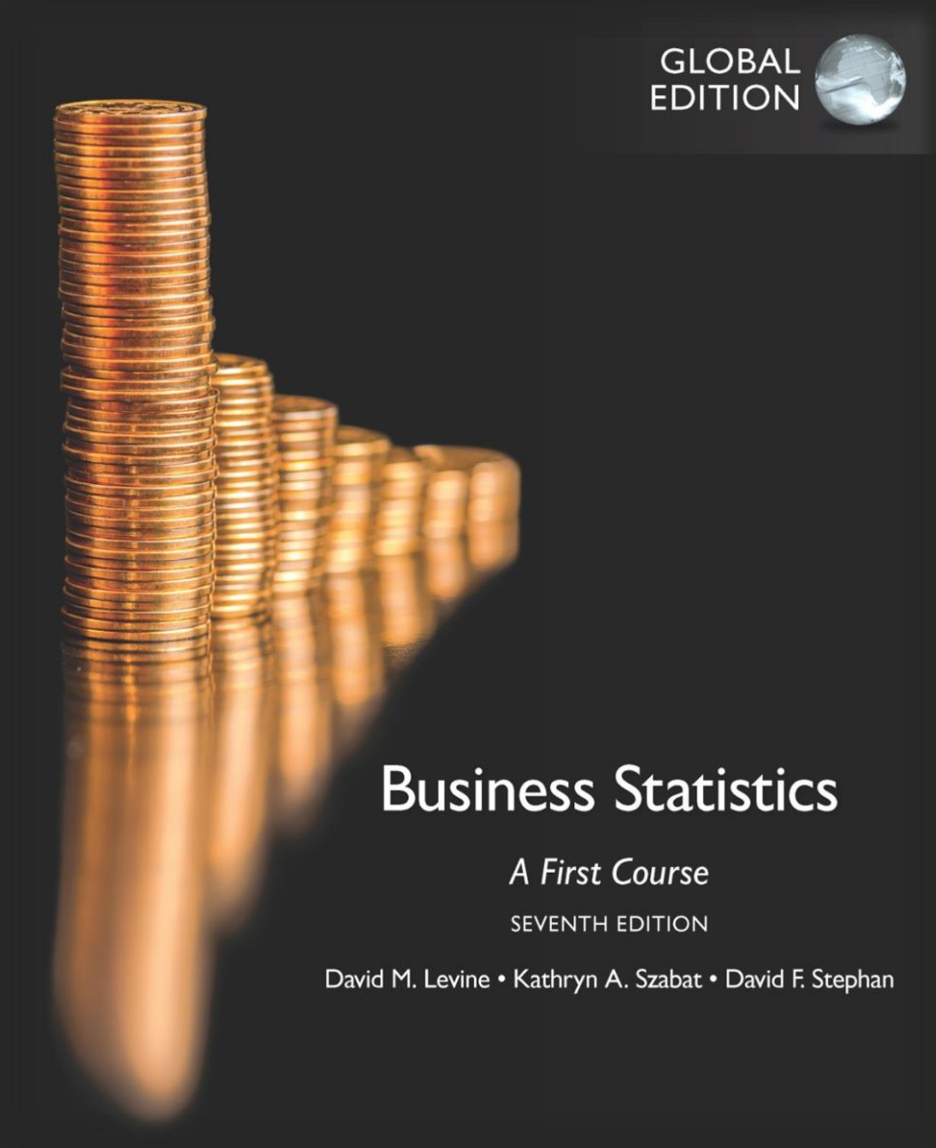


Chapter 4

Basic Probability

Objectives

The objectives for this chapter are:

- To understand basic probability concepts.
- To understand conditional probability
- To learn various counting rules

Basic Probability Concepts

- Probability – the chance that an uncertain event will occur (always between 0 and 1)
- Impossible Event – an event that has no chance of occurring (probability = 0)
- Certain Event – an event that is sure to occur (probability = 1)

Assessing Probability

There are three approaches to assessing the probability of an uncertain event:

Assuming all outcomes are equally likely

1. *Priori* probability

$$\text{probability of occurrence} = \frac{X}{T} = \frac{\text{number of ways in which the event occurs}}{\text{total number of possible outcomes}}$$

2. Empirical probability

$$\text{probability of occurrence} = \frac{\text{the number of times an event occurred}}{\text{the total number of trials or observations}}$$

3. Subjective probability

based on a combination of an individual's past experience, personal opinion, and analysis of a particular situation

Priori Probability

- A priori probability also known as theoretical probability (or classical probability), is calculated before any event has taken place.
- A priori probability is based on priori knowledge of the process involved.
- For instance, if you were to roll a pair of dice, you could work out the theoretical probability of rolling a four before any dice had been rolled at all

Example of *a priori* probability

When randomly selecting a day from the year 2015 what is the probability the day is in January?

$$\text{Probability of Day In January} = \frac{X}{T} = \frac{\text{number of days in January}}{\text{total number of days in 2015}}$$

$$\frac{X}{T} = \frac{31 \text{ days in January}}{365 \text{ days in 2015}} = \frac{31}{365}$$

Empirical Probability

- Empirical probability is calculated after the event has occurred. By observing the pattern of events and how often a certain outcome has been seen, mathematicians try to estimate how often they can expect to see a certain outcome in the future
- An empirical probability is based on observed data, not on priori knowledge

Example of empirical probability

Find the probability of selecting a male taking statistics from the population described in the following table:

	Taking Stats	Not Taking Stats	Total
Male	84	145	229
Female	76	134	210
Total	160	279	439

$$\text{Probability of male taking stats} = \frac{\text{number of males taking stats}}{\text{total number of people}} = \frac{84}{439} = 0.191$$

Subjective probability

- Subjective probability may differ from person to person
  - A media development team assigns a 60% probability of success to its new ad campaign.
  - The chief media officer of the company is less optimistic and assigns a 40% of success to the same campaign
- The assignment of a subjective probability is based on a person's experiences, opinions, and analysis of a particular situation
- Subjective probability is useful in situations when an empirical or a priori probability cannot be computed

Events

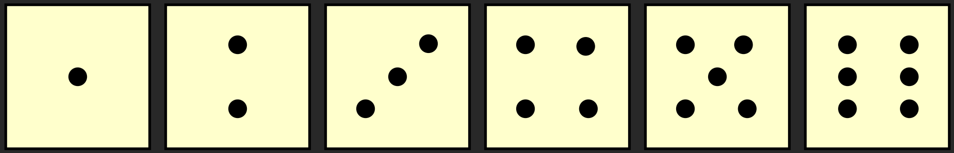
Each possible outcome of a variable is an event.

- Simple event
  - An event described by a single characteristic
  - e.g., A day in January from all days in 2015
- Joint event
  - An event described by two or more characteristics
  - e.g. A day in January that is also a Wednesday from all days in 2015
- Complement of an event A (denoted A')
  - All events that are not part of event A
  - e.g., All days from 2015 that are not in January

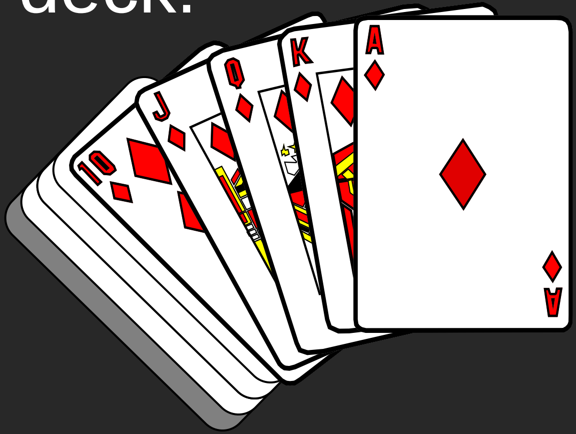
Sample Space

The Sample Space is the collection of all possible events

e.g. All 6 faces of a die:

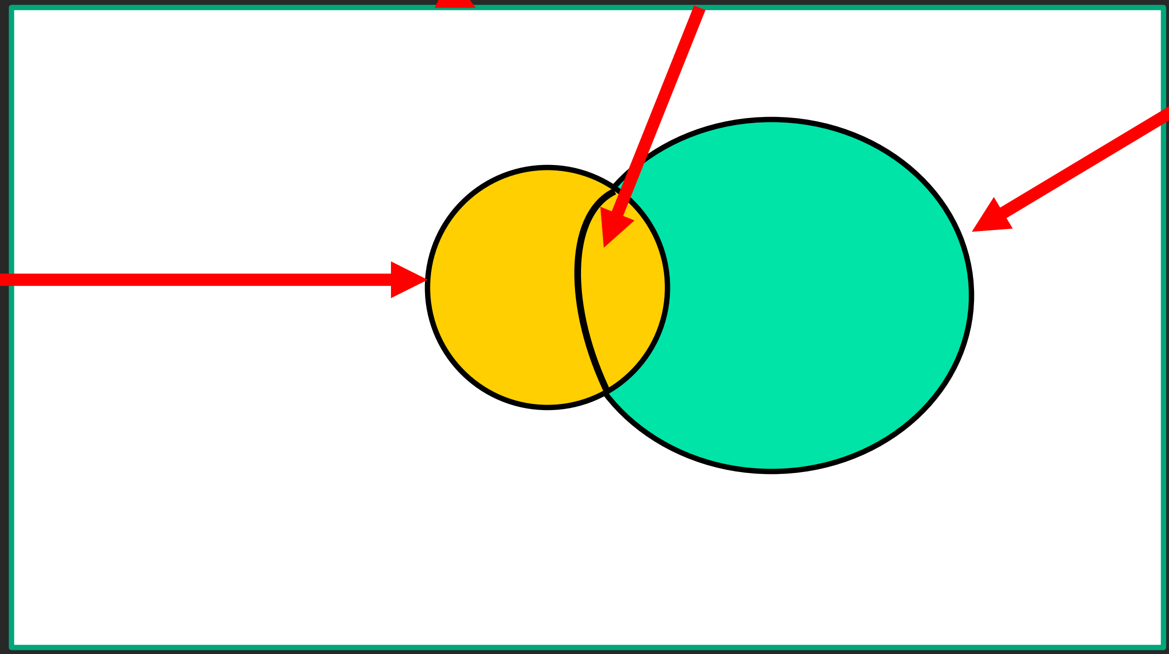


e.g. All 52 cards of a bridge deck:



Organizing & Visualizing Events

- Venn Diagram For All Days In 2015



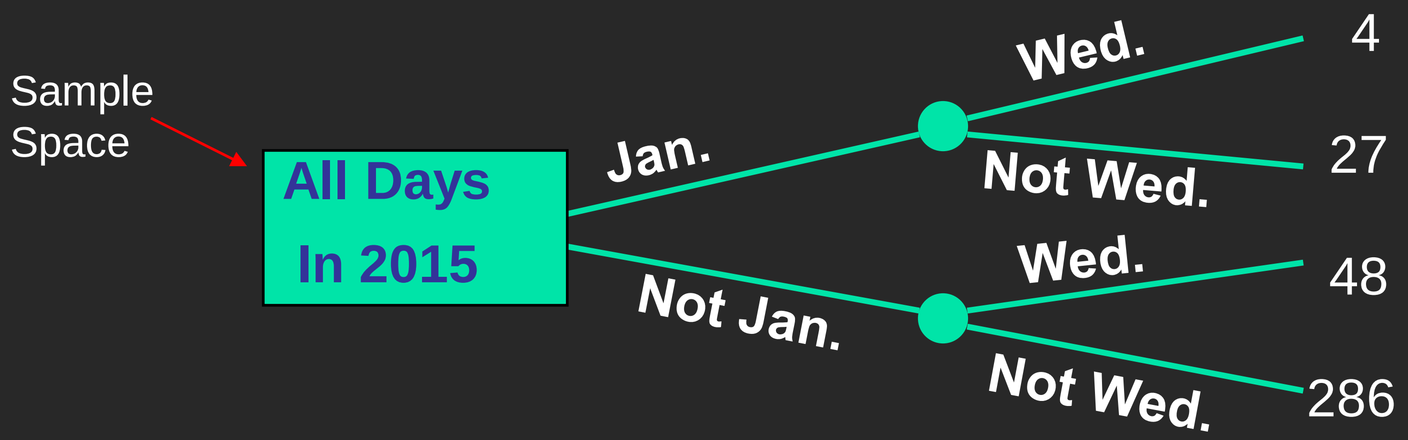
Organizing & Visualizing Events
(continued)

- Contingency Tables -- For All Days in 2015

	Jan.	Not Jan.	Total
Wed.	4	48	52
Not Wed.	27	286	313
Total	31	334	365

Total Number Of Sample Space Outcomes

- Decision Trees



Simple Probability

- Simple Probability refers to the probability of a simple event.
  - ex. P(Jan.)
  - ex. P(Wed.)

	Jan.	Not Jan.	Total
Wed.	4	48	52
Not Wed.	27	286	313
Total	31	334	365

P(Wed.) = 52 / 365

P(Jan.) = 31 / 365

Joint Probability

- Joint Probability refers to the probability of an occurrence of two or more events (joint event).
  - ex. P(Jan. and Wed.)
  - ex. P(Not Jan. and Not Wed.)

	Jan.	Not Jan.	Total
Wed.	4	48	52
Not Wed.	27	286	313
Total	31	334	365

P(Not Jan. and Not Wed.) = 286 / 365

P(Jan. and Wed.) = 4 / 365

Marginal Probability

- The marginal probability of an event consists of a set of joint probabilities.
- Example: the probability that a card drawn is red (p(red) = 0.5).
- Another example: the probability that a card drawn is a 4 (p(four)=4/52=1/13)

Mutually Exclusive Events

- Mutually exclusive events
  - Events that cannot occur simultaneously

Example: Randomly choosing a day from year 2015

A = day in January; B = day in February

- Events A and B are mutually exclusive

Collectively Exhaustive Events

- Collectively exhaustive events
  - One of the events must occur
  - The set of events covers the entire sample space

Example: Randomly choose a day from year 2015

A = Weekday; B = Weekend;
C = January; D = Spring;

  - Events A, B, C and D are collectively exhaustive (but not mutually exclusive – a weekday can be in January or in Spring)
  - Events A and B are collectively exhaustive and also mutually exclusive

Computing Joint and Marginal Probabilities

- The probability of a joint event, A and B:

$$P(\text{A and B}) = \frac{\text{number of outcomes satisfying A and B}}{\text{total number of elementary outcomes}}$$

- Computing a marginal (or simple) probability:

$$P(\text{A}) = P(\text{A and } B_1) + P(\text{A and } B_2) + \cdots + P(\text{A and } B_k)$$

  - Where B_1, B_2, …, B_k are k mutually exclusive and collectively exhaustive events

Joint Probability Example

P(Jan. and Wed.)

$$= \frac{\text{number of days that are in Jan. and are Wed.}}{\text{total number of days in 2015}} = \frac{4}{365}$$

	Jan.	Not Jan.	Total
Wed.	4	48	52
Not Wed.	27	286	313
Total	31	334	365

Marginal Probability Example

P(Wed.)

$$= P(\text{Jan. and Wed.}) + \text{P(Not Jan. and Wed.)} = \frac{4}{365} + \frac{48}{365} = \frac{52}{365}$$

	Jan.	Not Jan.	Total
Wed.	4	48	52
Not Wed.	27	286	313
Total	31	334	365

Marginal & Joint Probabilities In A Contingency Table

Event	Event		Total
	B_1	B_2	
A_1	$P(A_1 \text{ and } B_1)$	$P(A_1 \text{ and } B_2)$	$P(A_1)$
A_2	$P(A_2 \text{ and } B_1)$	$P(A_2 \text{ and } B_2)$	$P(A_2)$
Total	$P(B_1)$	$P(B_2)$	1

Joint Probabilities

Marginal (Simple) Probabilities

Probability Summary So Far

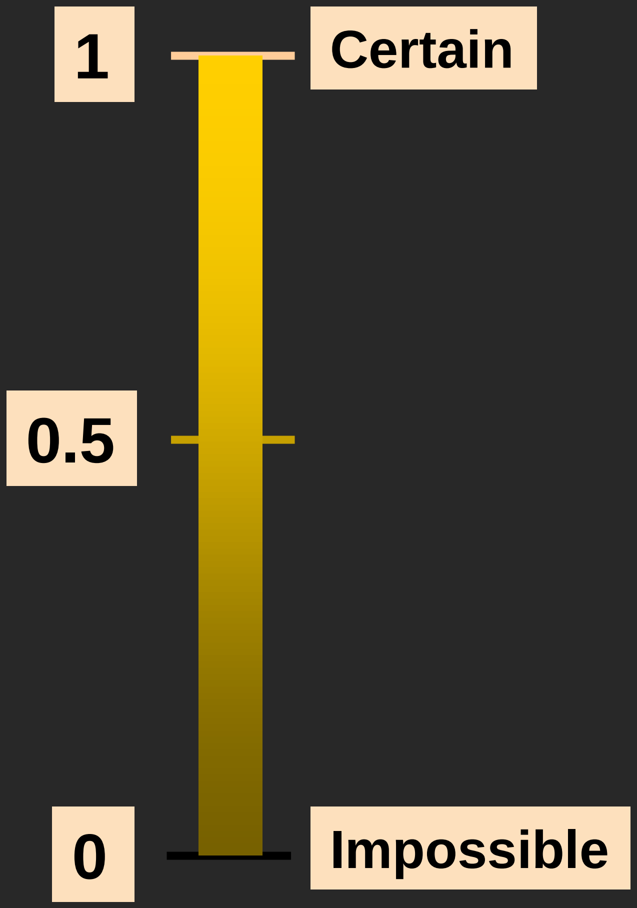
- Probability is the numerical measure of the likelihood that an event will occur
- The probability of any event must be between 0 and 1, inclusively

$$0 \le P(A) \le 1$$ For any event A

- The sum of the probabilities of all mutually exclusive and collectively exhaustive events is 1

$$P(A) + P(B) + P(C) = 1$$

If A, B, and C are mutually exclusive and collectively exhaustive



General Addition Rule

General Addition Rule:

P(A or B) = P(A) + P(B) - P(A and B)

If A and B are mutually exclusive, then P(A and B) = 0, so the rule can be simplified:

P(A or B) = P(A) + P(B)

For mutually exclusive events A and B

General Addition Rule Example

P(Jan. or Wed.) = P(Jan.) + P(Wed.) - P(Jan. and Wed.)

= 31/365 + 52/365 - 4/365 = 79/365

Don't count the four Wednesdays in January twice!

	Jan.	Not Jan.	Total
Wed.	4	48	52
Not Wed.	27	286	313
Total	31	334	365

Conditional Probabilities

- A conditional probability is the probability of one event, given that another event has occurred:

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

→ The conditional probability of A given that B has occurred

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

→ The conditional probability of B given that A has occurred

Where P(A and B) = joint probability of A and B

P(A) = marginal or simple probability of A

P(B) = marginal or simple probability of B

Conditional Probability Example

- Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a GPS. 20% of the cars have both.

- What is the probability that a car has a GPS, given that it has AC ?

i.e., we want to find   P(GPS | AC)

Conditional Probability Example

(continued)

- Of the cars on a used car lot, **70%** have air conditioning (AC) and **40%** have a GPS and **20%** of the cars have both.

	GPS	No GPS	Total
AC	0.2	0.5	0.7
No AC	0.2	0.1	0.3
Total	0.4	0.6	1.0

$$P(\text{GPS} \mid \text{AC}) = \frac{P(\text{GPS and AC})}{P(\text{AC})} = \frac{0.2}{0.7} = 0.2857$$

Conditional Probability Example

(continued)

- Given AC, we only consider the top row (70% of the cars). Of these, 20% have a GPS. 20% of 70% is about 28.57%.

	GPS	No GPS	Total
AC	0.2	0.5	0.7
No AC	0.2	0.1	0.3
Total	0.4	0.6	1.0

$$P(\text{GPS} \mid \text{AC}) = \frac{P(\text{GPS and AC})}{P(\text{AC})} = \frac{0.2}{0.7} = 0.2857$$

Independence

- Two events are independent if and only if:

$$P(A \mid B) = P(A)$$

- Events A and B are independent when the probability of one event is not affected by the fact that the other event has occurred

Multiplication Rules

- Multiplication rule for two events A and B:

$$P(A \text{ and } B) = P(A\,|\,B)\,P(B)$$

Note: If A and B are independent, then $P(A\,|\,B) = P(A)$ and the multiplication rule simplifies to

$$P(A \text{ and } B) = P(A)\,P(B)$$

Marginal Probability

- Marginal probability for event A:

$$P(A) = P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \cdots + P(A \mid B_k)P(B_k)$$

  - Where B_1, B_2, …, B_k are k mutually exclusive and collectively exhaustive events

Counting Rules Are Often Useful In Computing Probabilities

- In many cases, there are a large number of possible outcomes.

- Counting rules can be used in these cases to help compute probabilities.

Counting Rules

- Rules for counting the number of possible outcomes
- Counting Rule 1:
  - If any one of k different mutually exclusive and collectively exhaustive events can occur on each of n trials, the number of possible outcomes is equal to

$$k^n$$

  - Example
    - If you roll a fair die 3 times then there are $6^3 = 216$ possible outcomes

Counting Rules

(continued)

- Counting Rule 2:
  - If there are k_1 events on the first trial, k_2 events on the second trial, … and k_n events on the n^{th} trial, the number of possible outcomes is

$$(k_1)(k_2)\cdots(k_n)$$

  - Example:
    - You want to go to a park, eat at a restaurant, and see a movie. There are 3 parks, 4 restaurants, and 6 movie choices. How many different possible combinations are there?
    - Answer: (3)(4)(6) = 72 different possibilities

Counting Rules

(continued)

- Counting Rule 3:
  - The number of ways that n items can be arranged in order is

$$n! = (n)(n-1)\cdots(1)$$

  - Example:
    - You have five books to put on a bookshelf. How many different ways can these books be placed on the shelf?
    - Answer: $5! = (5)(4)(3)(2)(1) = 120$ different possibilities

Counting Rules

(continued)

- Counting Rule 4:
  - Permutations: The number of ways of arranging X objects selected from n objects in order is

$$ {}_nP_x = \frac{n!}{(n-X)!} $$

  - Example:
    - You have five books and are going to put three on a bookshelf. How many different ways can the books be ordered on the bookshelf?
    - Answer: $ {}_nP_x = \frac{n!}{(n-X)!} = \frac{5!}{(5-3)!} = \frac{120}{2} = 60 \text{ different possibilities}$

Counting Rules

(continued)

- Counting Rule 5:
  - Combinations: The number of ways of selecting X objects from n objects, irrespective of order, is

$$_{n}C_{x} = \frac{n!}{X!(n-X)!}$$

  - Example:
    - You have five books and are going to select three are to read. How many different combinations are there, ignoring the order in which they are selected?
    - Answer: $_{n}C_{x} = \frac{n!}{X!(n-X)!} = \frac{5!}{3!(5-3)!} = \frac{120}{(6)(2)} = 10$ different possibilities

Chapter Summary

In this chapter we covered:

- Understanding basic probability concepts.
- Understanding conditional probability
- Various counting rules