Homogeneous Differential Equation

A first order DE

$$\frac{dy}{dx} = f(x, y)$$

is said to be homogeneous if the function f is homogeneous of degree zero.

While the DE $M(x, y)dx + N(x, y)dy = 0$

is homogeneous if both M and N are homogeneous functions of the same degree.

A homogeneous DE is solved by reducing it to a separable DE using one of the substitutions

$$y = ux \Rightarrow dy = udx + xdu$$

$$or \quad x = vy \Rightarrow dx = vdy + ydv$$

Example 1

Solve the DE $xydy + (x^2 - y^2)dx = 0$.

The DE is homogeneous.

Let $y = ux \Rightarrow dy = udx + xdu$

$$\Rightarrow ux^2(udx + xdu) + (x^2 - u^2x^2)dx = 0$$

$$\Rightarrow ux^3du = -x^2dx$$

$$\Rightarrow udu = -\frac{1}{x}dx$$

$$\Rightarrow \frac{1}{2}u^2 = -\ln x + c_1$$

Thus, the solution of the DE is given by

$$y^2 = -2x^2 \ln x + cx^2$$

Example 2

Solve the following differential equation.

$$x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}\,,\ x > 0.$$

Solution.

The DE is homogeneous.

Thus, let $y = ux \Rightarrow dy = udx + xdu$

Hence, we have

$$xudx + x^2du = \left(xu + x\sqrt{1+u^2}\right)dx$$

$$\Rightarrow \frac{1}{\sqrt{1+u^2}}du = \frac{1}{x}\,dx \ \Rightarrow \int\frac{1}{\sqrt{1+u^2}}du = \int\frac{1}{x}dx$$

$$\Rightarrow \sinh^{-1}u = \ln x + c \ \Rightarrow \sinh^{-1}\left(\tfrac{y}{x}\right) = \ln x + c\,.$$

Example 3

Solve the initial value problem

$$x\frac{dy}{dx} = y + xe^{\frac{y}{x}}, \;\; y(1) = 1.$$

Solution

The DE is homogeneous, thus, let

$$y = ux \Rightarrow dy = udx + xdu.$$

Hence, we have

$$xudx + x^2du = \left(xu + xe^u\right)dx$$

$$\Rightarrow e^{-u}du = \frac{1}{x}\,dx$$

$$\Rightarrow -e^{-u} = \ln|x| + c$$

$$\Rightarrow -e^{\frac{-y}{x}} = \ln|x| + c$$

Since $y(1) = 1 \Rightarrow c = -e^{-1}$

Hence , the solution is

$$\Rightarrow -e^{\frac{-y}{x}} = \ln|x| - e^{-1}$$

Homework

Solve the following DE

$$\frac{dy}{dx}+\frac{x+y}{x-y}=0$$

Differential Equations with linear coefficients

Consider the first order DE

$$\frac{dy}{dx}=\frac{ax+by+c}{hx+ky+l},$$

where a,b,c,h,k,l are constants and $h,\ k\neq 0.$

If $\frac{a}{h}=\frac{b}{k}$, then the above differential equation can be reduced to a separable DE using the substitution $u=ax+by$ *or* $v=hx+ky.$

If $\frac{a}{h}\neq\frac{b}{k}$, then it can be converted to a homogeneous DE as follows:

Put $ax + by + c = 0$

and $hx + ky + l = 0$

then, solve these two equations simultaneously, assume the solution is $x = x_0$ *and* $y = y_0$.

Now, let

$$x = X + x_0 \quad and \quad y = Y + y_0$$

$$\Rightarrow dy = dY, \; dx = dX.$$

This substitution will reduce the equation to a homogeneous DE, then it can be solved by reducing it to a separable DE.

Example 1

Solve the DE: $\frac{dy}{dx} = \frac{2+x+y}{1-2x-2y}$

Here $a = b = 1,\ h = k = -2.$ Hence $\frac{a}{h} = \frac{b}{k} = \frac{1}{-2}$, therefore put

$$u = x + y \Rightarrow \frac{dy}{dx} = \frac{du}{dx} - 1.$$

Hence the DE becomes

$$\frac{du}{dx} - 1 = \frac{2+u}{1-2u} \Rightarrow \frac{du}{dx} = \frac{3-u}{1-2u}$$

Which is a separable DE.

Example 2

Solve the DE: $\dfrac{dy}{dx} = \dfrac{3+x+y}{1-x+y}$

Here $a = b = 1,\ \ h = -1,\ k = 1.$ Hence $\dfrac{a}{h} \neq \dfrac{b}{k}$, therefore, solve the two equations $3+x+y=0 \ \ and \ \ 1-x+y=0$

to obtain $x = -1,\ \ y = -2.$

Now let $x = X + (-1) \ \ and \ \ y = Y + (-2)$.

Hence the DE becomes

$$\frac{dY}{dX} = \frac{X+Y}{-X+Y}$$

Which is a homogeneous DE.