GLOBAL EDITION

Chapter 6

The Normal Distribution

Business Statistics

A First Course

SEVENTH EDITION

David M. Levine • Kathryn A. Szabat • David F. Stephan

Objectives

In this chapter, you learn:

- To compute probabilities from the normal distribution

- How to use the normal distribution to solve business problems

- To use the normal probability plot to determine whether a set of data is approximately normally distributed

Continuous Probability Distributions

- A continuous variable is a variable that can assume any value on a continuum (can assume an uncountable number of values)
  - thickness of an item
  - time required to complete a task
  - temperature of a solution
  - height, in inches

- These can potentially take on any value depending only on the ability to precisely and accurately measure
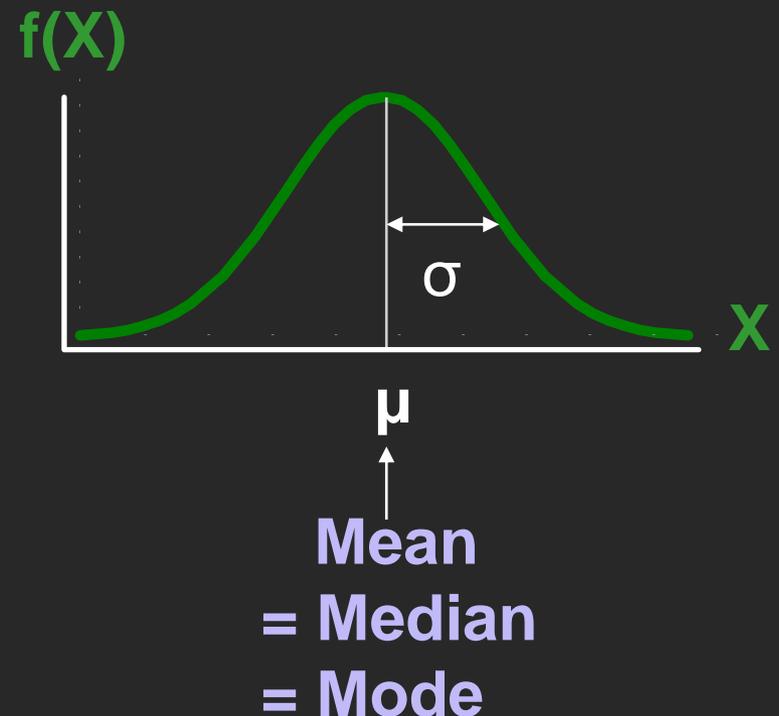
The Normal Distribution

- Bell Shaped
- Symmetrical
- Mean, Median and Mode
  are Equal

Location is determined by the mean, μ

Spread is determined by the standard deviation, σ

The random variable has an infinite theoretical range:
+ ∞ to − ∞

f(X)

σ

μ

X

Mean
= Median
= Mode

The Normal Distribution Density Function

- The formula for the normal probability density function is

$$f(X) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{(X-\mu)}{\sigma}\right)^2}$$

Where   e = the mathematical constant approximated by 2.71828
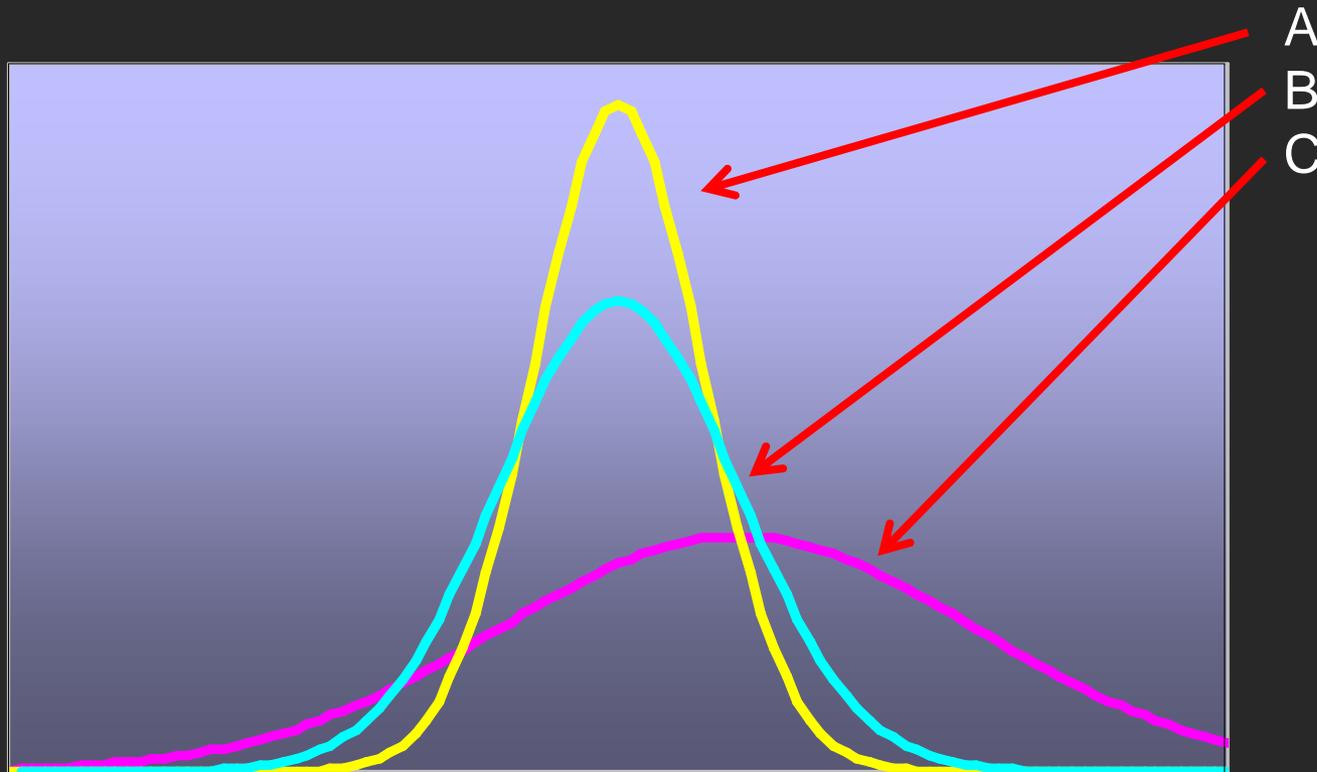
π = the mathematical constant approximated by 3.14159

μ = the population mean

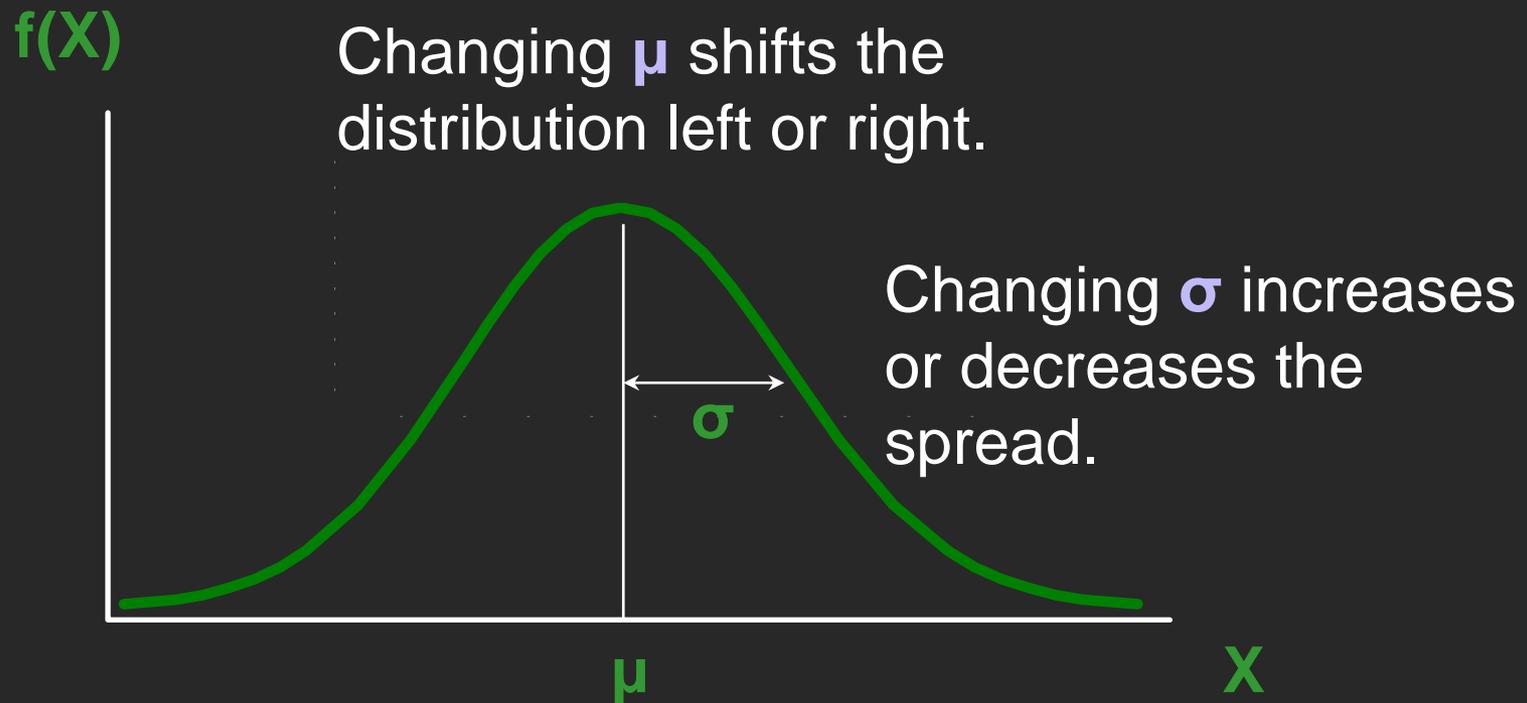σ = the population standard deviation

X = any value of the continuous variable

By varying the parameters μ and σ, we obtain different normal distributions

A and B have the same mean but different standard deviations.

B and C have different means and different standard deviations.

The Normal Distribution Shape

f(X)

Changing **μ** shifts the distribution left or right.

Changing **σ** increases or decreases the spread.

σ

μ

X

The Standardized Normal

- **Any** normal distribution (with any mean and standard deviation combination) can be transformed into the standardized normal distribution (Z)

- To compute normal probabilities need to transform X units into Z units

- The standardized normal distribution (Z) has a mean of 0 and a standard deviation of 1

Translation to the Standardized Normal Distribution

- Translate from X to the standardized normal (the "Z" distribution) by subtracting the mean of X and dividing by its standard deviation:

$$Z = \frac{X - \mu}{\sigma}$$

The Z distribution always has mean = 0 and standard deviation = 1

The Standardized Normal Probability Density Function

- The formula for the standardized normal probability density function is
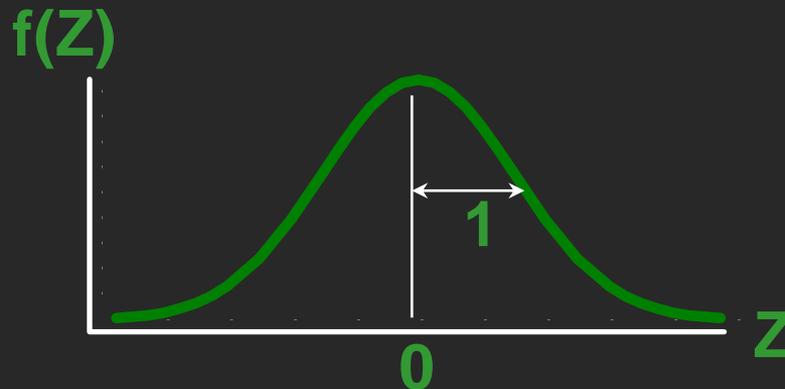
$$f(Z) = \frac{1}{\sqrt{2\pi}} e^{-(1/2)Z^2}$$

Where e = the mathematical constant approximated by 2.71828

π = the mathematical constant approximated by 3.14159

Z = any value of the standardized normal distribution

The Standardized Normal Distribution

- Also known as the "Z" distribution

- Mean is 0

- Standard Deviation is 1

Values above the mean have positive Z-values.
Values below the mean have negative Z-values.

Example

- If X is distributed normally with mean of $100 and standard deviation of $50, the Z value for X = $200 is

$$Z = \frac{X - \mu}{\sigma} = \frac{\$200 - \$100}{\$50} = 2.0$$

- This says that X = $200 is two standard deviations (2 increments of $50 units) above the mean of $100.

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Comparing X and Z units

$100 $200 $X (μ = $100, σ = $50)
  0   2.0   Z (μ = 0, σ = 1)

Note that the shape of the distribution is the same, only the scale has changed. We can express the problem in the original units (X in dollars) or in standardized units (Z)
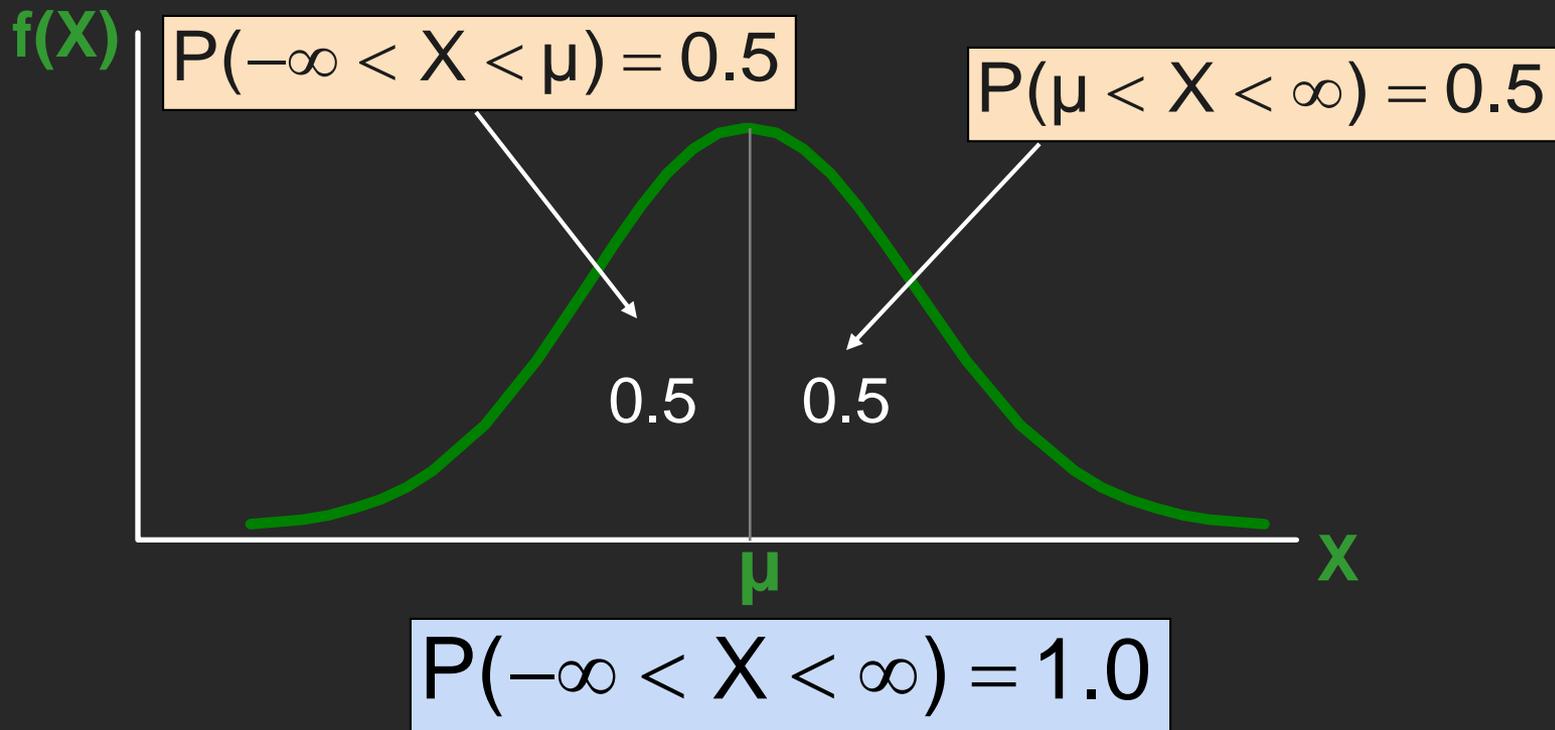
Finding Normal Probabilities

Probability is measured by the area under the curve

f(X)

$$P(a \leq X \leq b)$$

$$= P(a < X < b)$$

(Note that the probability of any individual value is zero)

a b

X
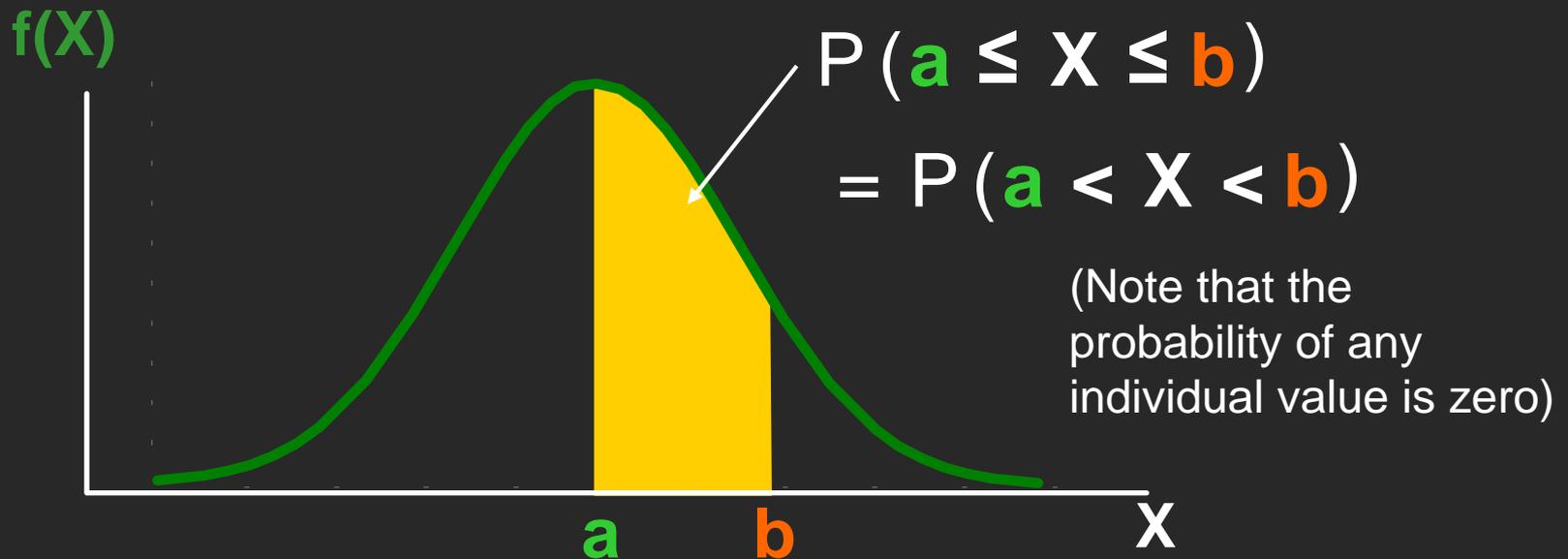
Probability as Area Under the Curve

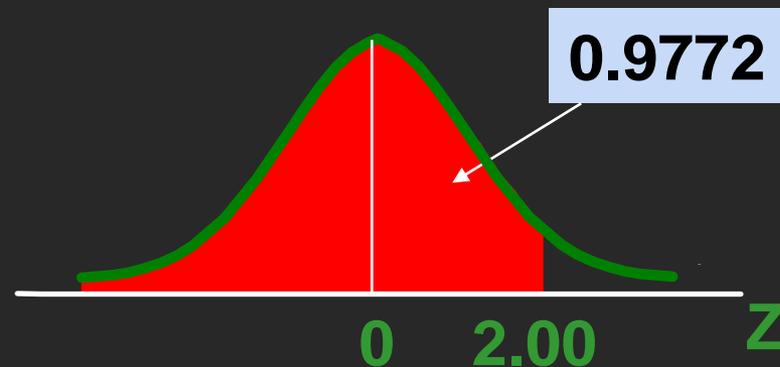The total area under the curve is 1.0, and the curve is symmetric, so half is above the mean, half is below

$$P(-\infty < X < \mu) = 0.5$$

$$P(\mu < X < \infty) = 0.5$$

0.5   0.5

$$P(-\infty < X < \infty) = 1.0$$

The Standardized Normal Table

- The Cumulative Standardized Normal table in the textbook (Appendix table E.2) gives the probability less than a desired value of Z (i.e., from negative infinity to Z)

Example:

P(Z < 2.00) = 0.9772

0.9772

0   2.00   Z

The Standardized Normal Table

(continued)

The column gives the value of Z to the second decimal point

Z	0.00	0.01	0.02 …

The row shows the value of Z to the first decimal point

0.0	
0.1	
.	
.	
.	
2.0	.9772

The value within the table gives the probability from Z = – ∞ up to the desired Z value

P(Z < 2.00) = 0.9772

General Procedure for Finding Normal Probabilities

To find P(a < X < b) when X is distributed normally:

- Draw the normal curve for the problem in terms of X

- Translate X-values to Z-values

- Use the Standardized Normal Table

Finding Normal Probabilities

- Let X represent the time it takes (in seconds) to download an image file from the internet.

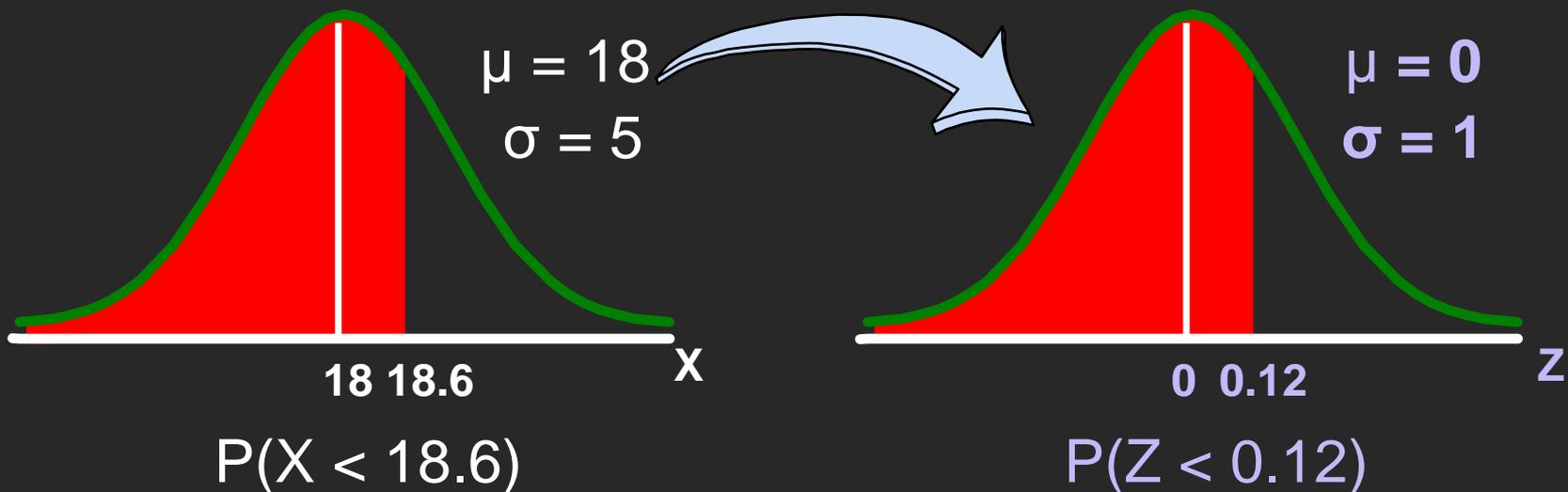- Suppose X is normal with a mean of 18.0 seconds and a standard deviation of 5.0 seconds. Find $P(X < 18.6)$

18.0

18.6

X
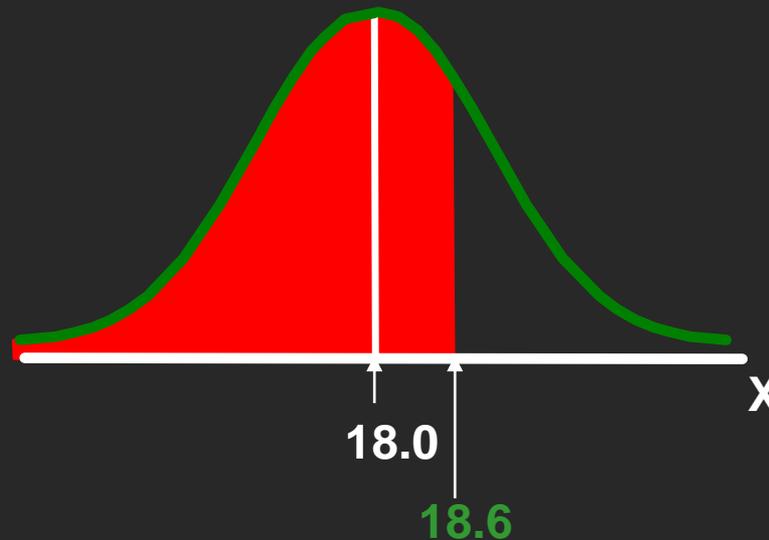
Finding Normal Probabilities

- Let X represent the time it takes, in seconds to download an image file from the internet.
- Suppose X is normal with a mean of 18.0 seconds and a standard deviation of 5.0 seconds. Find P(X < 18.6)

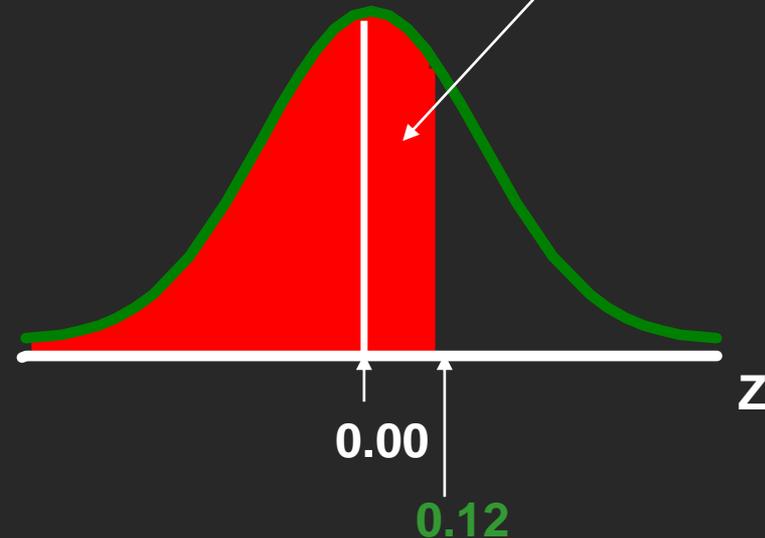$$Z = \frac{X - \mu}{\sigma} = \frac{18.6 - 18.0}{5.0} = 0.12$$

μ = 18
σ = 5

μ = 0
σ = 1

18 18.6   X

0 0.12   Z

P(X < 18.6)

P(Z < 0.12)

Solution: Finding P(Z < 0.12)

Standardized Normal Probability Table (Portion)

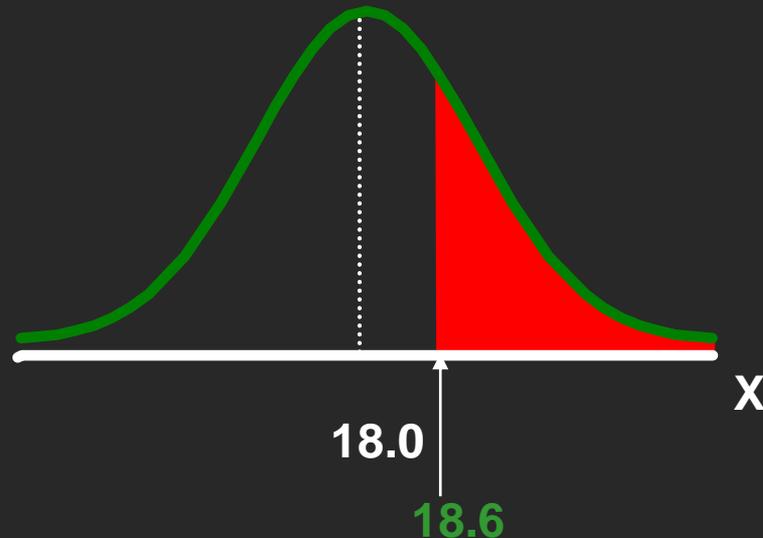| Z	.00	.01	.02
0.0	.5000	.5040	.5080
0.1	.5398	.5438	.5478
0.2	.5793	.5832	.5871
0.3	.6179	.6217	.6255

P(X < 18.6)
= P(Z < 0.12)

0.5478

z

0.00

0.12

Finding Normal Upper Tail Probabilities

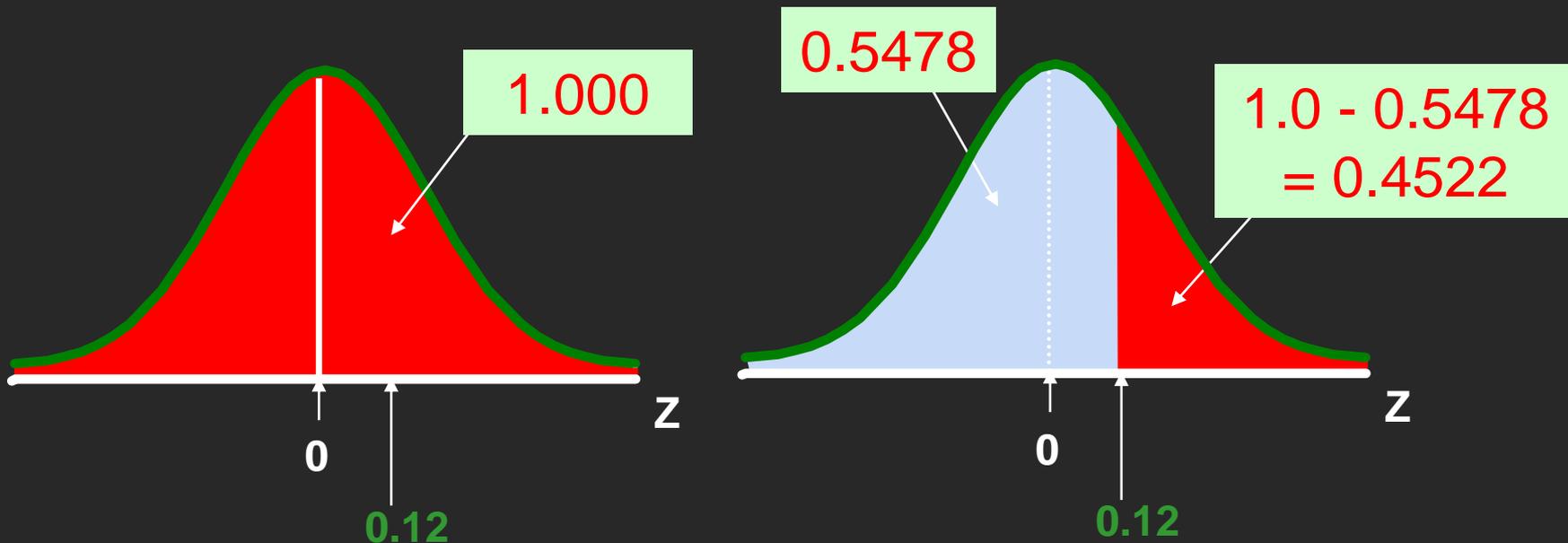- Suppose X is normal with mean 18.0 and standard deviation 5.0.

- Now Find P(X > 18.6)

18.0

18.6

X

Finding Normal Upper Tail Probabilities

■ Now Find P(X > 18.6)…

P(X > 18.6) = P(Z > 0.12) = 1.0 - P(Z ≤ 0.12)

= 1.0 - 0.5478 = 0.4522

1.000

0.5478

1.0 - 0.5478
= 0.4522

Z

0

0.12

Z

0

0.12

Finding a Normal Probability Between Two Values

- Suppose X is normal with mean 18.0 and standard deviation 5.0. Find P(18 < X < 18.6)

Calculate Z-values:

$$Z = \frac{X - \mu}{\sigma} = \frac{18 - 18}{5} = 0$$

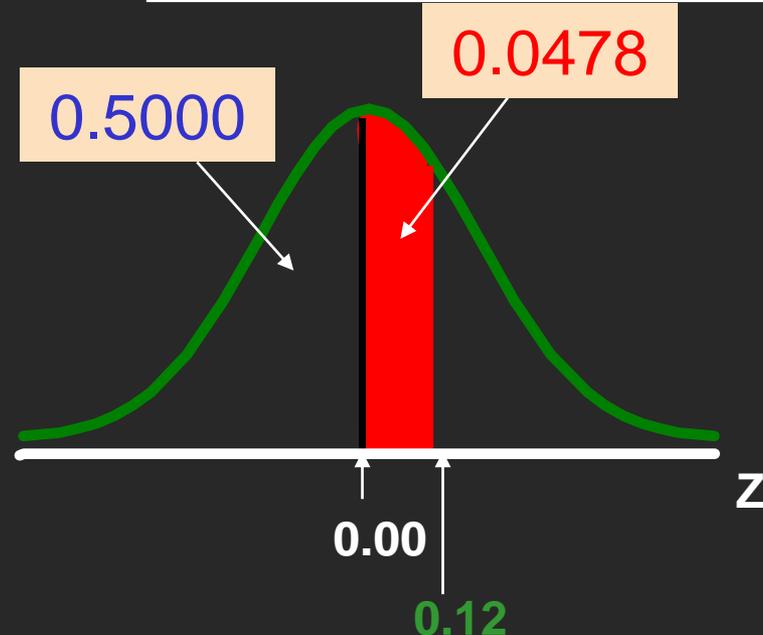$$Z = \frac{X - \mu}{\sigma} = \frac{18.6 - 18}{5} = 0.12$$

18	18.6	X
0	0.12	Z

P(18 < X < 18.6)

= P(0 < Z < 0.12)

Solution: Finding P(0 < Z < 0.12)
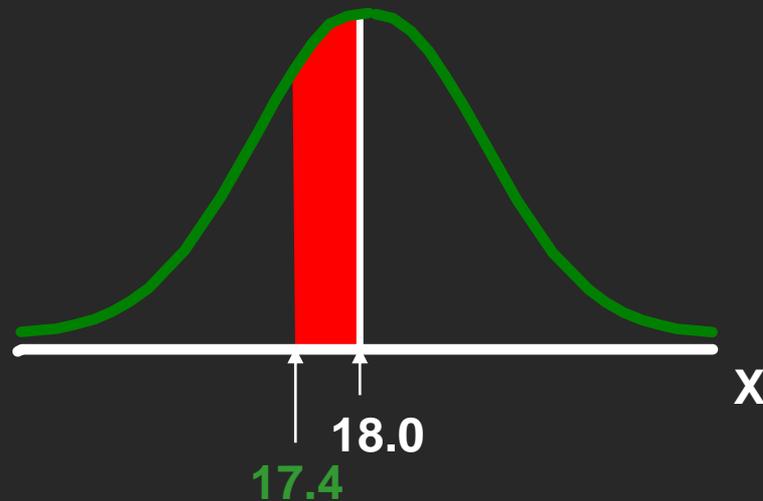
Standardized Normal Probability Table (Portion)

Z	.00	.01	.02
0.0	.5000	.5040	.5080
0.1	.5398	.5438	.5478
0.2	.5793	.5832	.5871
0.3	.6179	.6217	.6255

P(18 < X < 18.6)
= P(0 < Z < 0.12)
= P(Z < 0.12) − P(Z ≤ 0)
= 0.5478 - 0.5000 = 0.0478

0.0478

0.5000

0.00

0.12

z

Probabilities in the Lower Tail

- Suppose X is normal with mean 18.0 and standard deviation 5.0.
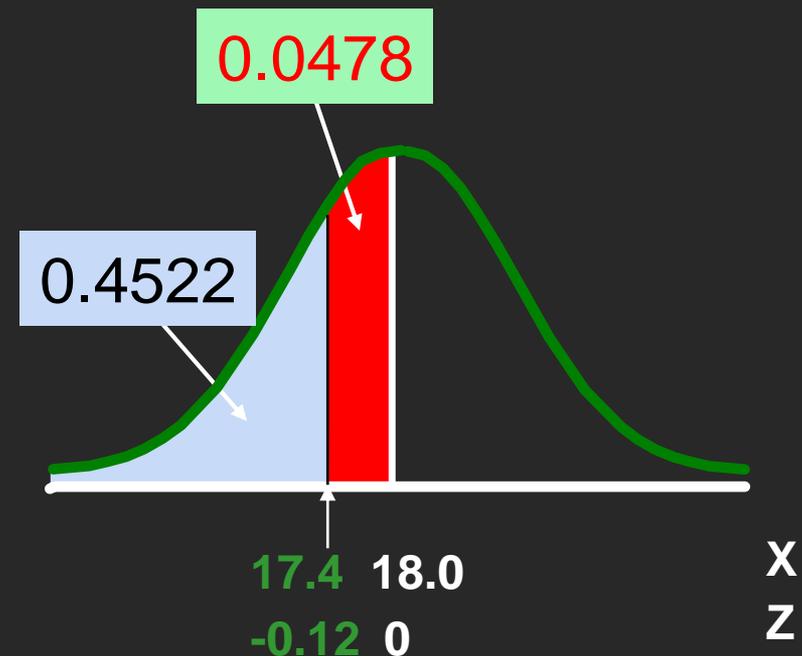- Now Find P(17.4 < X < 18)

X

18.0

17.4

Probabilities in the Lower Tail

Now Find P(17.4 < X < 18)…

P(17.4 < X < 18)

= P(-0.12 < Z < 0)

= P(Z < 0) − P(Z ≤ -0.12)

= 0.5000 - 0.4522 = 0.0478

0.0478

0.4522

The Normal distribution is symmetric, so this probability is the same as P(0 < Z < 0.12)

17.4 **18.0**
-0.12 **0**

X
Z

Empirical Rule
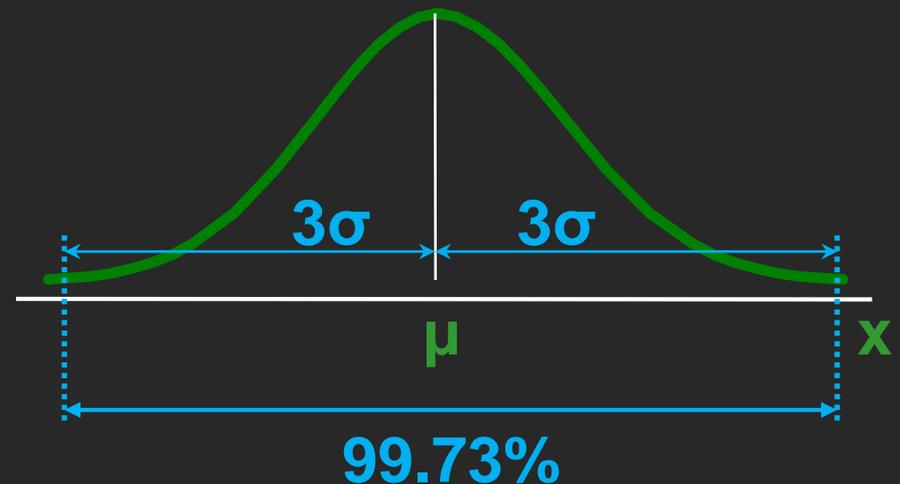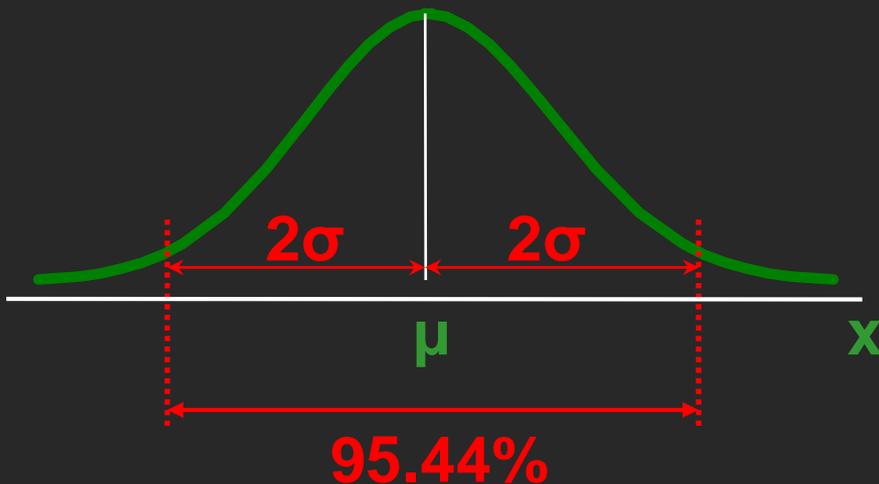
What can we say about the distribution of values around the mean? For any normal distribution:

$\mu \pm 1\sigma$ encloses about 68.26% of X's
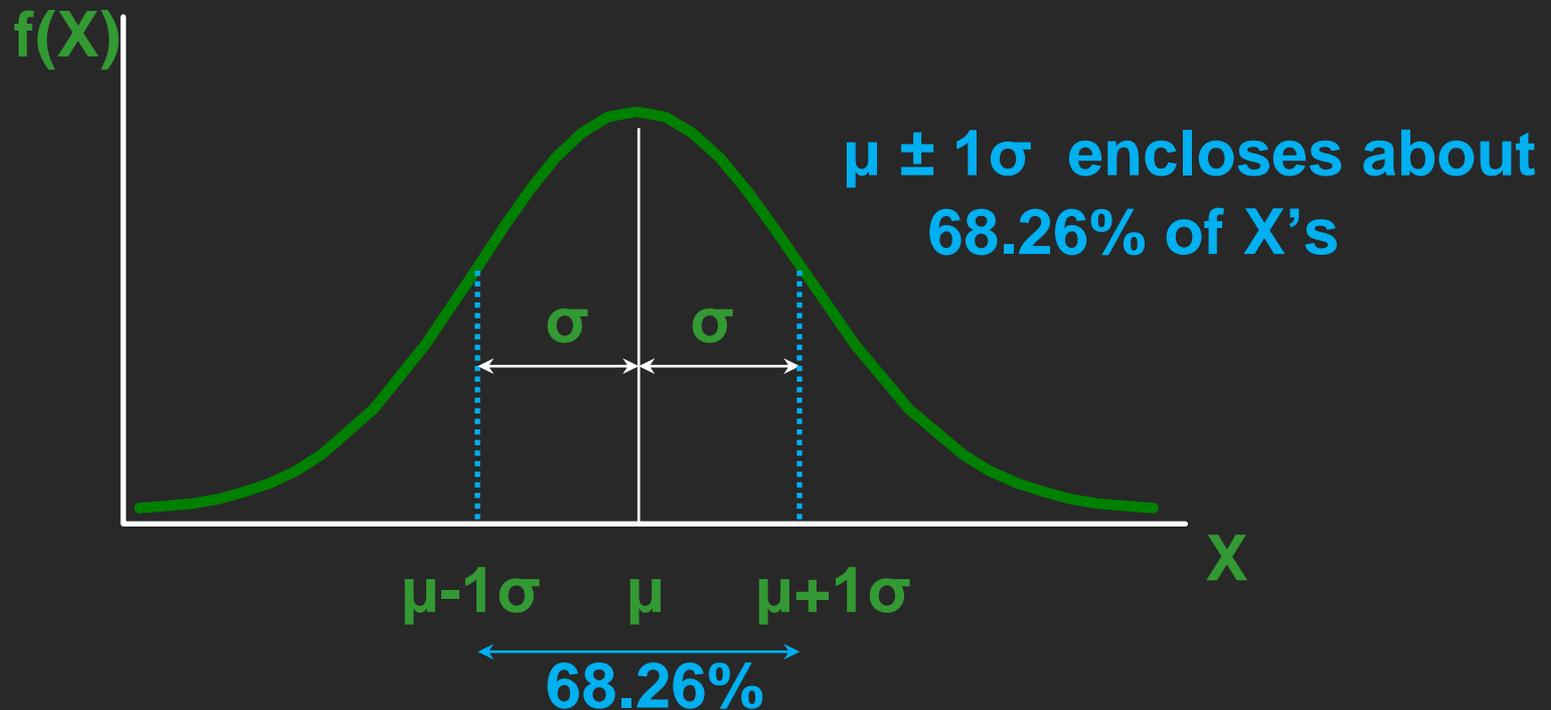
The Empirical Rule

- μ ± **2σ** covers about **95.44%** of X's

- μ ± **3σ** covers about **99.73%** of X's

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Chapter 6, Slide 29

Given a Normal Probability Find the X Value

- Steps to find the X value for a known probability:

  1. Find the Z value for the known probability
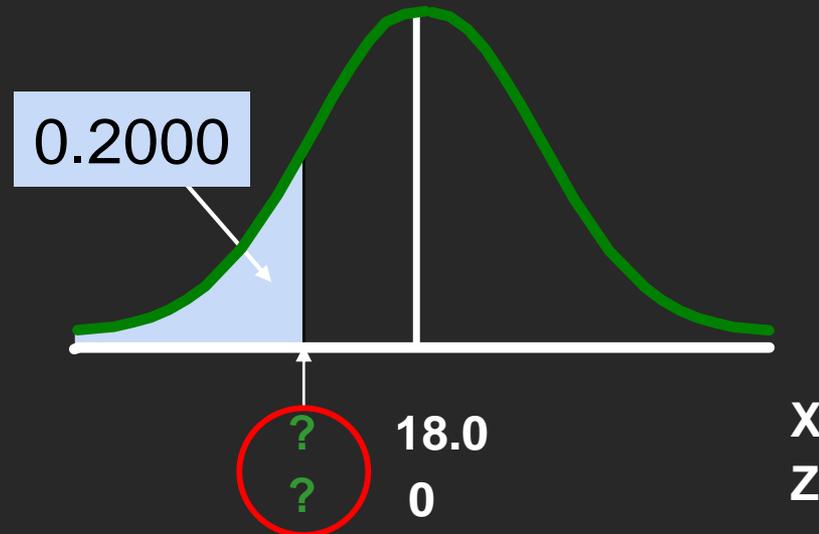
  2. Convert to X units using the formula:

$$X = \mu + Z\sigma$$

Finding the X value for a Known Probability

(continued)

Example:

- Let X represent the time it takes (in seconds) to download an image file from the internet.

- Suppose X is normal with mean 18.0 and standard deviation 5.0

- Find X such that 20% of download times are less than X.

0.2000
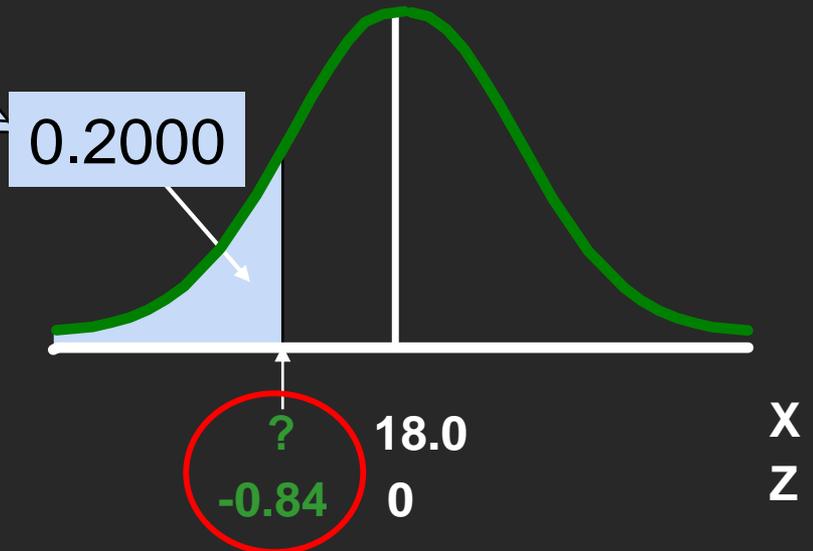
? ?   **18.0**   **X**

0   **Z**

Find the Z value for 20% in the Lower Tail

1. Find the Z value for the known probability

Standardized Normal Probability Table (Portion)

■ 20% area in the lower tail is consistent with a Z value of -0.84

Z	…	.03	.04	.05
-0.9	…	.1762	.1736	.1711
-0.8	…	.2033	.2005	.1977
-0.7	…	.2327	.2296	.2266

0.2000

? 18.0
-0.84 0

X
Z

Finding the X value

2. Convert to X units using the formula:

$$X = \mu + Z\sigma$$
$$= 18.0 + (-0.84)5.0$$
$$= 13.8$$

So 20% of the values from a distribution with mean 18.0 and standard deviation 5.0 are less than 13.80

Chapter Summary

In this chapter we discussed:

- Computing probabilities from the normal distribution

- Using the normal distribution to solve business problems

- Using the normal probability plot to determine whether a set of data is approximately normally distributed