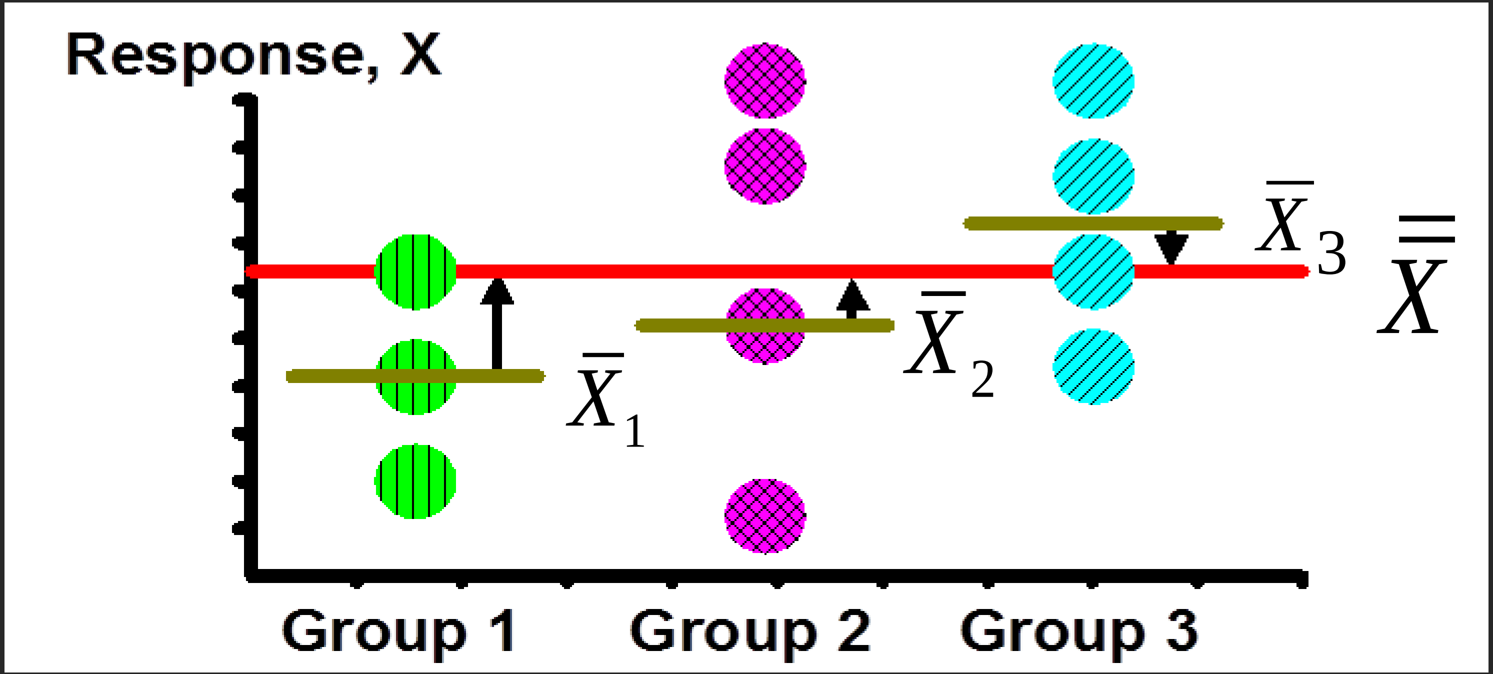
Mean Square Among =
SSA/degrees of freedom

Among-Group Variation

DCOVA

(continued)

$$SSA = n_1(\bar{X}_1 - \bar{\bar{X}})^2 + n_2(\bar{X}_2 - \bar{\bar{X}})^2 + \cdots + n_c(\bar{X}_c - \bar{\bar{X}})^2$$



Within-Group Variation

DCOVA

$$SST = SSA + \boxed{SSW}$$

$$SSW = \sum_{j=1}^{c} \sum_{i=1}^{n_j} (X_{ij} - \bar{X}_j)^2$$

Where:

SSW = Sum of squares within groups

c = number of groups

n_j = sample size from group j

$\bar{X}_j$ = sample mean from group j

X_{ij} = i^{th} observation in group j

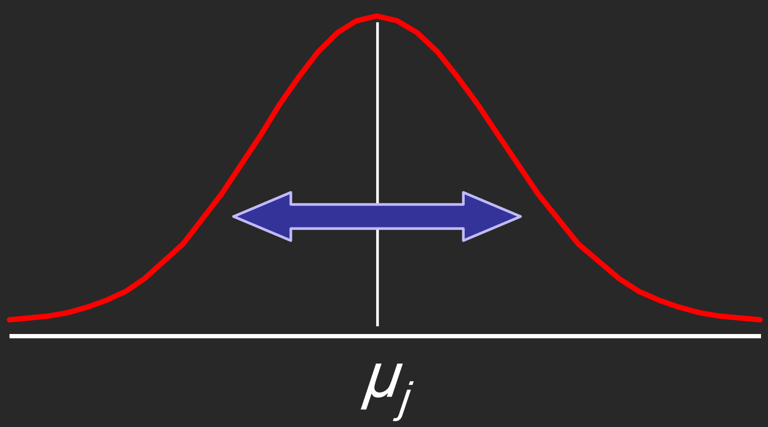
Within-Group Variation

(continued)

DCOVA

$$SSW = \sum_{j=1}^{c} \sum_{i=1}^{n_j} (X_{ij} - \bar{X}_j)^2$$

Summing the variation within each group and then adding over all groups



$$MSW = \frac{SSW}{n - c}$$

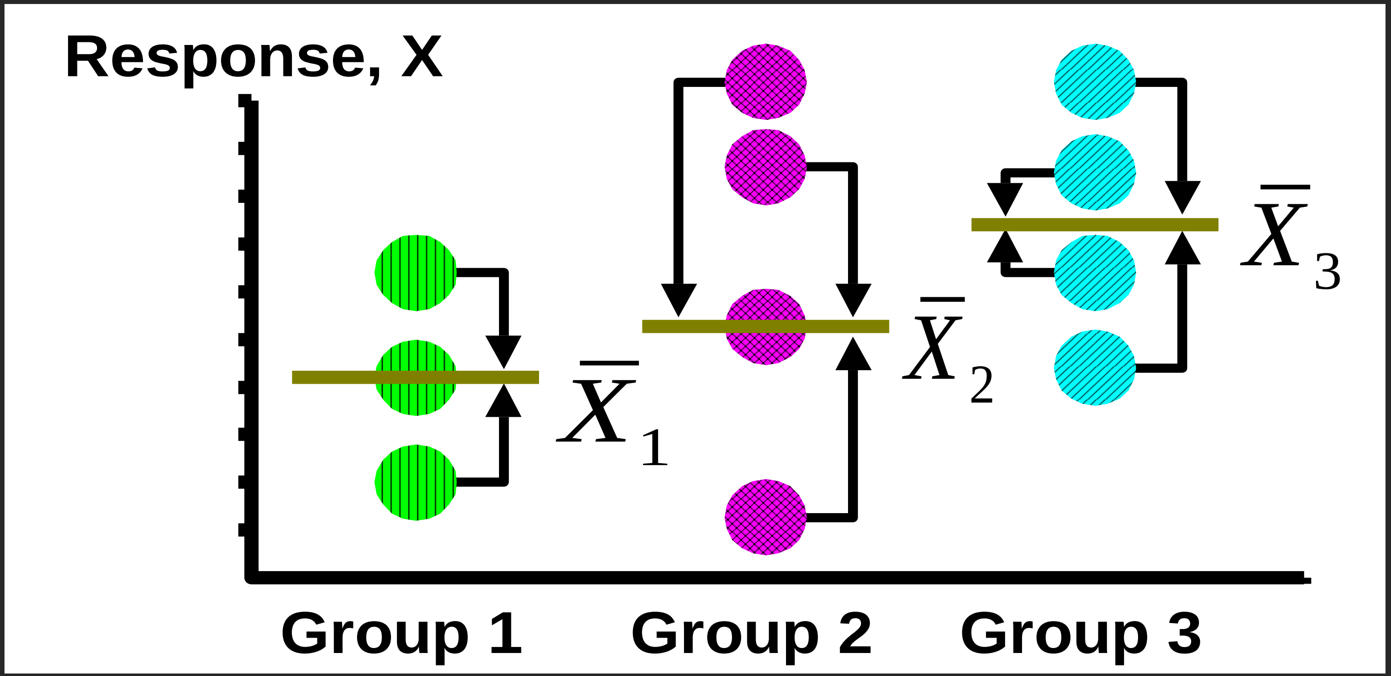
Mean Square Within = SSW/degrees of freedom

Within-Group Variation

DCOVA

(continued)

$$SSW = (X_{11} - \bar{X}_1)^2 + (X_{12} - \bar{X}_2)^2 + \cdots + (X_{cn_c} - \bar{X}_c)^2$$



Obtaining the Mean Squares

DCOVA

The Mean Squares are obtained by dividing the various sum of squares by their associated degrees of freedom

$$MSA = \frac{SSA}{c-1}$$

Mean Square Among (d.f. = c-1)

$$MSW = \frac{SSW}{n-c}$$

Mean Square Within (d.f. = n-c)

$$MST = \frac{SST}{n-1}$$

Mean Square Total (d.f. = n-1)

One-Way ANOVA Table

DCOVA

Source of Variation	Degrees of Freedom	Sum Of Squares	Mean Square (Variance)	F
Among Groups	c - 1	SSA	$MSA = \frac{SSA}{c - 1}$	$F_{STAT} = \frac{MSA}{MSW}$
Within Groups	n - c	SSW	$MSW = \frac{SSW}{n - c}$	
Total	n – 1	SST		

c = number of groups

n = sum of the sample sizes from all groups

df = degrees of freedom

One-Way ANOVA
F Test Statistic

DCOVA

H_0: $\mu_1 = \mu_2 = \cdots = \mu_c$

H_1: At least two population means are different

- Test statistic

$$F_{STAT} = \frac{MSA}{MSW}$$

MSA is mean squares among groups

MSW is mean squares within groups

- Degrees of freedom
  - $df_1 = c - 1$ (c = number of groups)
  - $df_2 = n - c$ (n = sum of sample sizes from all populations)

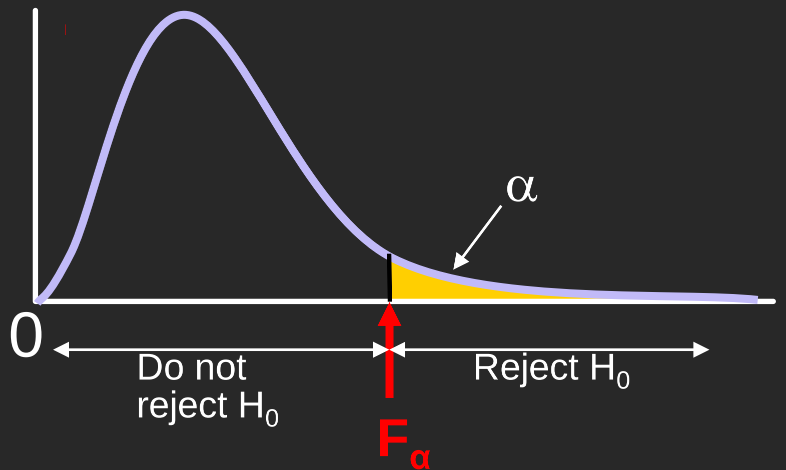
Interpreting One-Way ANOVA F Statistic

DCOVA

- The F statistic is the ratio of the among estimate of variance and the within estimate of variance
  - The ratio must always be positive
  - $df_1 = c - 1$ will typically be small
  - $df_2 = n - c$ will typically be large

Decision Rule:

- Reject H_0 if $F_{STAT} > F_\alpha$, otherwise do not reject H_0

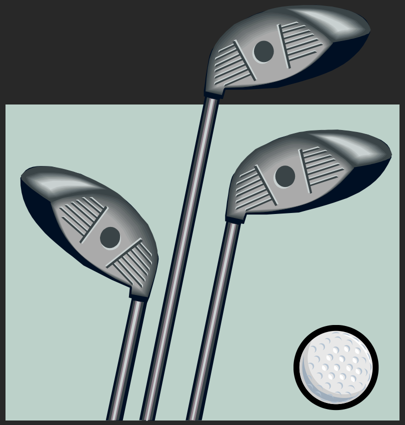


One-Way ANOVA F Test Example

DCOVA

You want to see if three different golf clubs yield different distances. You randomly select five measurements from trials on an automated driving machine for each club. At the 0.05 significance level, is there a difference in mean distance?

Club 1	Club 2	Club 3
254	234	200
263	218	222
241	235	197
237	227	206
251	216	204



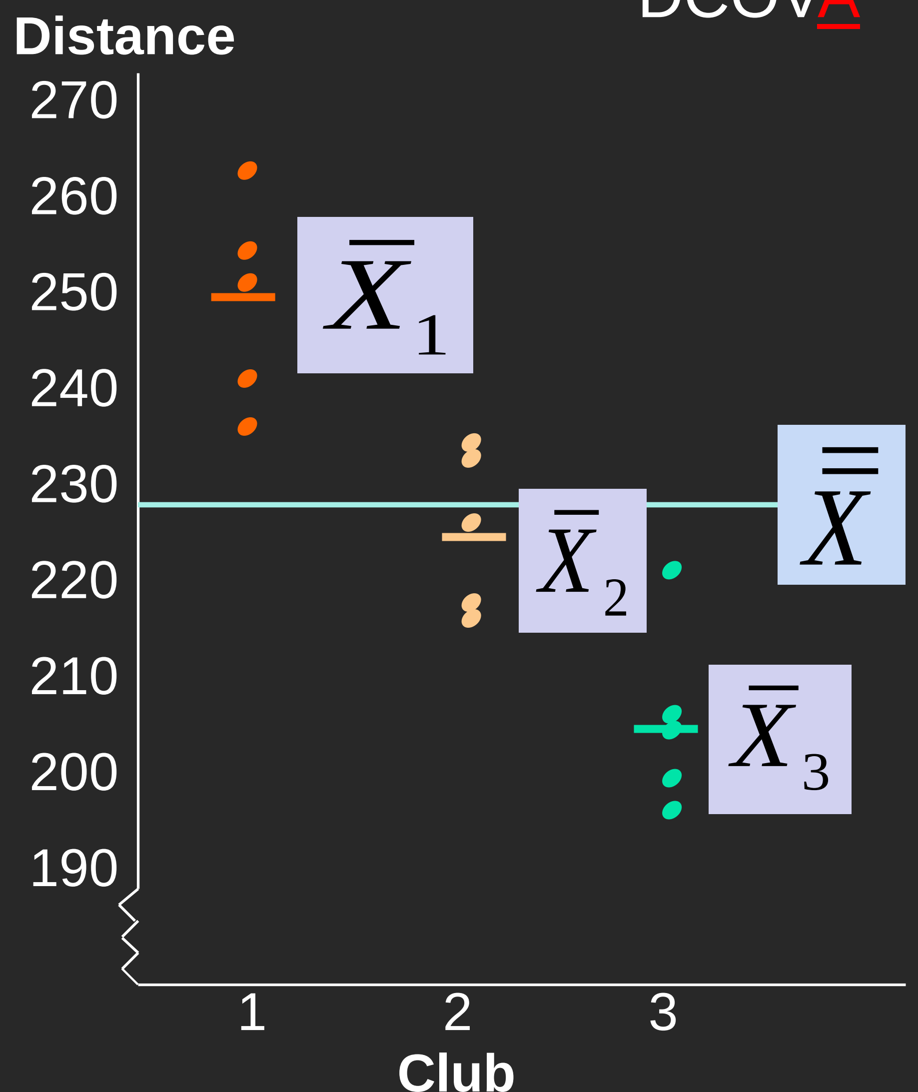
One-Way ANOVA Example: Scatter Plot

DCOVA

Club 1	Club 2	Club 3
254	234	200
263	218	222
241	235	197
237	227	206
251	216	204

$\bar{x}_1 = 249.2$	$\bar{x}_2 = 226.0$	$\bar{x}_3 = 205.8$

$\bar{\bar{x}} = 227.0$



One-Way ANOVA Example Computations

DCOVA

Club 1	Club 2	Club 3
254	234	200
263	218	222
241	235	197
237	227	206
251	216	204

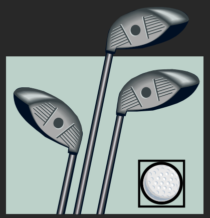
$\bar{X}_1 = 249.2$ $n_1 = 5$

$\bar{X}_2 = 226.0$ $n_2 = 5$

$\bar{X}_3 = 205.8$ $n_3 = 5$

$\bar{\bar{X}} = 227.0$ $n = 15$

$c = 3$



$SSA = 5\,(249.2 - 227)^2 + 5\,(226 - 227)^2 + 5\,(205.8 - 227)^2 = 4716.4$

$SSW = (254 - 249.2)^2 + (263 - 249.2)^2 + \cdots + (204 - 205.8)^2 = 1119.6$

$MSA = 4716.4 / (3\text{-}1) = 2358.2$

$MSW = 1119.6 / (15\text{-}3) = 93.3$

$$F_{STAT} = \frac{2358.2}{93.3} = 25.275$$

One-Way ANOVA Example Solution

DCOVA

H_0: $\mu_1 = \mu_2 = \mu_3$

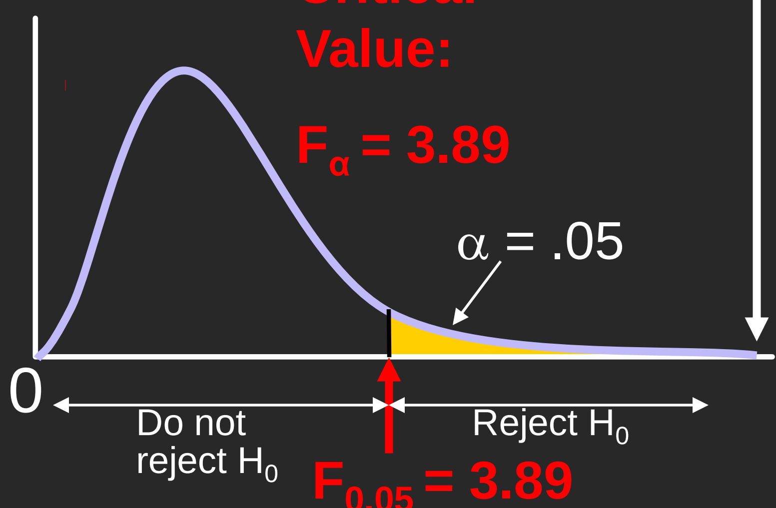
H_1: μ_j not all equal

$\alpha = 0.05$

$df_1 = 2$ $df_2 = 12$

Critical Value:

$F_\alpha = 3.89$



Test Statistic:

$$F_{STAT} = \frac{MSA}{MSW} = \frac{2358.2}{93.3} = 25.275$$

Decision:

Reject H_0 at $\alpha = 0.05$

Conclusion:

There is evidence that at least one μ_j differs from the rest

One-Way ANOVA Excel Output

DCOVA

SUMMARY						
Groups	*Count*	*Sum*	*Average*	*Variance*		
Club 1	5	1246	249.2	108.2		
Club 2	5	1130	226	77.5		
Club 3	5	1029	205.8	94.2		
ANOVA						
Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	**4716.4**	**2**	**2358.2**	**25.275**	0.0000	**3.89**
Within Groups	**1119.6**	**12**	**93.3**			
Total	5836.0	14				

One-Way ANOVA Minitab Output

DCOVA

One-way ANOVA: Distance versus Club

Source	DF	SS	MS	F	P
Club	2	4716.4	2358.2	25.28	0.000
Error	12	1119.6	93.3		
Total	14	5836.0			

S = 9.659   R-Sq = 80.82%   R-Sq(adj) = 77.62%

```
                              Individual 95% CIs For Mean Based on Pooled StDev

Level  N    Mean   StDev  -------+---------+---------+---------+--
1      5  249.20  10.40                         (-----*-----)
2      5  226.00   8.80               (-----*-----)
3      5  205.80   9.71   (-----*-----)
                          -------+---------+---------+---------+--
                               208       224       240       256
Pooled StDev = 9.66
```

ANOVA Assumptions

DCOVA

- Randomness and Independence
  - Select random samples from the c groups (or randomly assign the levels)
- Normality
  - The sample values for each group are from a normal population
- Homogeneity of Variance
  - All populations sampled from have the same variance
  - Can be tested with Levene's Test

ANOVA Assumptions
Levene's Test

DCOVA

- Tests the assumption that the variances of each population are equal.
- First, define the null and alternative hypotheses:
  - H_0: $\sigma^2_1 = \sigma^2_2 = \cdots = \sigma^2_c$
  - H_1: Not all σ^2_j are equal
- Second, compute the absolute value of the difference between each value and the median of each group.
- Third, perform a one-way ANOVA on these absolute differences.

Levene Homogeneity Of Variance Test Example

DCOVA

$H0: \sigma^2_1 = \sigma^2_2 = \sigma^2_3$

$H1$: Not all σ^2_j are equal

Calculate Medians

Club 1	Club 2	Club 3	
237	216	197	
241	218	200	
251	227	204	**Median**
254	234	206	
263	235	222	

Calculate Absolute Differences

Club 1	Club 2	Club 3
14	11	7
10	9	4
0	0	0
3	7	2
12	8	18

Levene Homogeneity Of Variance Test Example

(continued)

DCOVA

Anova: Single Factor

SUMMARY

Groups	*Count*	*Sum*	*Average*	*Variance*
Club 1	5	39	7.8	36.2
Club 2	5	35	7	17.5
Club 3	5	31	6.2	50.2

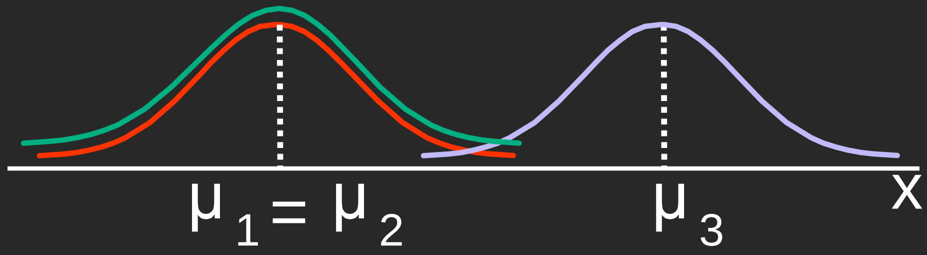
Source of Variation	*SS*	*df*	*MS*	*F*	*P-value*	*F crit*
Between Groups	6.4	2	3.2	0.092	0.912	3.885
Within Groups	415.6	12	34.6			
Total	422	14				

Since the p-value is greater than 0.05 there is insufficient evidence of a difference in the variances

The Tukey-Kramer Procedure

DCOVA

- Tells which population means are significantly different
  - e.g.: $\mu_1 = \mu_2 \neq \mu_3$
  - Done after rejection of equal means in ANOVA
- Allows paired comparisons
  - Compare absolute mean differences with critical range



Tukey-Kramer Critical Range

DCOVA

$$\text{Critical Range} = Q_{\alpha}\sqrt{\frac{MSW}{2}\left(\frac{1}{n_j}+\frac{1}{n_{j'}}\right)}$$

where:

Q_α = Upper Tail Critical Value from Studentized Range Distribution with c and n - c degrees of freedom (see appendix E.7 table)

MSW = Mean Square Within

n_j and $n_{j'}$ = Sample sizes from groups j and j'

The Tukey-Kramer Procedure: Example

DCOVA

Club 1	Club 2	Club 3
254	234	200
263	218	222
241	235	197
237	227	206
251	216	204

1. Compute absolute mean differences:

$$|\bar{x}_1 - \bar{x}_2| = |249.2 - 226.0| = 23.2$$

$$|\bar{x}_1 - \bar{x}_3| = |249.2 - 205.8| = 43.4$$

$$|\bar{x}_2 - \bar{x}_3| = |226.0 - 205.8| = 20.2$$

2. Find the Q_α value from the table in appendix E.7 with $c = 3$ and $(n - c) = (15 - 3) = 12$ degrees of freedom:

$$Q_\alpha = 3.77$$

The Tukey-Kramer Procedure: Example

(continued)

DCOVA

3. Compute Critical Range:

$$\text{Critical Range} = Q_{\alpha}\sqrt{\frac{MSW}{2}\left(\frac{1}{n_j}+\frac{1}{n_{j'}}\right)} = 3.77\sqrt{\frac{93.3}{2}\left(\frac{1}{5}+\frac{1}{5}\right)} = 16.285$$

4. Compare:

$$|\bar{x}_1 - \bar{x}_2| = 23.2$$

$$|\bar{x}_1 - \bar{x}_3| = 43.4$$

$$|\bar{x}_2 - \bar{x}_3| = 20.2$$

5. All of the absolute mean differences are greater than the critical range. Therefore there is a significant difference between each pair of means at 5% level of significance.

Thus, with 95% confidence we conclude that the mean distance for club 1 is greater than club 2 and 3, and club 2 is greater than club 3.

Chapter Summary

In this chapter we discussed:

- How to use hypothesis testing for comparing the difference between
  - The means of two independent populations
  - The means of two related populations
  - The proportions of two independent populations
  - The variances of two independent populations
  - The means of more than two populations