Control Systems

Time Domain Analysis of 2nd Order Systems

Introduction

- We have already discussed the affect of location of poles and zeros on the transient response of 1^{st} order systems.

- Compared to the simplicity of a first-order system, a second-order system exhibits a wide range of responses that must be analyzed and described.

- Varying a first-order system's parameter (T, K) simply changes the speed and offset of the response

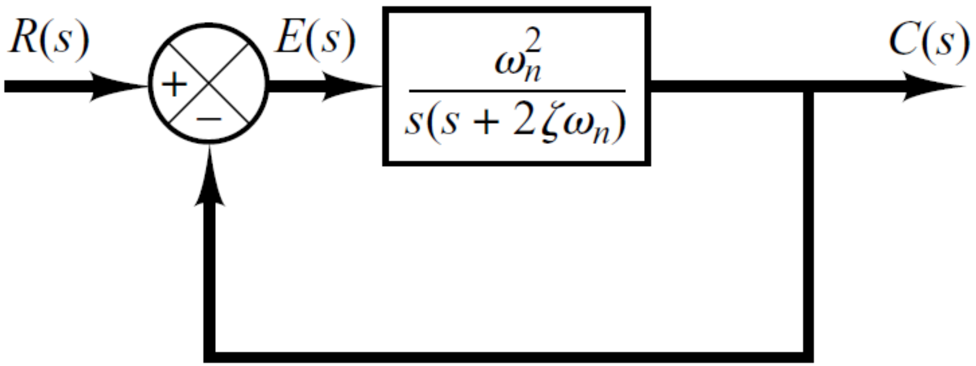
- Whereas, changes in the parameters of a second-order system can change the *form of* the response.

- A second-order system can display characteristics much like a first-order system or, depending on component values, display damped or *pure oscillations* for its *transient response*.

Introduction

- A general second-order system is characterized by the following transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



ω_n ⟶ un-damped natural frequency of the second order system, which is the frequency of oscillation of the system without damping.

ζ ⟶ damping ratio of the second order system, which is a measure of the degree of resistance to change in the system output.

Example#1

- Determine the un-damped natural frequency and damping ratio of the following second order system.

$$\frac{C(s)}{R(s)} = \frac{4}{s^2 + 2s + 4}$$

- Compare the numerator and denominator of the given transfer function with the general 2nd order transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n^2 = 4 \quad \Rightarrow \omega_n = 2\ rad/\sec$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2s + 4$$

$$\Rightarrow 2\zeta\omega_n s = 2s$$

$$\Rightarrow \zeta\omega_n = 1$$

$$\Rightarrow \zeta = 0.5$$

Introduction

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Two poles of the system are

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

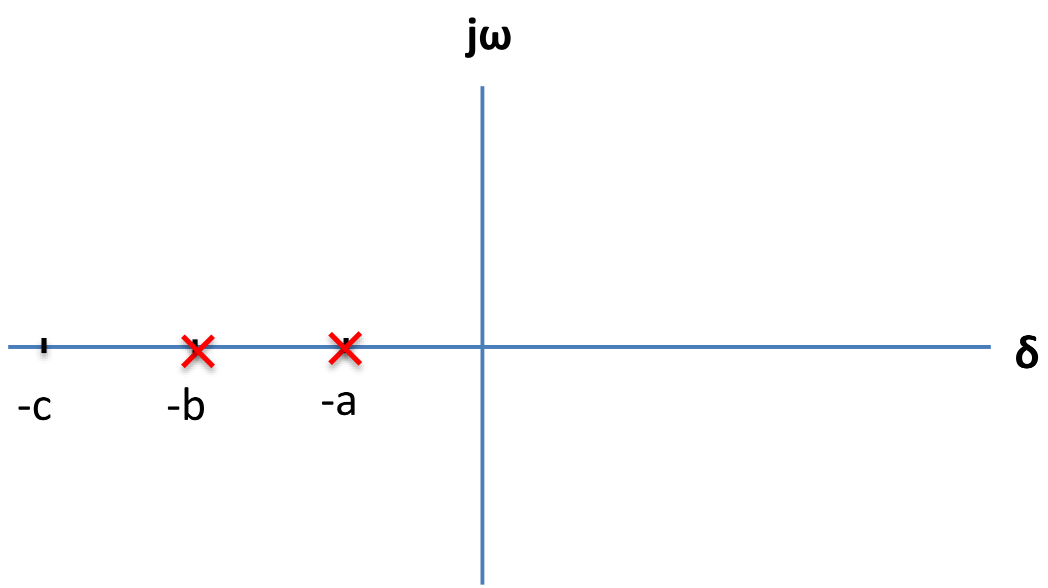
Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories:

  1. Overdamped - when the system has two real distinct poles (ζ >1).



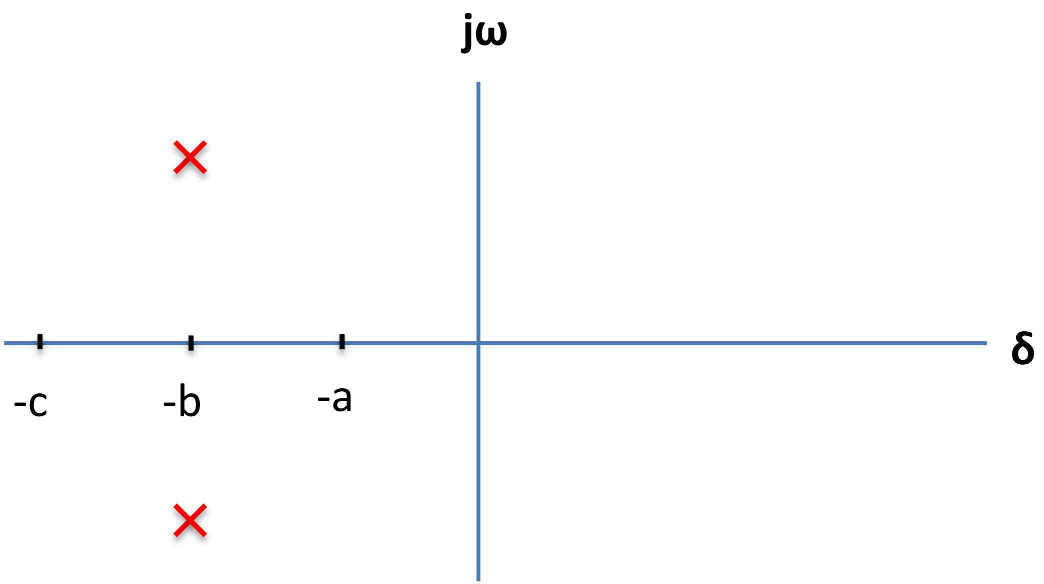
Introduction

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories:

2. *Underdamped* - when the system has two complex conjugate poles ($0 < \zeta < 1$)



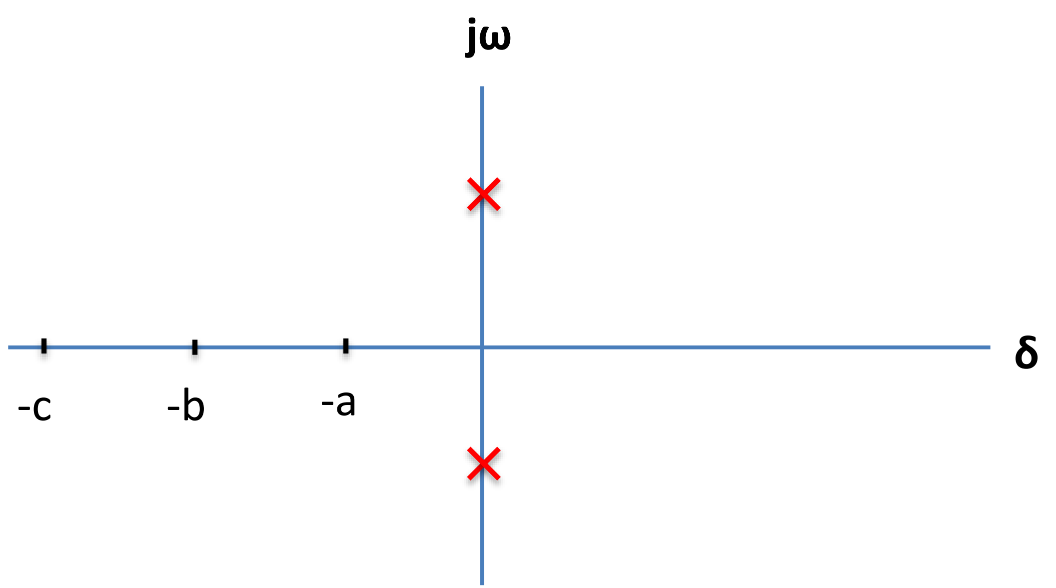
Introduction

$$-\omega_n\zeta+\omega_n\sqrt{\zeta^2-1}$$

$$-\omega_n\zeta-\omega_n\sqrt{\zeta^2-1}$$

- According the value of ζ, a second-order system can be set into one of the four categories:

3. *Undamped* - when the system has two imaginary poles (ζ = 0).



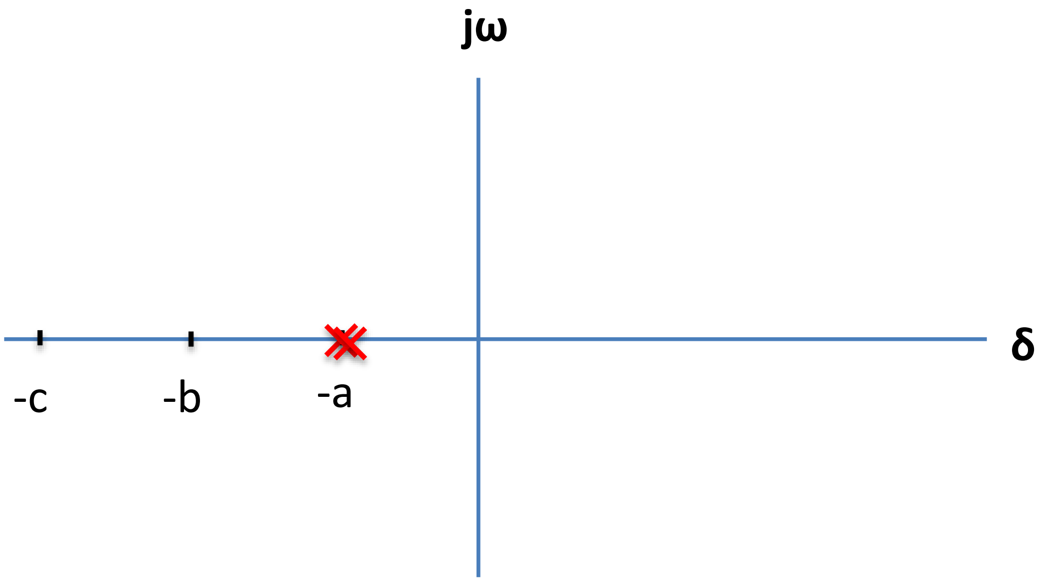
Introduction

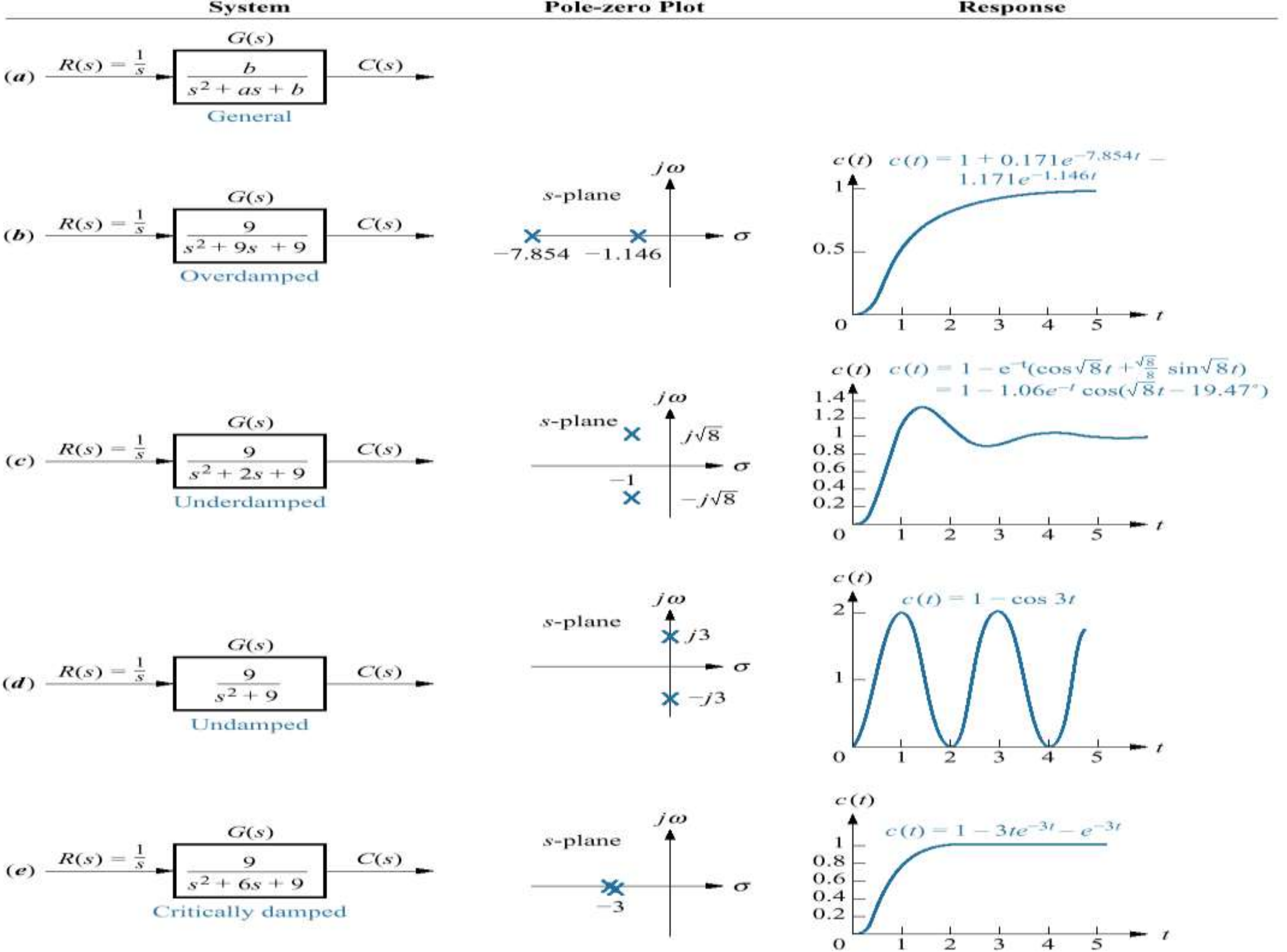
$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$

- According the value of ζ , a second-order system can be set into one of the four categories:

4. *Critically damped* - when the system has two real but equal poles (ζ = 1).

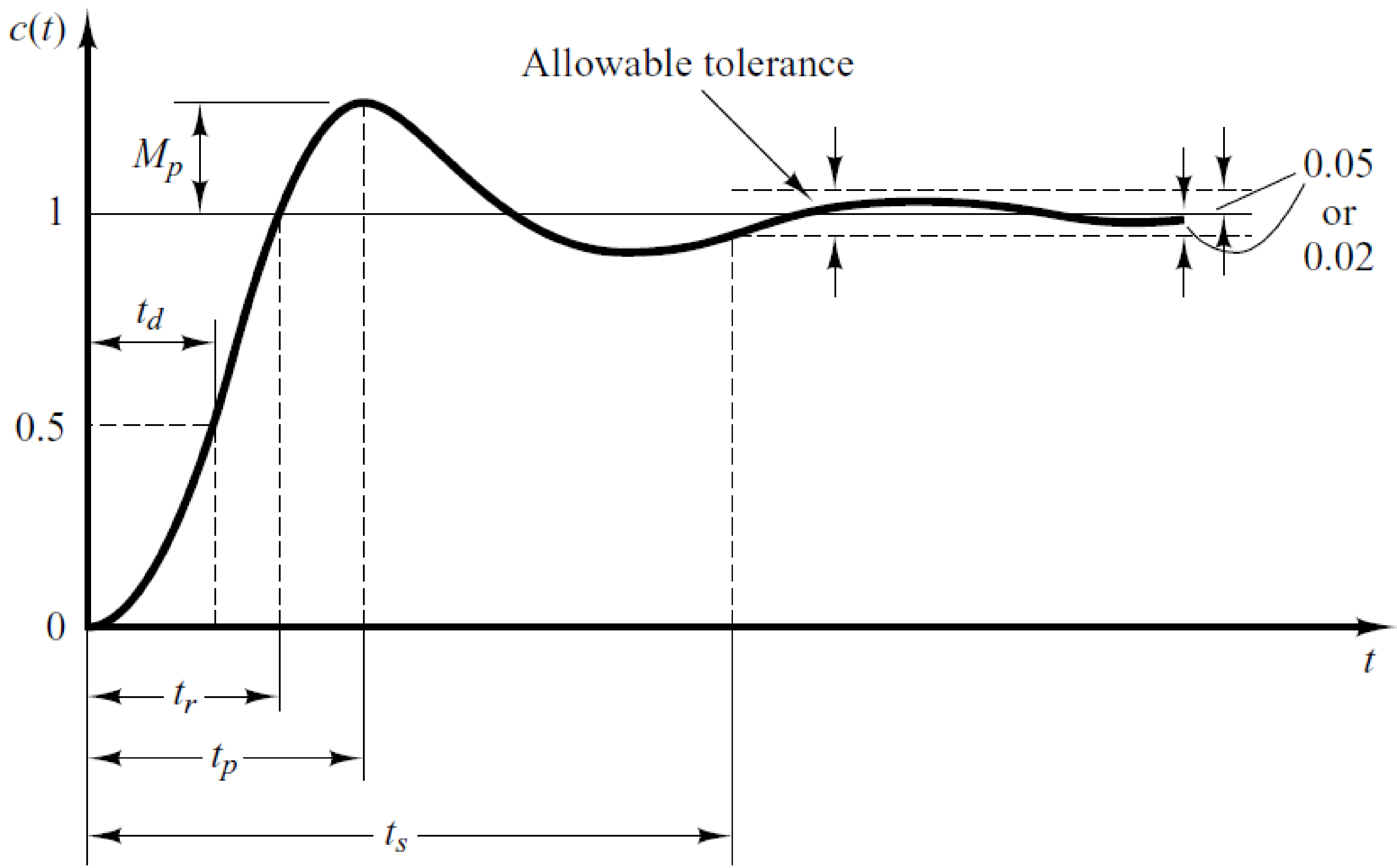




System
Pole-zero Plot
Response
(a)
R(s) = 1/s
G(s)
b / (s^2 + as + b)
C(s)
General
(b)
R(s) = 1/s
G(s)
9 / (s^2 + 9s + 9)
C(s)
Overdamped
jω
s-plane
σ
−7.854
−1.146
c(t)
c(t) = 1 + 0.171e^{−7.854t} − 1.171e^{−1.146t}
1
0.5
0
1
2
3
4
5
t
(c)
R(s) = 1/s
G(s)
9 / (s^2 + 2s + 9)
C(s)
Underdamped
jω
s-plane
j√8
σ
−1
−j√8
c(t)
c(t) = 1 − e^{−t}(cos √8 t + (√8/8) sin √8 t)
= 1 − 1.06e^{−t} cos(√8 t − 19.47°)
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
2
3
4
5
t
(d)
R(s) = 1/s
G(s)
9 / (s^2 + 9)
C(s)
Undamped
jω
s-plane
j3
σ
−j3
c(t)
c(t) = 1 − cos 3t
2
1
0
1
2
3
4
5
t
(e)
R(s) = 1/s
G(s)
9 / (s^2 + 6s + 9)
C(s)
Critically damped
jω
s-plane
σ
−3
c(t)
c(t) = 1 − 3te^{−3t} − e^{−3t}
1
0.8
0.6
0.4
0.2
0
1
2
3
4
5
t

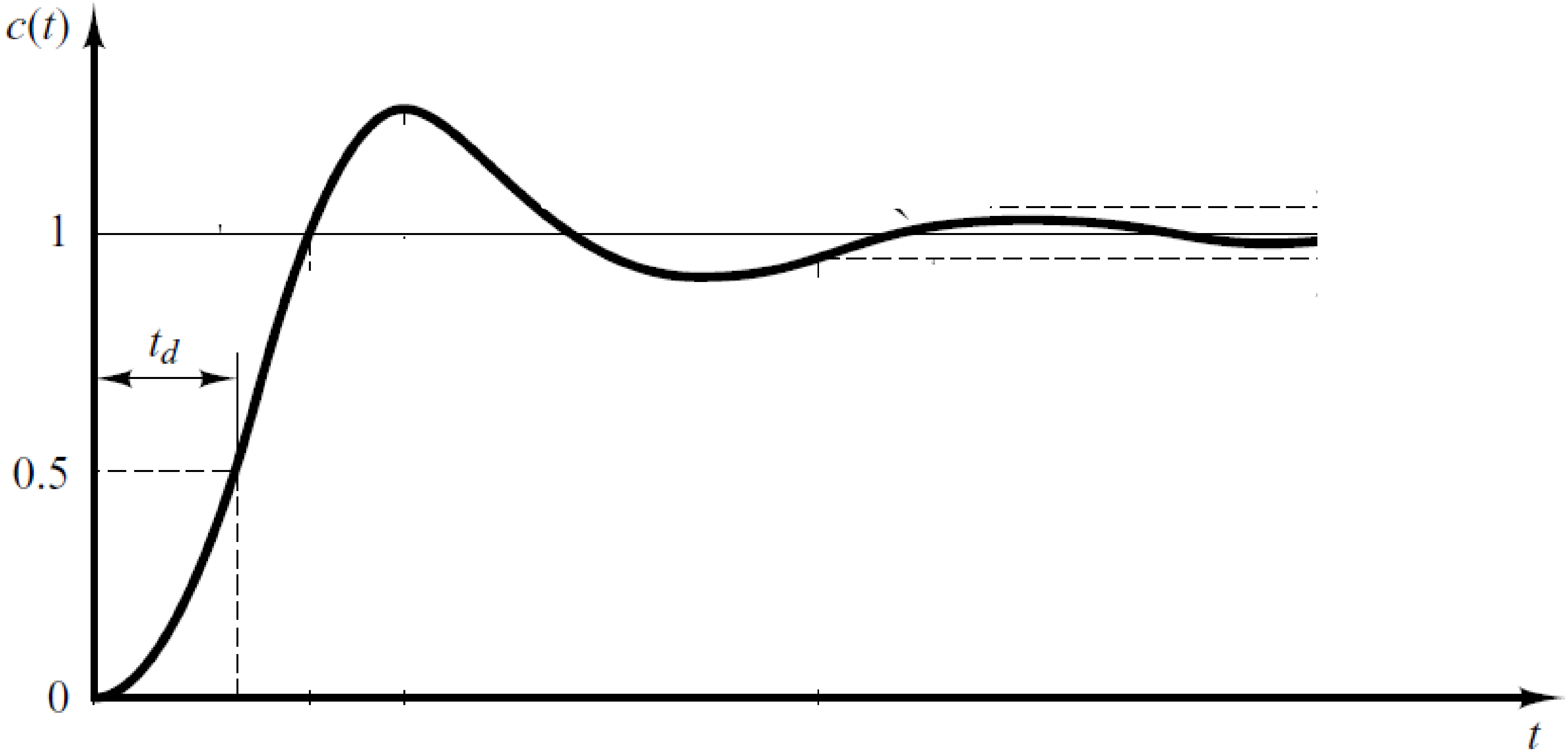
Time-Domain Specification

For $0<\zeta<1$ and $\omega_n > 0$, the 2nd order system's response due to a unit step input looks like



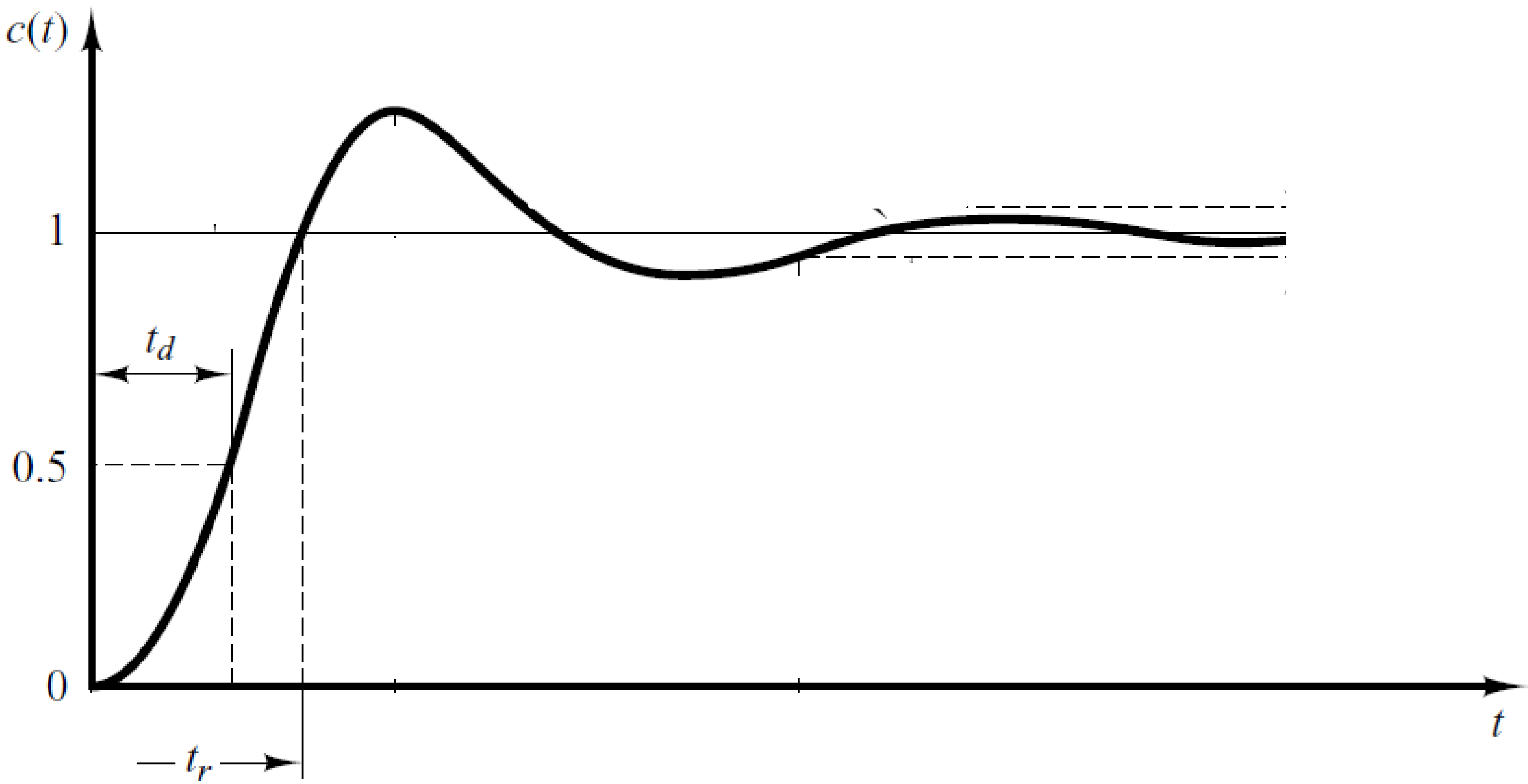
Time-Domain Specification

- The delay (t_d) time is the time required for the response to reach half the final value the very first time.



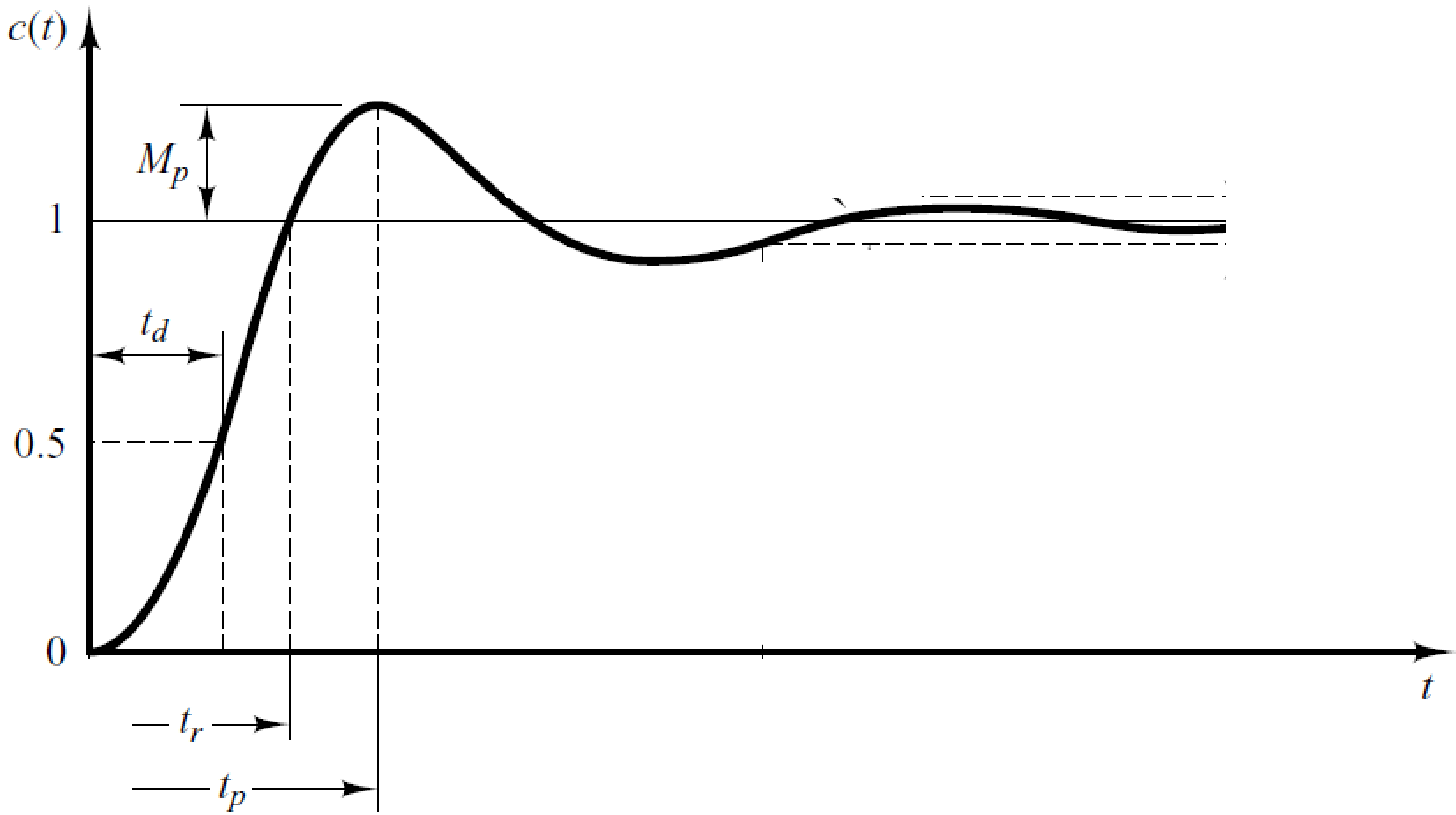
Time-Domain Specification

- The rise time is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value.
- For underdamped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.



Time-Domain Specification

- The peak time is the time required for the response to reach the first peak of the overshoot.



Time-Domain Specification

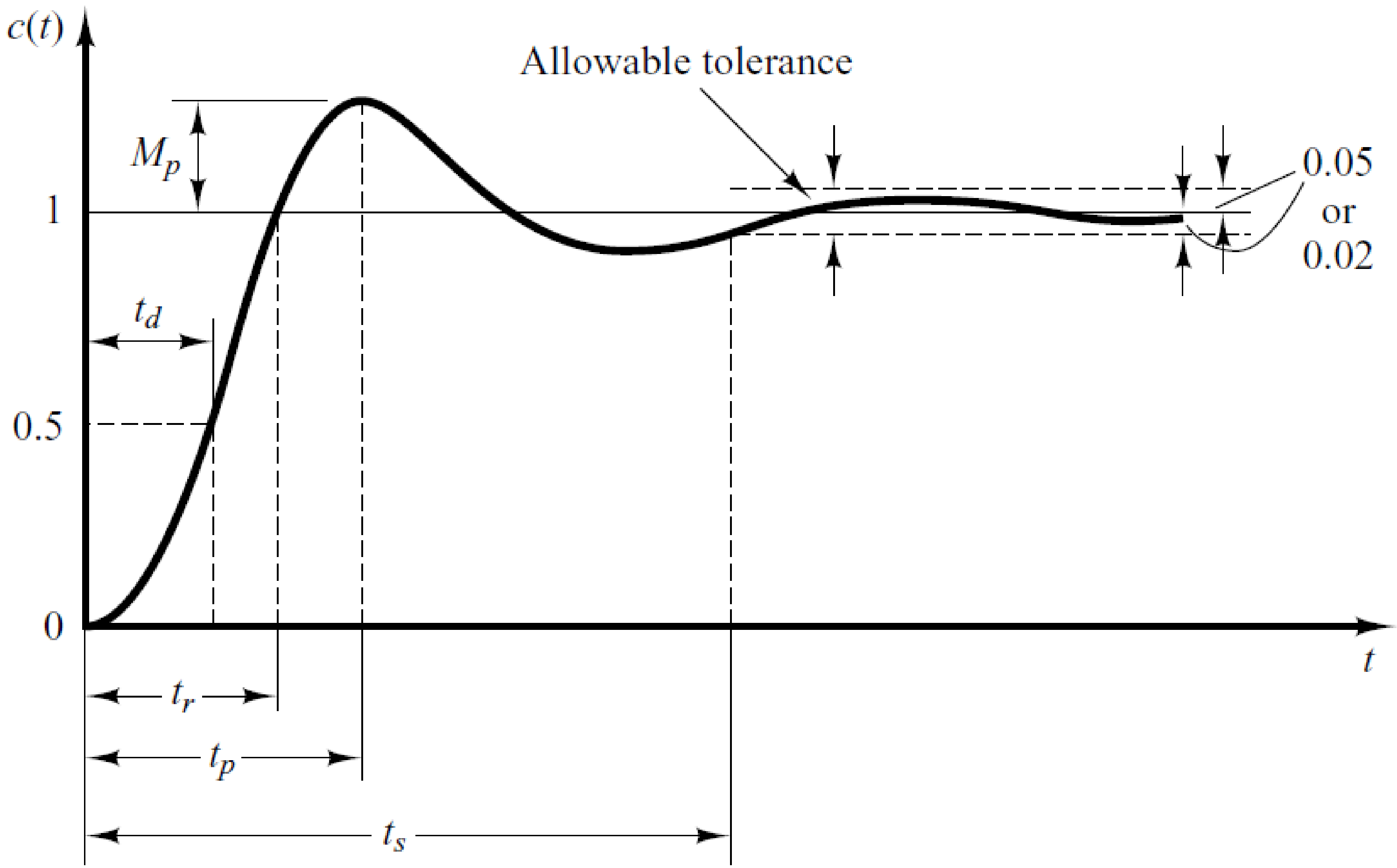
The maximum overshoot is the maximum peak value of the response curve measured from unity. If the final steady-state value of the response differs from unity, then it is common to use the maximum percent overshoot. It is defined by

$$\text{Maximum percent overshoot} = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system.

Time-Domain Specification

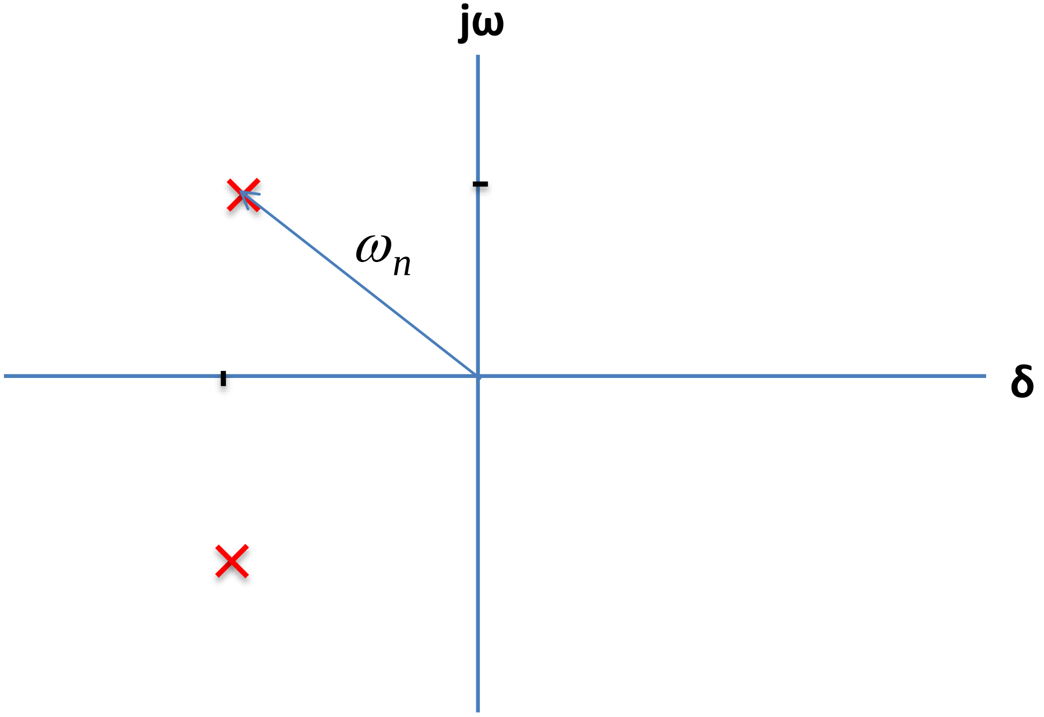
- The settling time is the time required for the response curve to reach and stay within a range about the final value of size specified by absolute percentage of the final value (usually 2% or 5%).



S-Plane

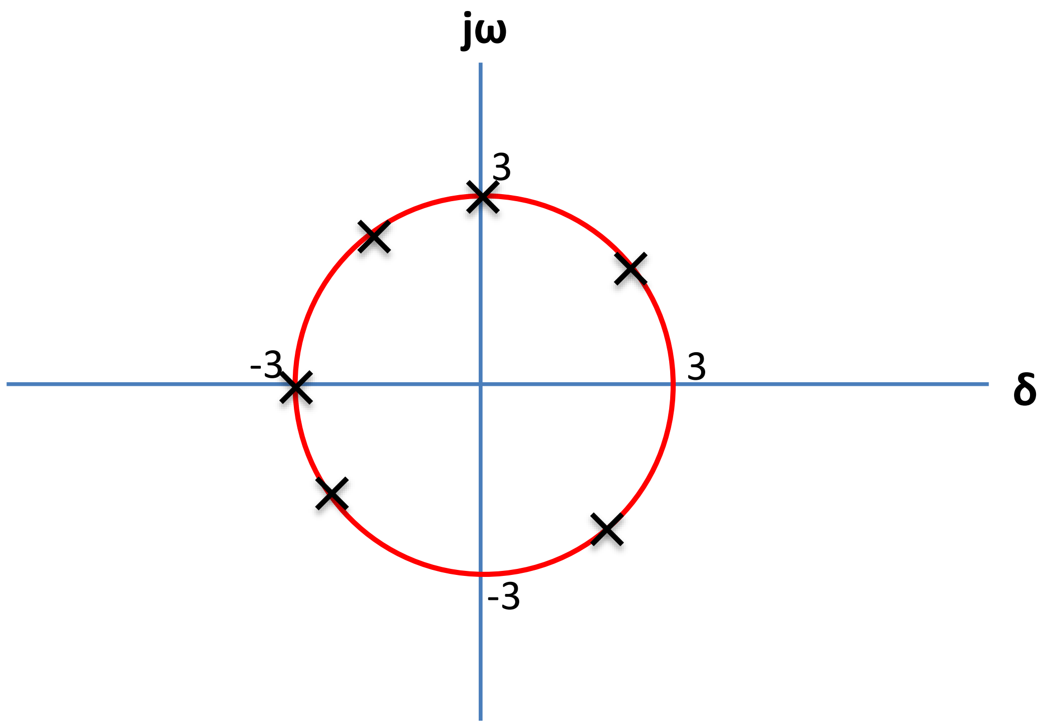
- Natural Undamped Frequency.

- Distance from the origin of s-plane to pole is natural undamped frequency in rad/sec.



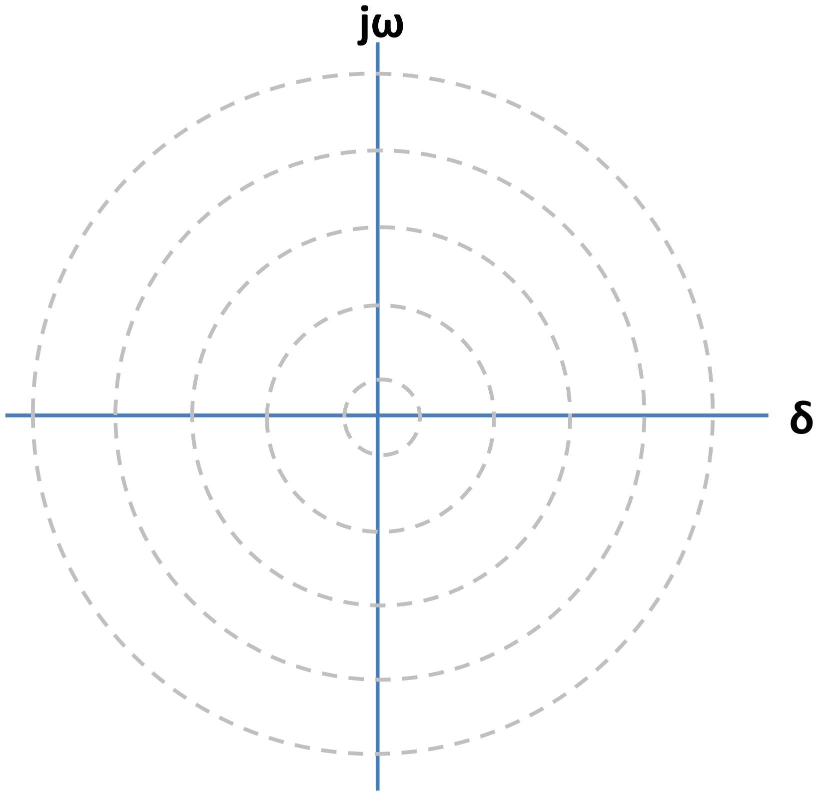
S-Plane

- Let us draw a circle of radius 3 in s-plane.
- If a pole is located anywhere on the circumference of the circle the natural undamped frequency would be *3 rad/sec*.



S-Plane

- Therefore the s-plane is divided into Constant Natural Undamped Frequency (ω_n) Circles.

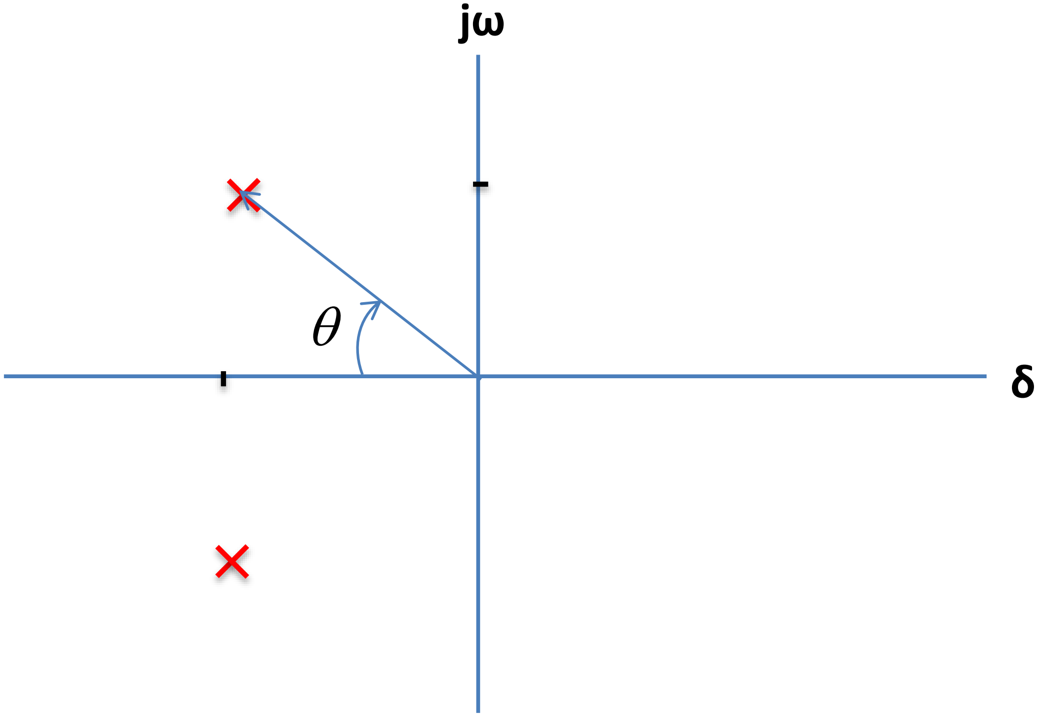


S-Plane

- Damping ratio.

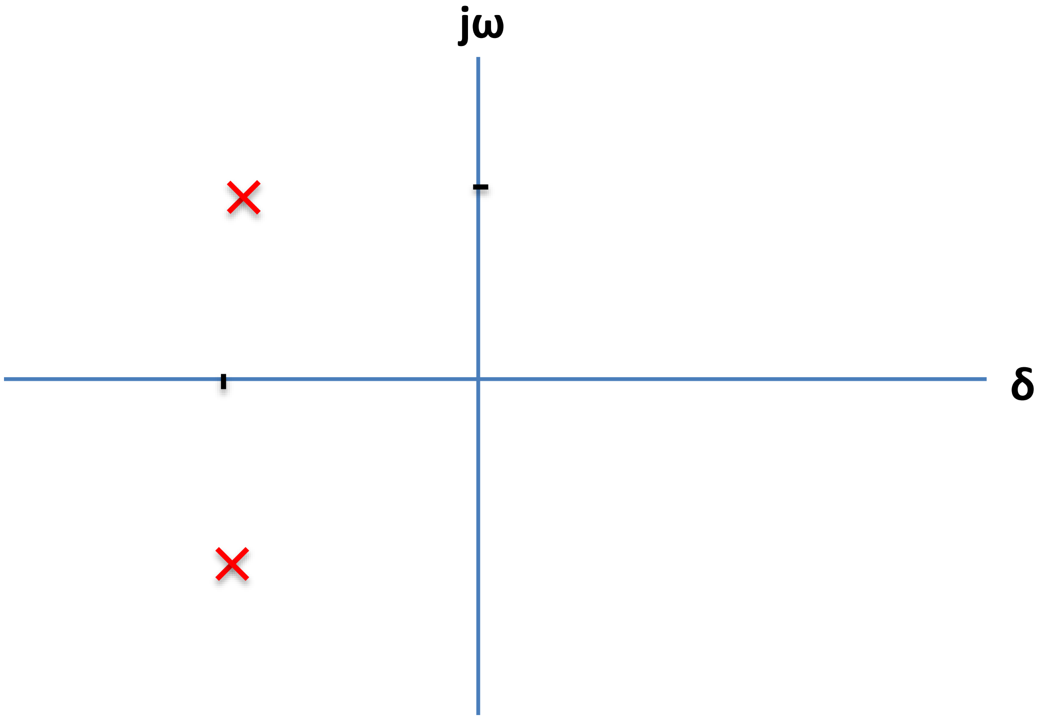
- Cosine of the angle between vector connecting origin and pole and –ve real axis yields damping ratio.

$$\zeta = \cos\theta$$



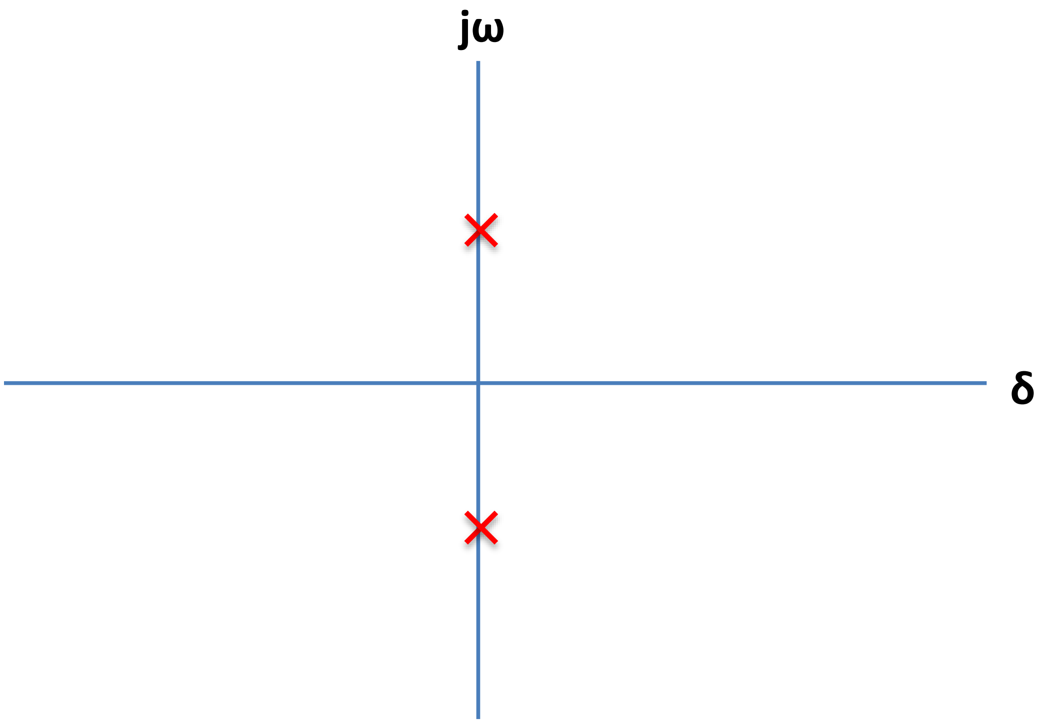
S-Plane

- For Underdamped system $0^\circ < \theta < 90^\circ$ therefore, $0 < \zeta < 1$



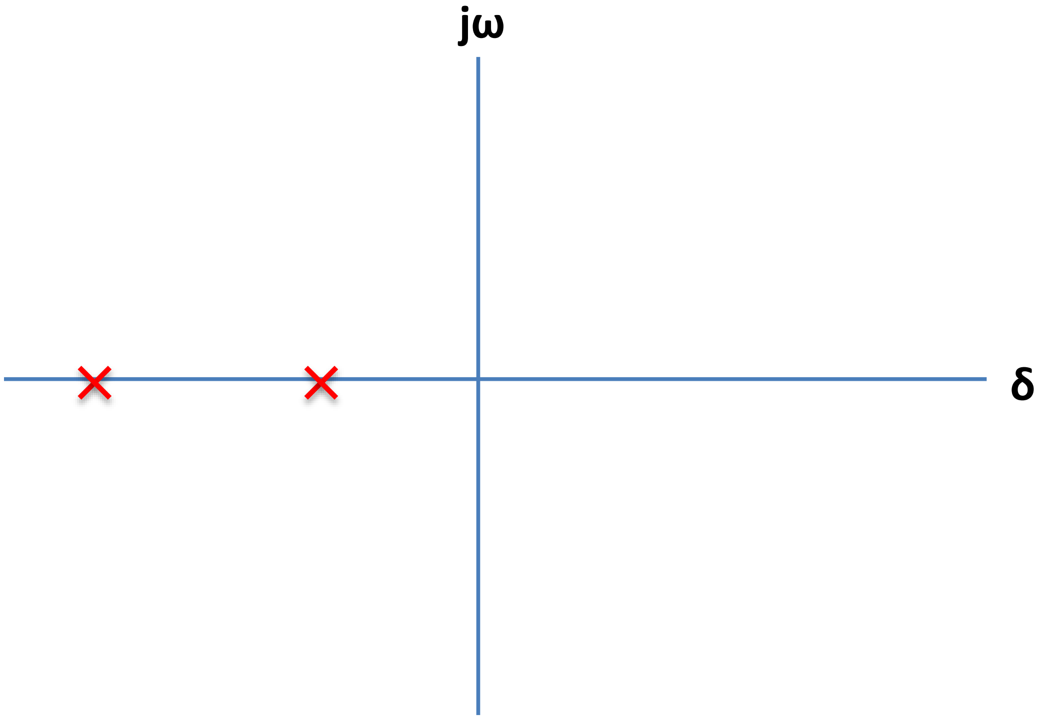
S-Plane

- For Undamped system $\theta = 90^\circ$ therefore, $\zeta = 0$



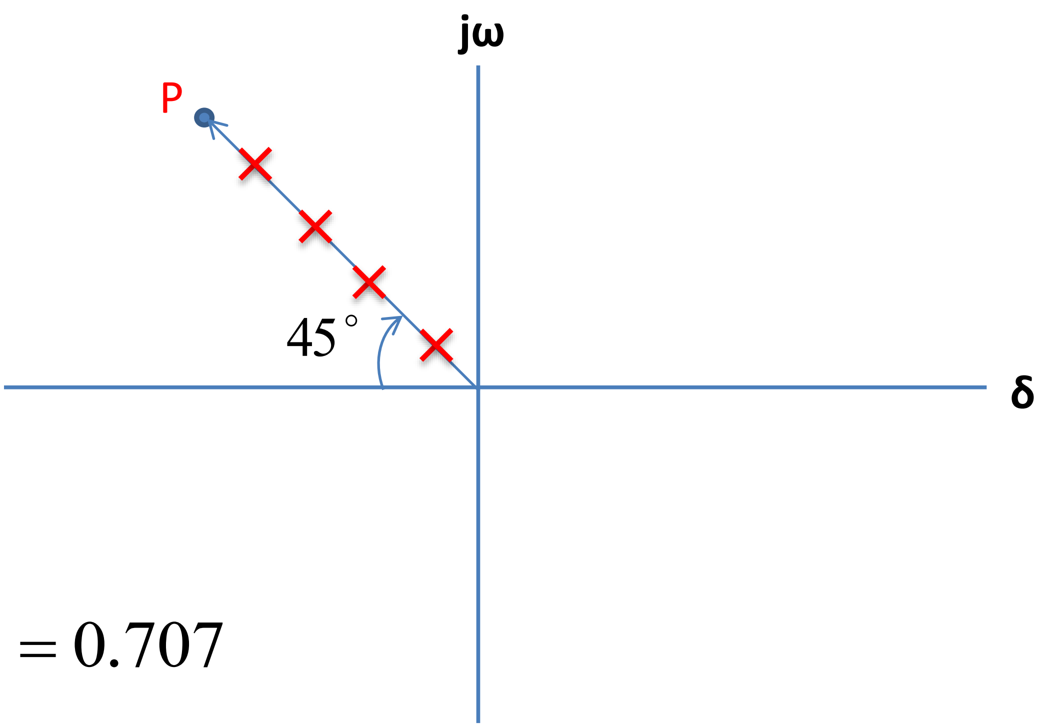
S-Plane

- For overdamped and critically damped systems $\theta = 0^{\circ}$ therefore, $\zeta \geq 0$



S-Plane

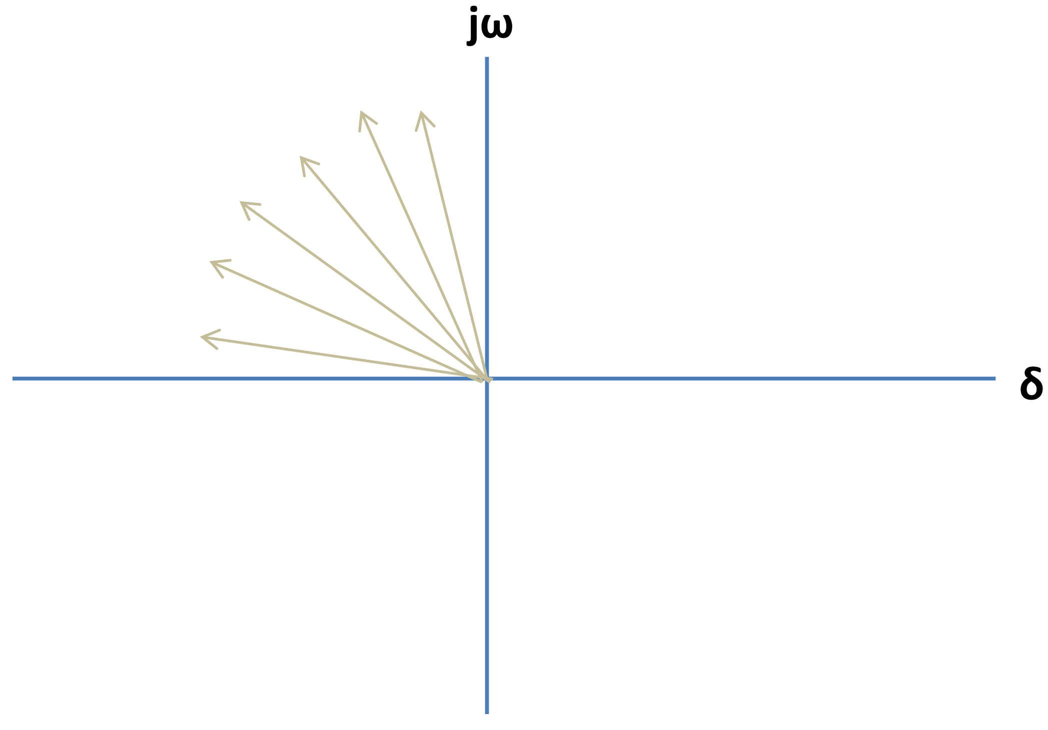
- Draw a vector connecting origin of s-plane and some point P.



$$\zeta = \cos 45^\circ = 0.707$$

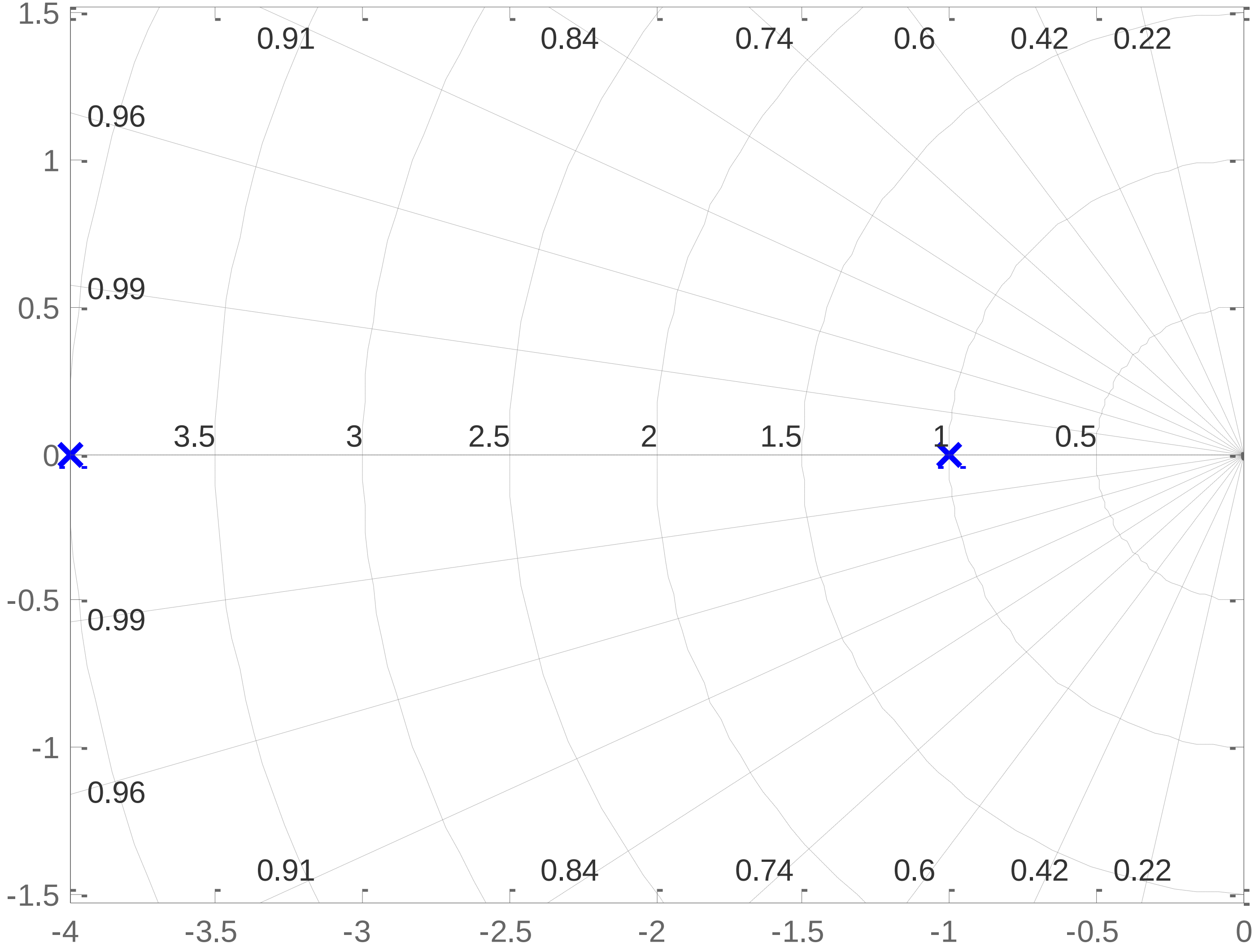
S-Plane

- Therefore, s-plane is divided into sections of constant damping ratio lines.



Example-2

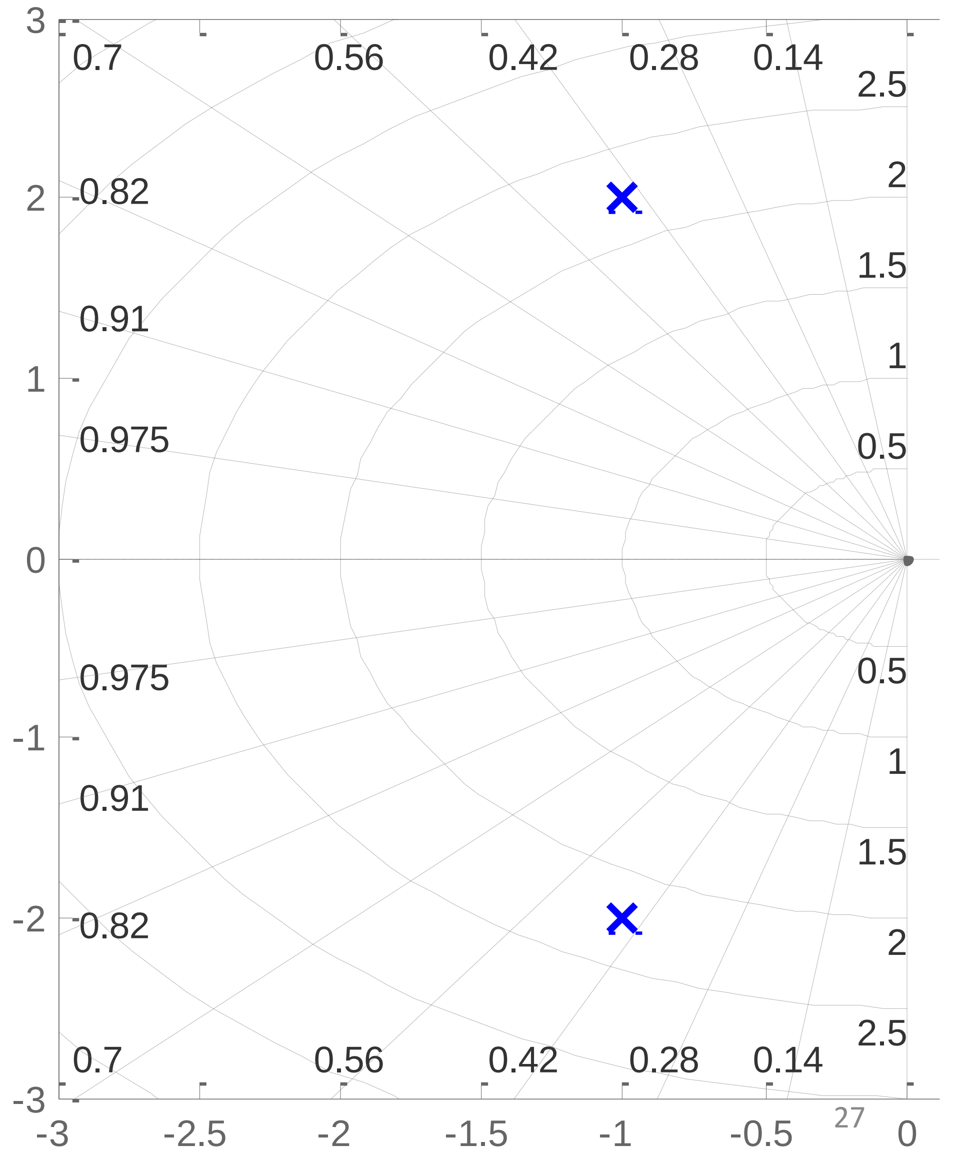
- Determine the natural frequency and damping ratio of the poles from the following pz-map.



Example-3

- Determine the natural frequency and damping ratio of the poles from the given pz-map.

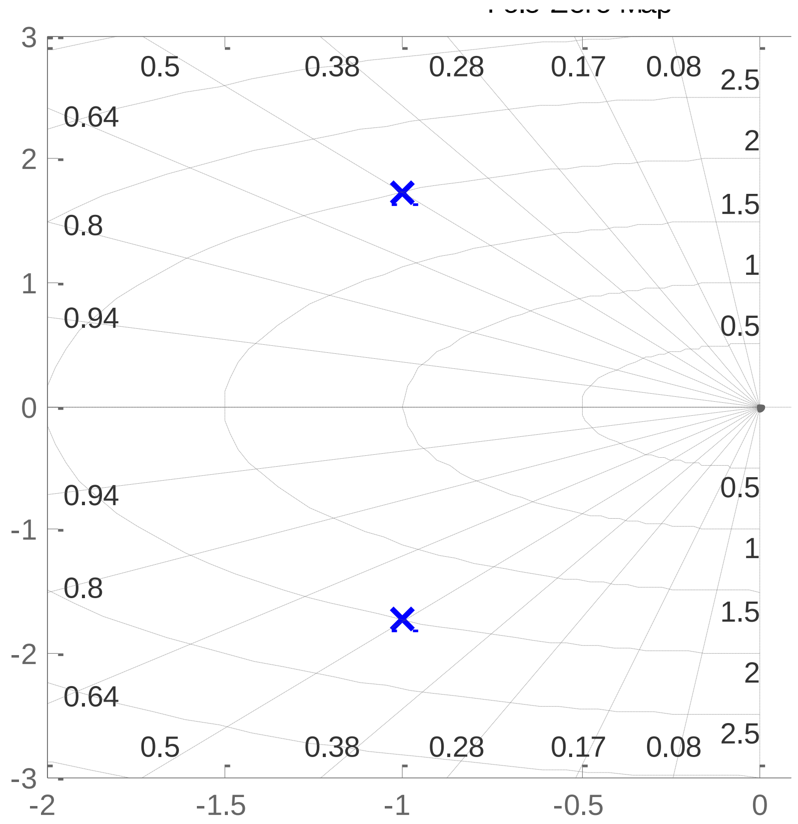
- Also determine the transfer function of the system and state whether system is underdamped, overdamped, undamped or critically damped.



Example-4

- The natural frequency of closed loop poles of 2nd order system is 2 rad/sec and damping ratio is 0.5.

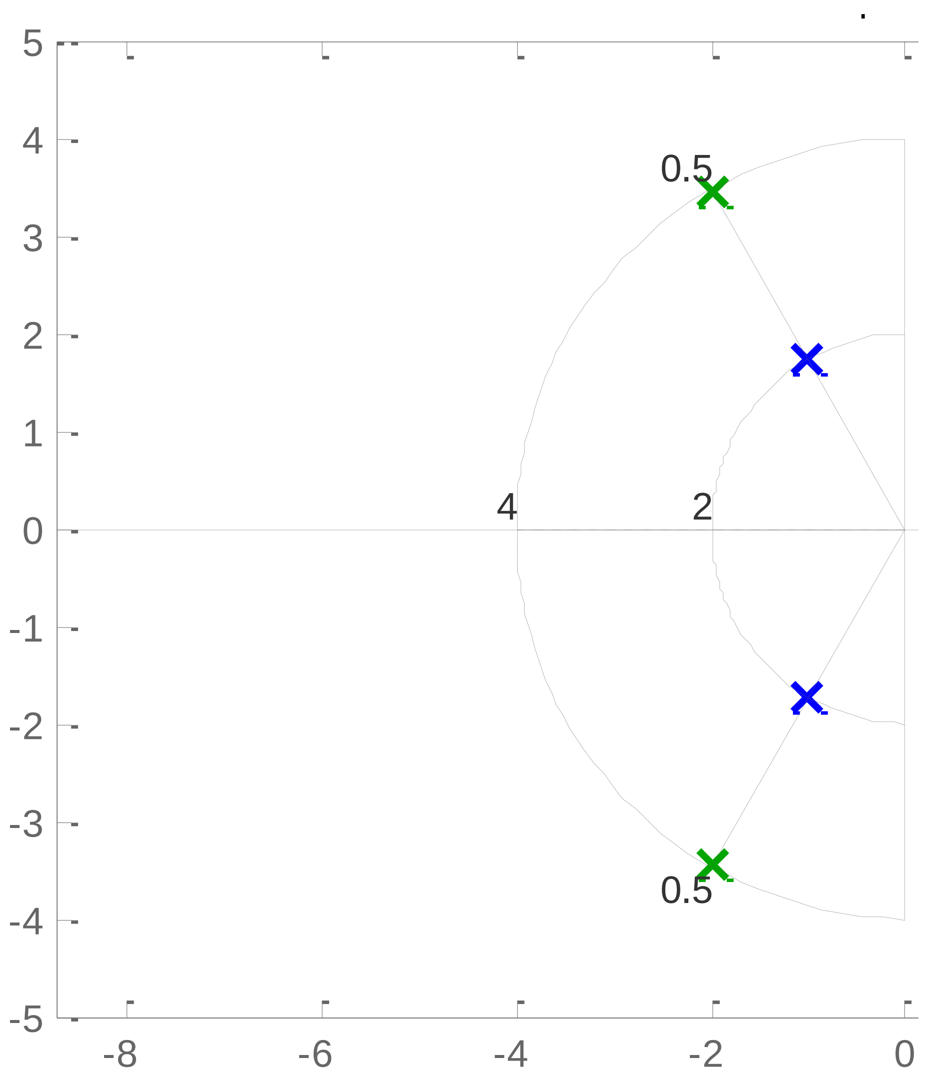
- Determine the location of closed loop poles so that the damping ratio remains same but the natural undamped frequency is doubled.



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{4}{s^2 + 2s + 4}$$

Example-4

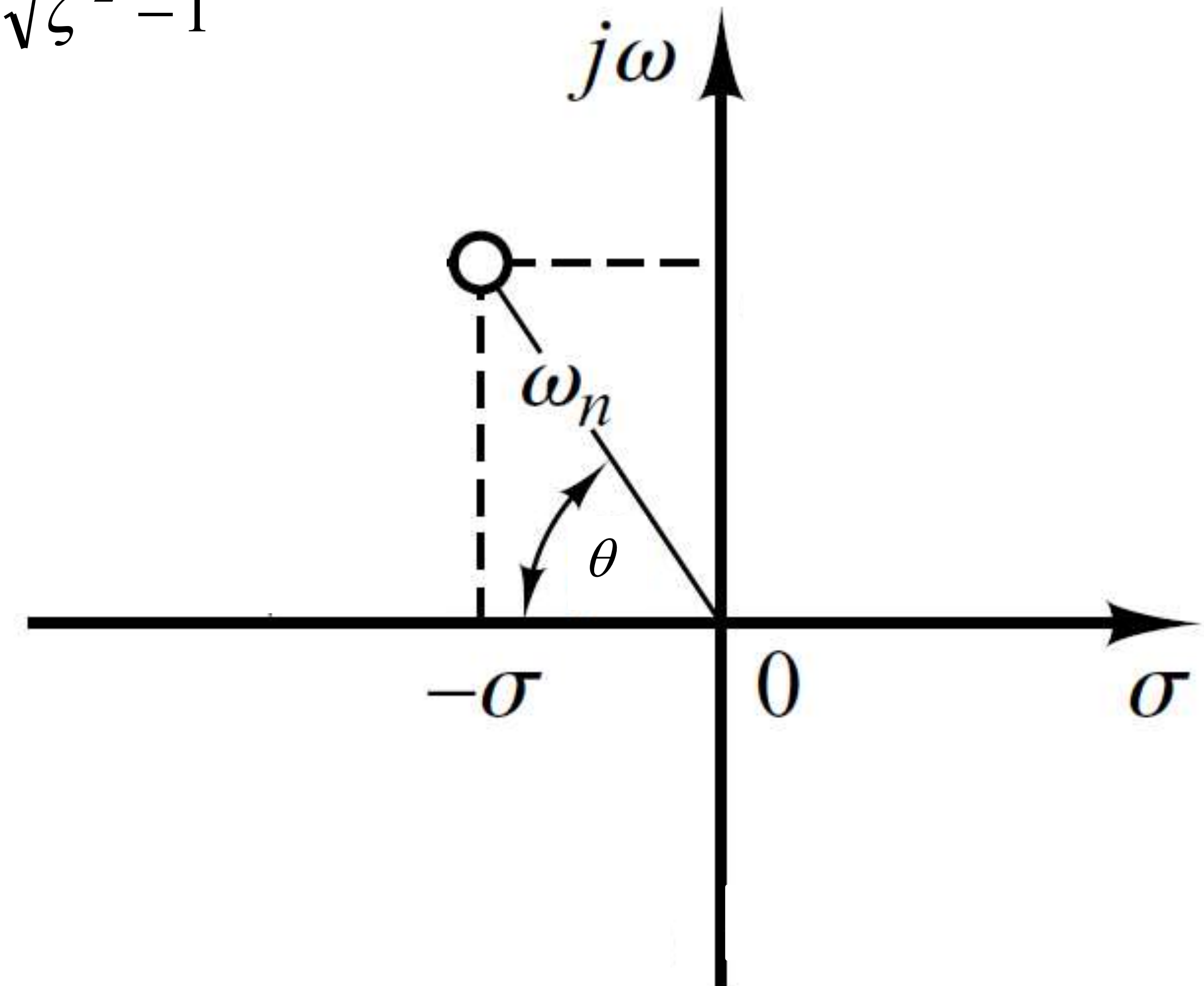
Determine the location of closed loop poles so that the damping ratio remains same but the natural undamped frequency is doubled.



S-Plane

$$-\omega_n\zeta + \omega_n\sqrt{\zeta^2 - 1}$$

$$-\omega_n\zeta - \omega_n\sqrt{\zeta^2 - 1}$$



Step Response of underdamped System

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \xrightarrow{\text{Step Response}} C(s) = \frac{\omega_n^2}{s\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)}$$

- The partial fraction expansion of above equation is given as

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- C(s) can be written in the following form:

$$C(s) = \frac{1}{s} - \frac{s + \zeta\omega_n}{\left(s + \zeta\omega_n\right)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\left(s + \zeta\omega_n\right)^2 + \omega_d^2}$$

Step Response of underdamped System

- The inverse Laplace transform of above equation can be obtained easily

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

- When $\zeta = 0$

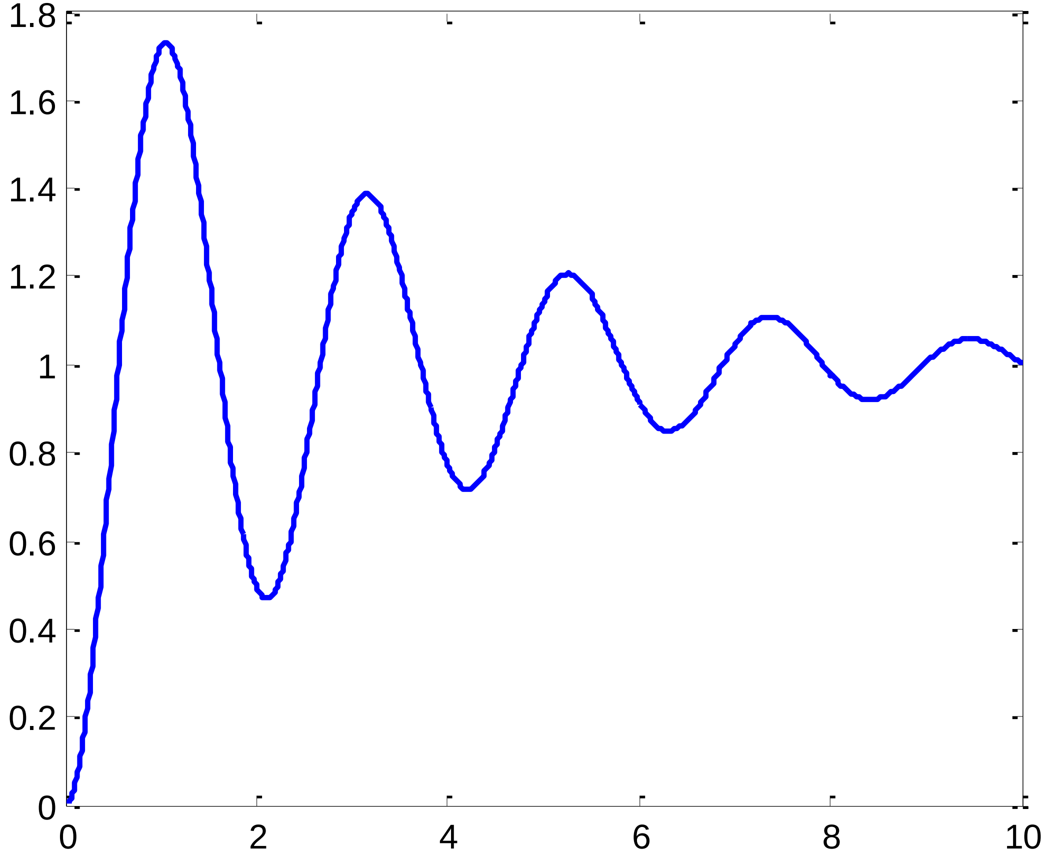
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$
$$= \omega_n$$

$$c(t) = 1 - \cos\omega_n t$$

Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

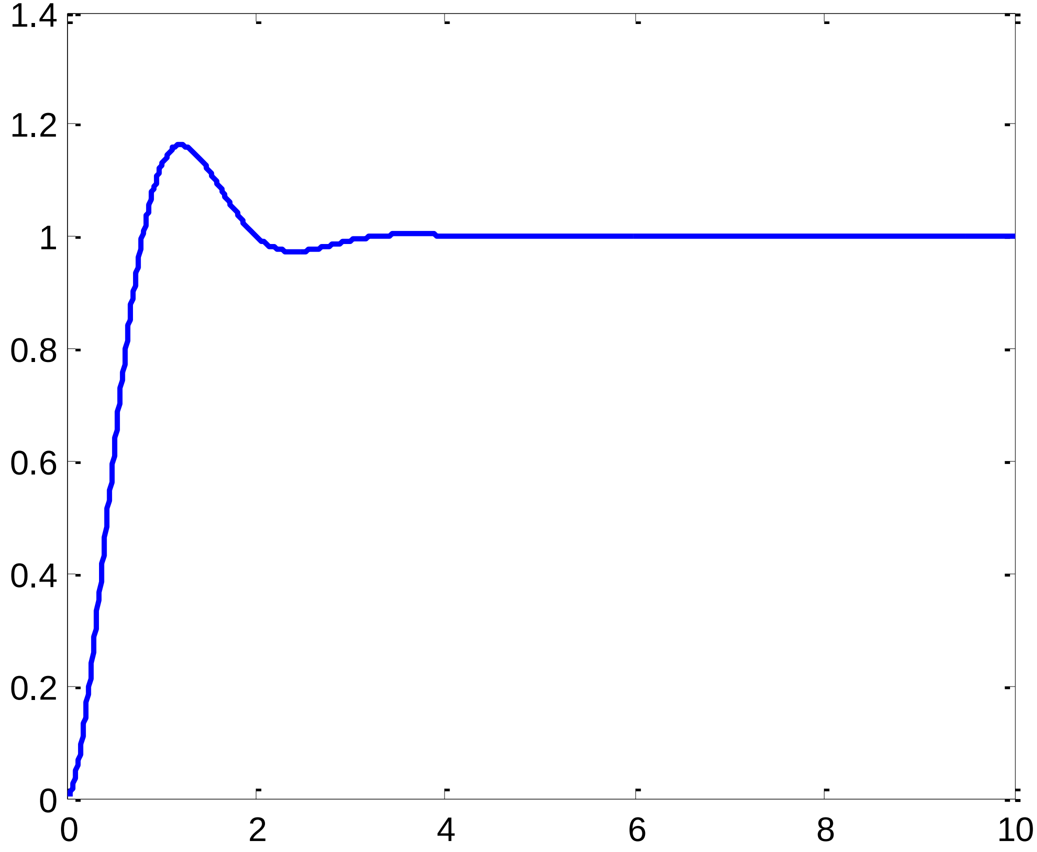
if $\zeta = 0.1$ and $\omega_n = 3 \ rad/\sec$



Step Response of underdamped System

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

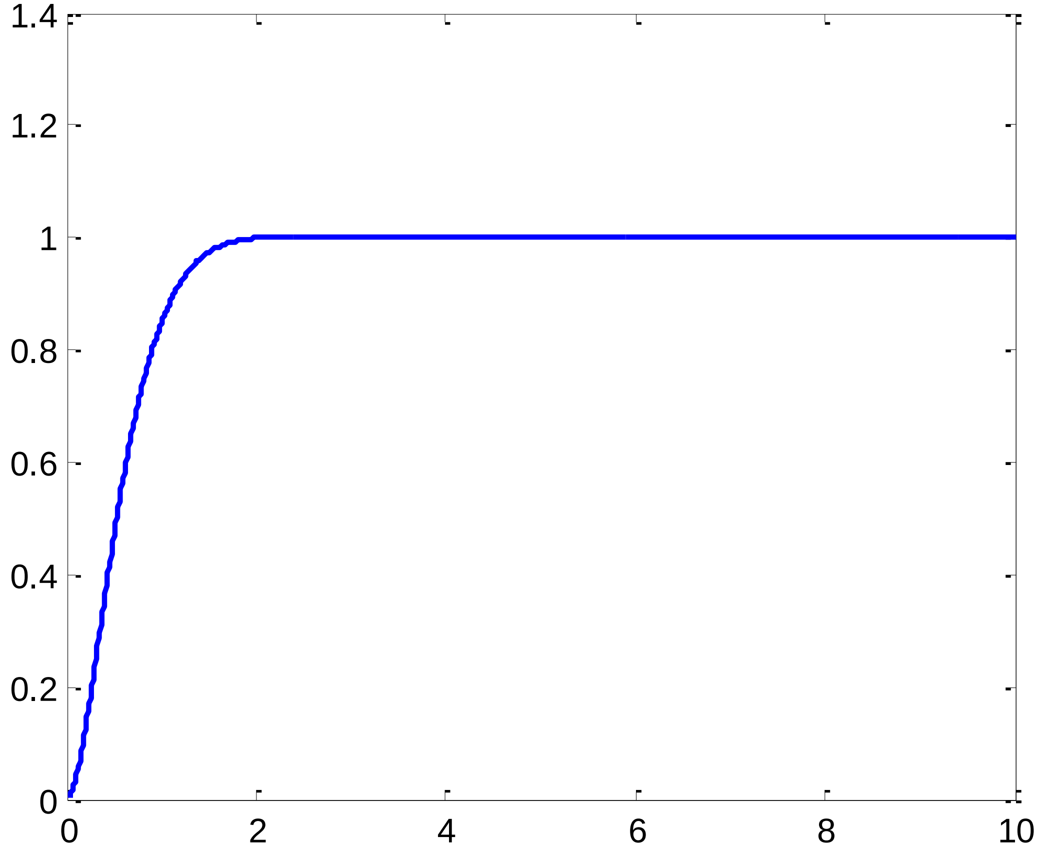
if $\zeta = 0.5$ and $\omega_n = 3 \ rad/\sec$



Step Response of underdamped System

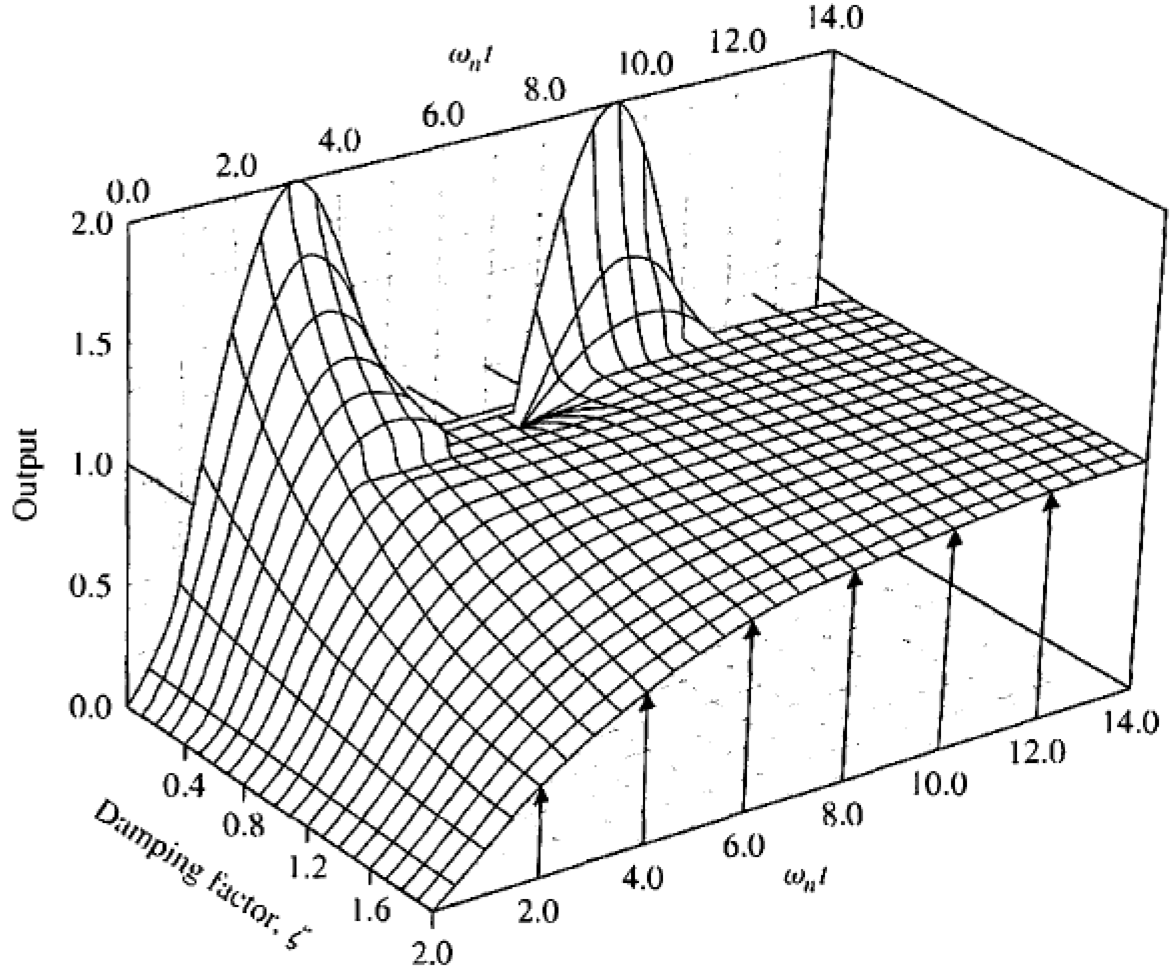
$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$

if $\zeta = 0.9$ and $\omega_n = 3 \ rad/\sec$

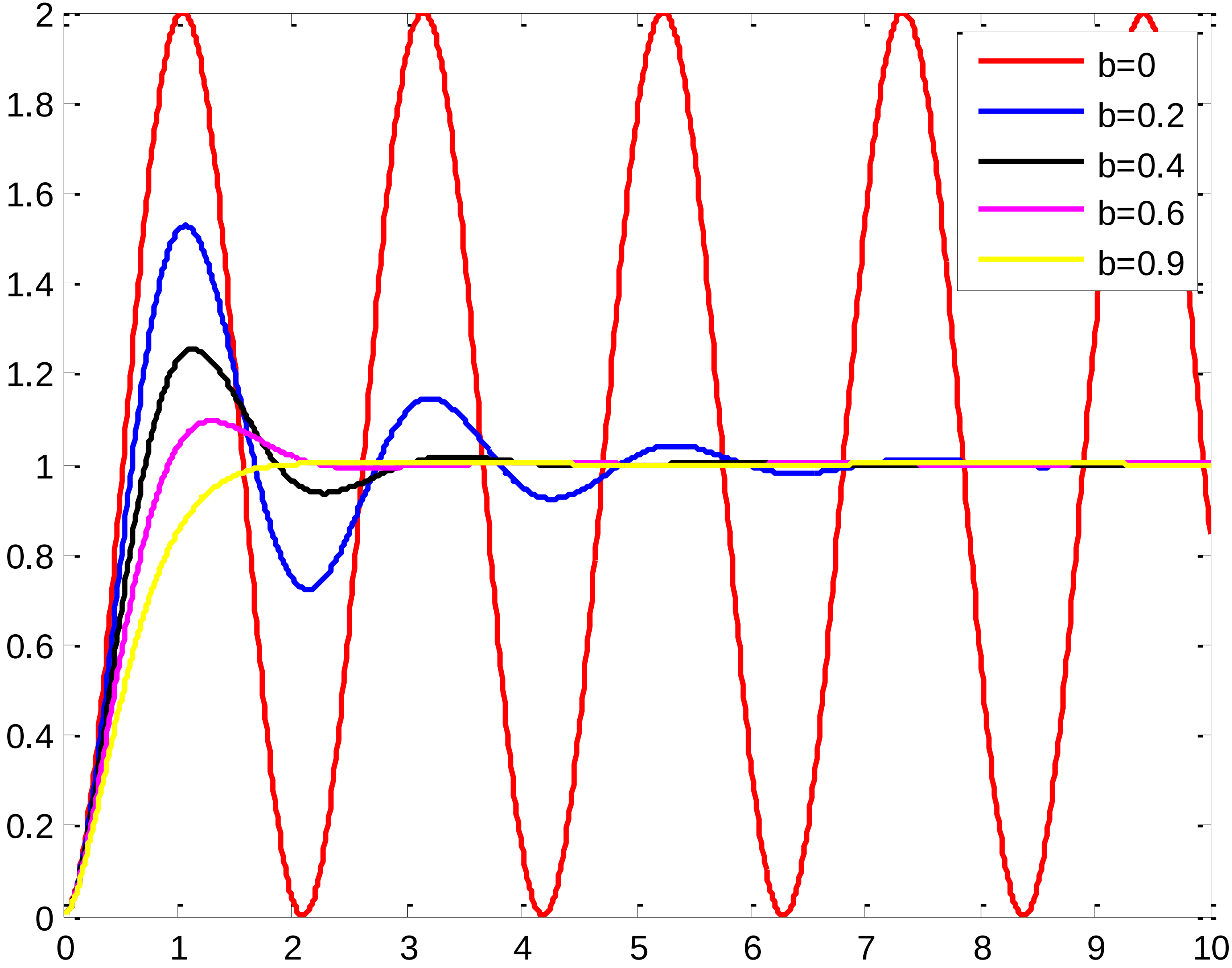


Step Response of underdamped System

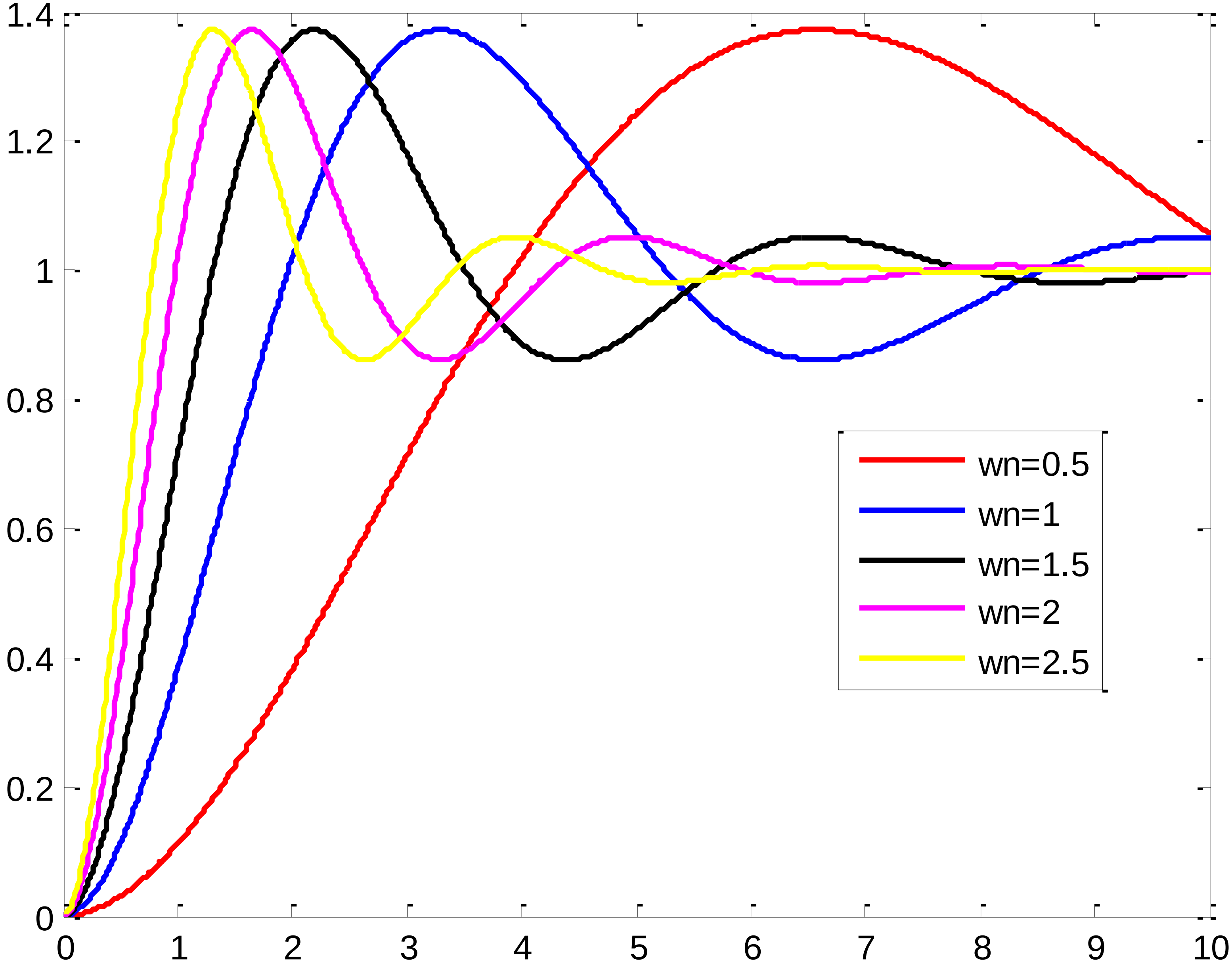
$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t\right]$$



Step Response of underdamped System



Step Response of underdamped System



Time Domain Specifications

Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \theta}{\omega_n \sqrt{1 - \zeta^2}}$$

Peak Time

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n \sqrt{1 - \zeta^2}}$$

Settling Time (2%)

$$t_s = 4T = \frac{4}{\zeta \omega_n}$$

$$t_s = 3T = \frac{3}{\zeta \omega_n}$$

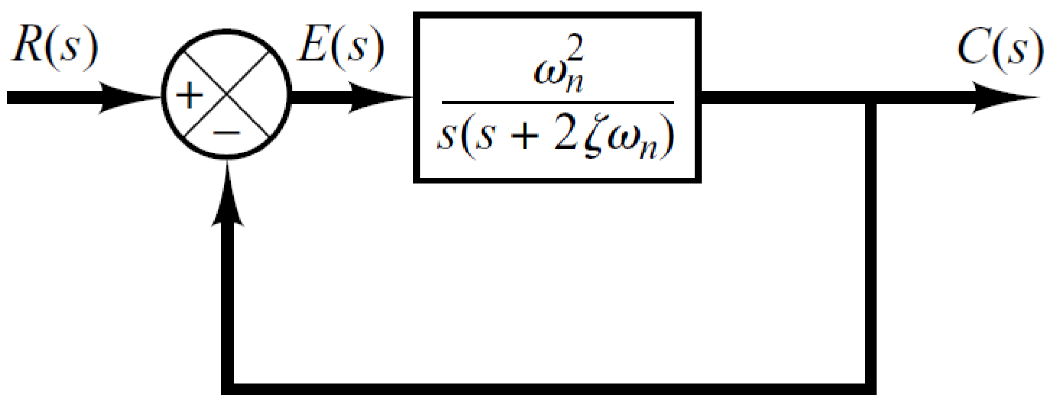
Settling Time (4%)

Maximum Overshoot

$$M_p = e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}} \times 100$$

Example#5

- Consider the system shown in following figure, where damping ratio is 0.6 and natural undamped frequency is 5 rad/sec. Obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time 2% and 5% criterion t_s when the system is subjected to a unit-step input.



Example#5

Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d}$$

Peak Time

$$t_p = \frac{\pi}{\omega_d}$$

Settling Time (2%)

$$t_s = 4T = \frac{4}{\zeta\omega_n}$$

$$t_s = 3T = \frac{3}{\zeta\omega_n}$$

Settling Time (4%)

Maximum Overshoot

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

Example#5

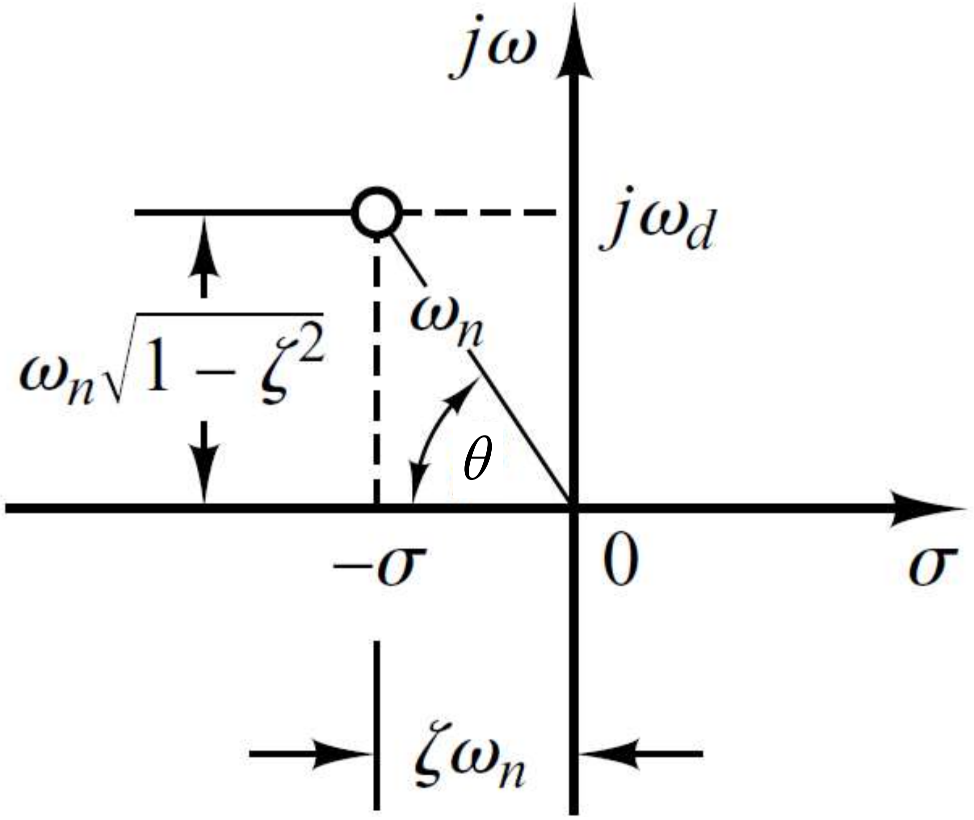
Rise Time

$$t_r = \frac{\pi - \theta}{\omega_d}$$

$$t_r = \frac{3.141 - \theta}{\omega_n \sqrt{1 - \zeta^2}}$$

$$\theta = \cos^{-1}\zeta = 53.3^{\circ} = 0.93\,\text{rad}$$

$$t_r = \frac{3.141 - 0.93}{5\sqrt{1 - 0.6^2}} = 0.55s$$



Example#5

Peak Time

$$t_p = \frac{\pi}{\omega_d}$$

$$t_p = \frac{3.141}{4} = 0.785\,s$$

Settling Time (2%)

$$t_s = \frac{4}{\zeta \omega_n}$$

$$t_s = \frac{4}{0.6 \times 5} = 1.33\,s$$

Settling Time (4%)

$$t_s = \frac{3}{\zeta \omega_n}$$

$$t_s = \frac{3}{0.6 \times 5} = 1s$$

Example#5

Maximum Overshoot

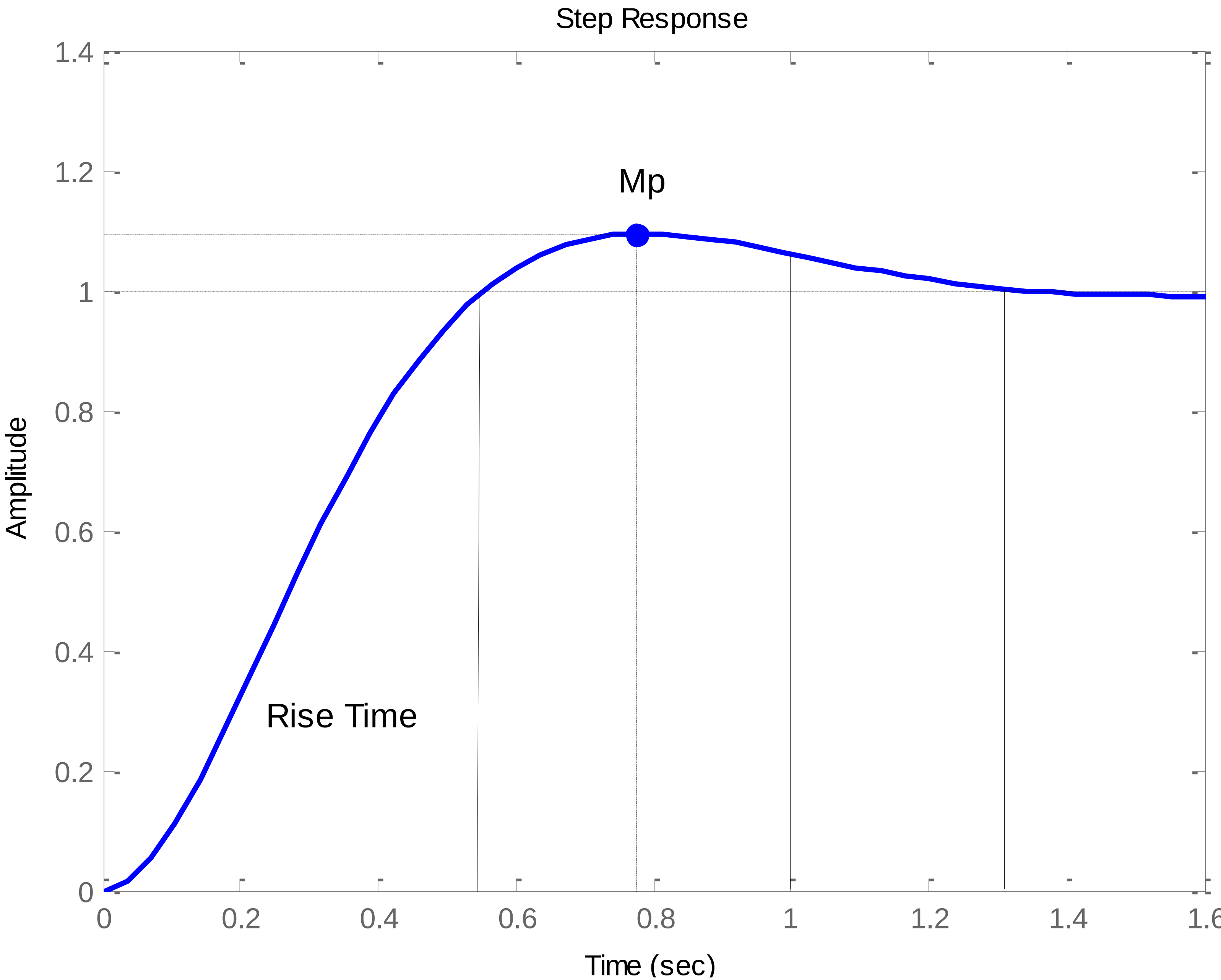
$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

$$M_p = e^{-\frac{3.141 \times 0.6}{\sqrt{1-0.6^2}}} \times 100$$

$$M_p = 0.095 \times 100$$

$$M_p = 9.5\%$$

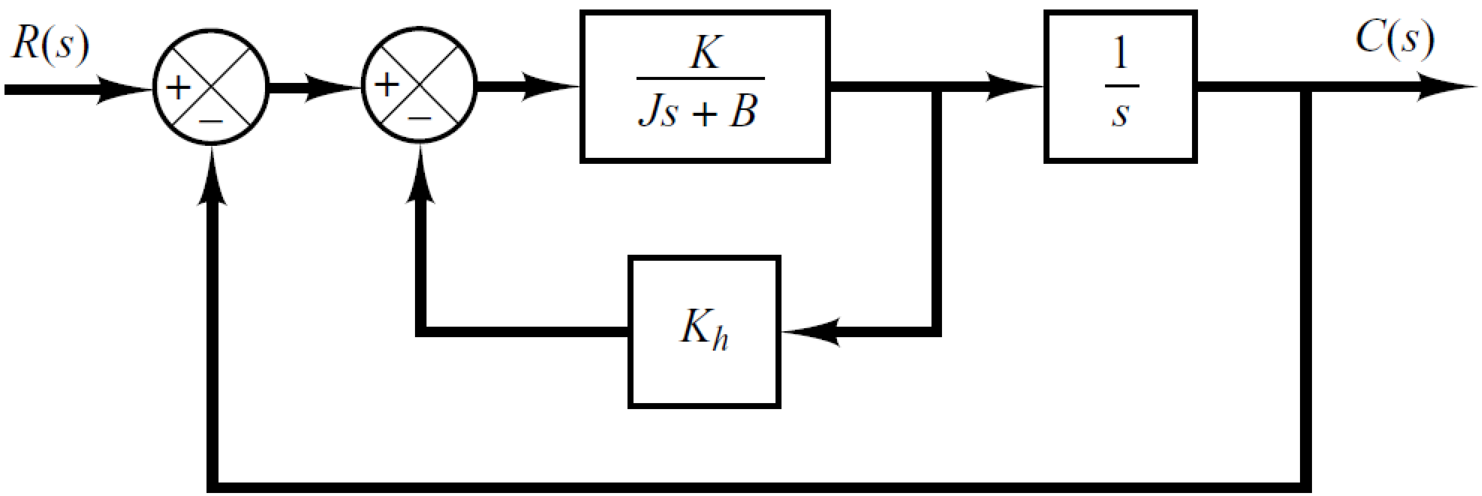
Example#5



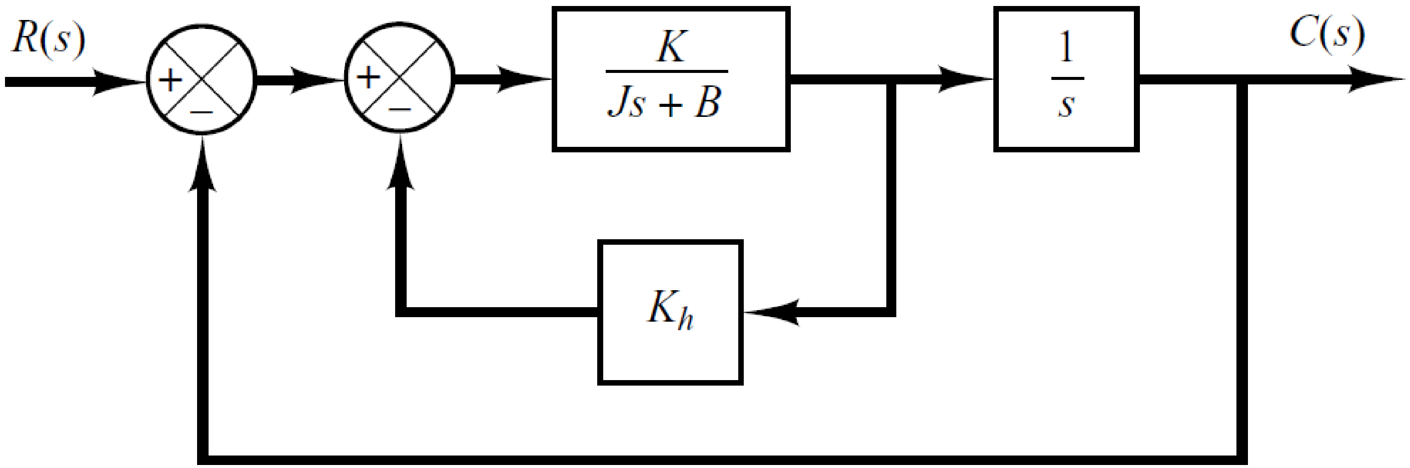
Step Response
Mp
Rise Time
Amplitude
Time (sec)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6

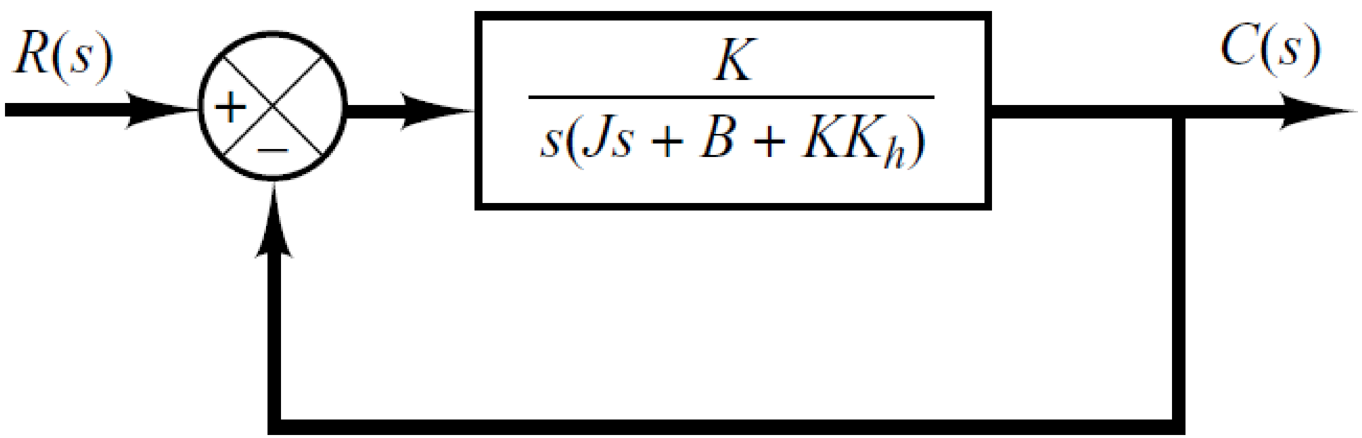
Example#6

- For the system shown in Figure-(a), determine the values of gain K and velocity-feedback constant K_h so that the maximum overshoot in the unit-step response is 0.2 and the peak time is 1 sec. With these values of K and K_h, obtain the rise time and settling time. Assume that $J=1$ kg-m^2 and $B=1$ N-m/rad/sec.



Example#6





$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + KK_h)s + K}$$

Example#6

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + KK_h)s + K}$$

Since $J = 1\ kgm^2$ and $B = 1$ Nm/rad/sec

$$\frac{C(s)}{R(s)} = \frac{K}{s^2 + (1 + KK_h)s + K}$$

- Comparing above T.F with general 2[nd] order T.F

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n = \sqrt{K} \qquad \zeta = \frac{(1 + KK_h)}{2\sqrt{K}}$$

Example#6

$$\omega_n = \sqrt{K}$$

$$\zeta = \frac{(1 + KK_h)}{2\sqrt{K}}$$

- Maximum overshoot is 0.2.

$$M_p = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$$

$$e^{-(\zeta/\sqrt{1-\zeta^2})\pi} = 0.2$$

$$\ln(e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}) = \ln(0.2)$$

$$\frac{\zeta\pi}{\sqrt{1 - \zeta^2}} = 1.61$$

$$\zeta = 0.456$$

- The peak time is 1 sec

$$t_p = \frac{\pi}{\omega_d}$$

$$1 = \frac{3.141}{\omega_n\sqrt{1-\zeta^2}}$$

$$\omega_n = \frac{3.141}{\sqrt{1-0.456^2}}$$

$$\omega_n = 3.53$$

Example#6

$\zeta = 0.456$

$\omega_n = 3.96$

$\omega_n = \sqrt{K}$

$3.53 = \sqrt{K}$

$3.53^2 = K$

$K = 12.5$

$$\zeta = \frac{(1 + KK_h)}{2\sqrt{K}}$$

$$0.456 \times 2\sqrt{12.5} = (1 + 12.5K_h)$$

$$K_h = 0.178$$

Example#6

$\zeta = 0.456$

$\omega_n = 3.96$

$$t_r = \frac{\pi - \theta}{\omega_n \sqrt{1-\zeta^2}}$$

$$t_s = \frac{4}{\zeta \omega_n}$$

$t_r = 0.65 s$

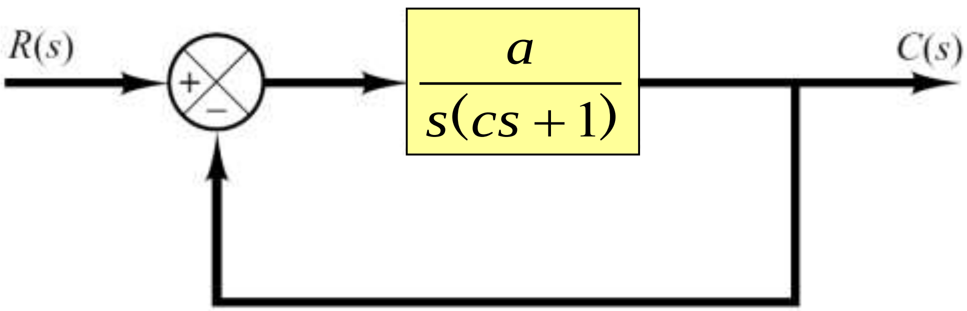
$t_s = 2.48 s$

$$t_s = \frac{3}{\zeta \omega_n}$$

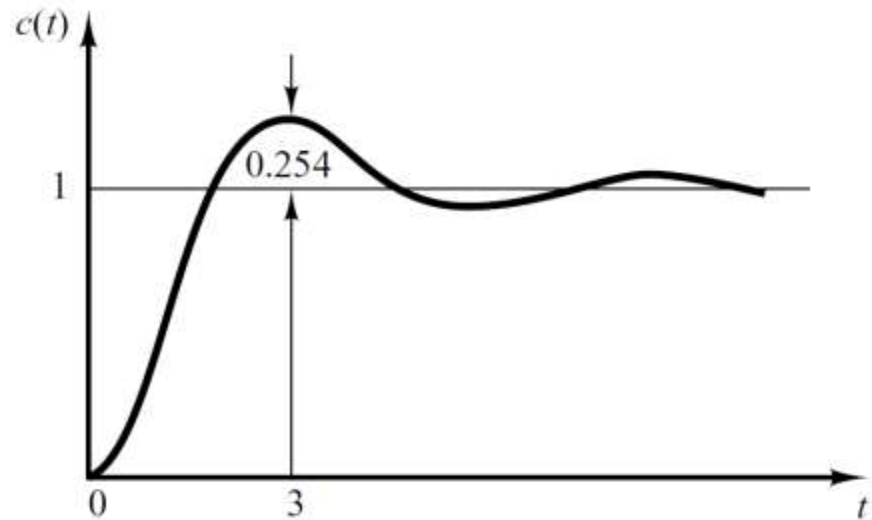
$t_s = 1.86 s$

Example#7

When the system shown in Figure(a) is subjected to a unit-step input, the system output responds as shown in Figure(b). Determine the values of *a* and *c* from the response curve.



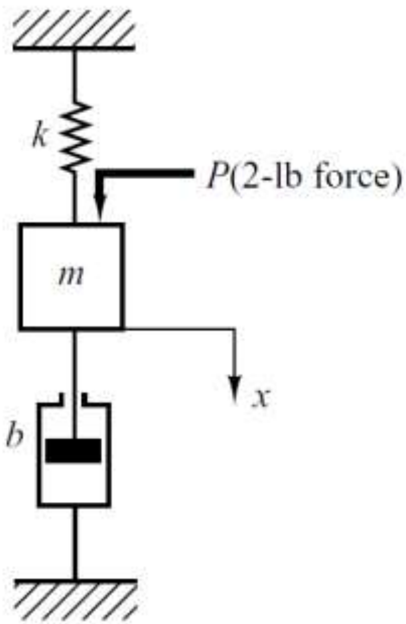
(a)



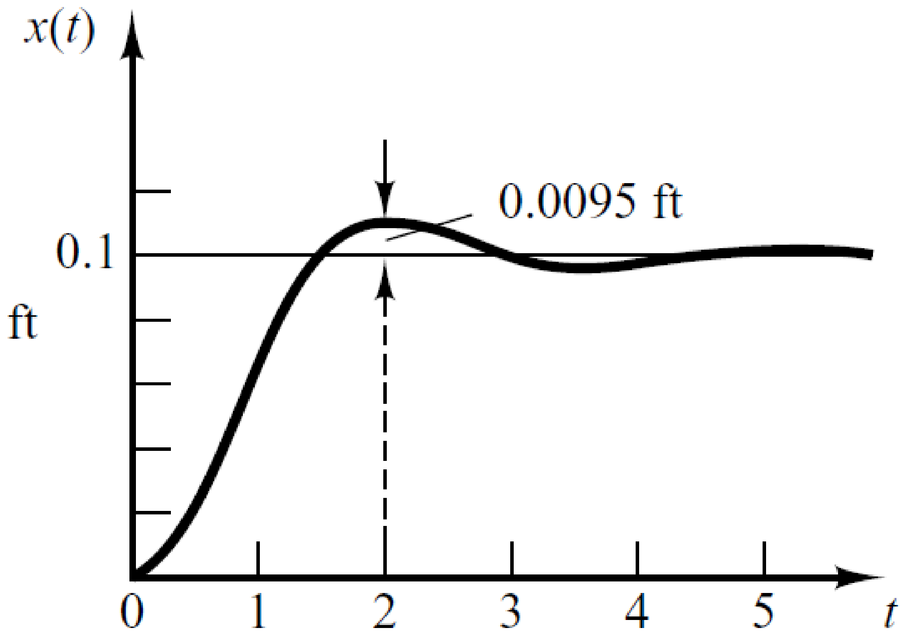
(b)

Example#8

Figure (a) shows a mechanical vibratory system. When 2 lb of force (step input) is applied to the system, the mass oscillates, as shown in Figure (b). Determine m, b, and k of the system from this response curve.



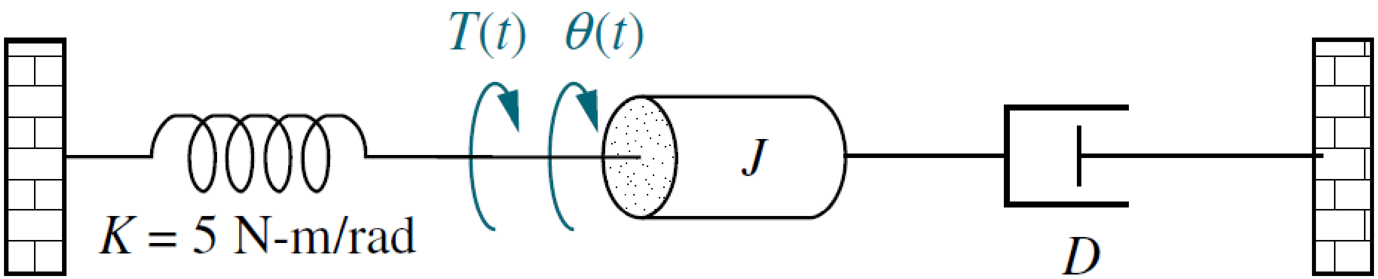
(a)



(b)

Example#9

Given the system shown in following figure, find *J and D to yield 20%* overshoot and a settling time of 2 seconds for a step input of torque *T(t)*.



$$G(s) = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}}$$

$$2\zeta\omega_n = 4$$

$$\omega_n = \sqrt{\frac{K}{J}}$$

$$\zeta = \frac{4}{2\omega_n} = 2\sqrt{\frac{J}{K}}$$

$$T_s = 2 = \frac{4}{\zeta\omega_n}$$

Example#9

$$\omega_n = \sqrt{\frac{K}{J}} \qquad \zeta = 2\sqrt{\frac{J}{K}}$$

20% overshoot implies $\zeta = 0.456$. Therefore,

$$\zeta = 2\sqrt{\frac{J}{K}} = 0.456$$

Hence,

$$\frac{J}{K} = 0.052$$

From the problem statement, $K = 5$ N-m/rad.

$$J = 0.26 \text{ kg-m}^2$$

Example#9

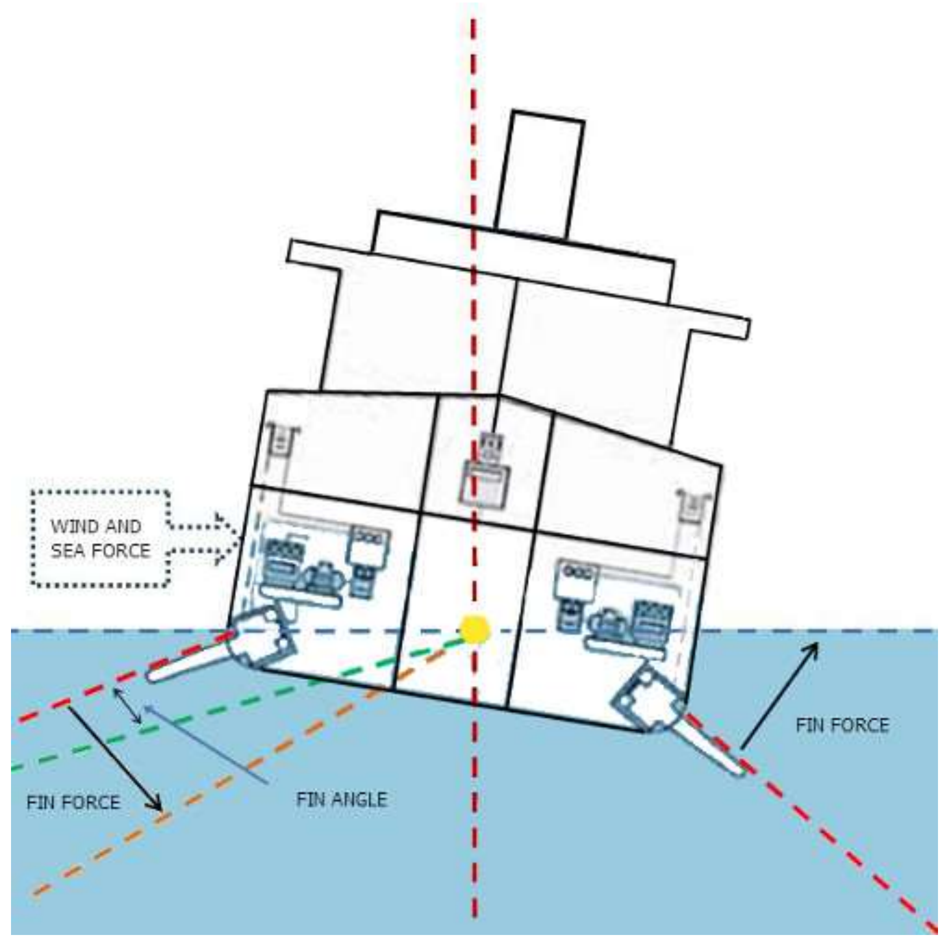
$$G(s) = \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}}$$

$$2\zeta\omega_n = \frac{D}{J}$$

$$D = 1.04 \text{ N-m-s/rad}$$

Example # 10

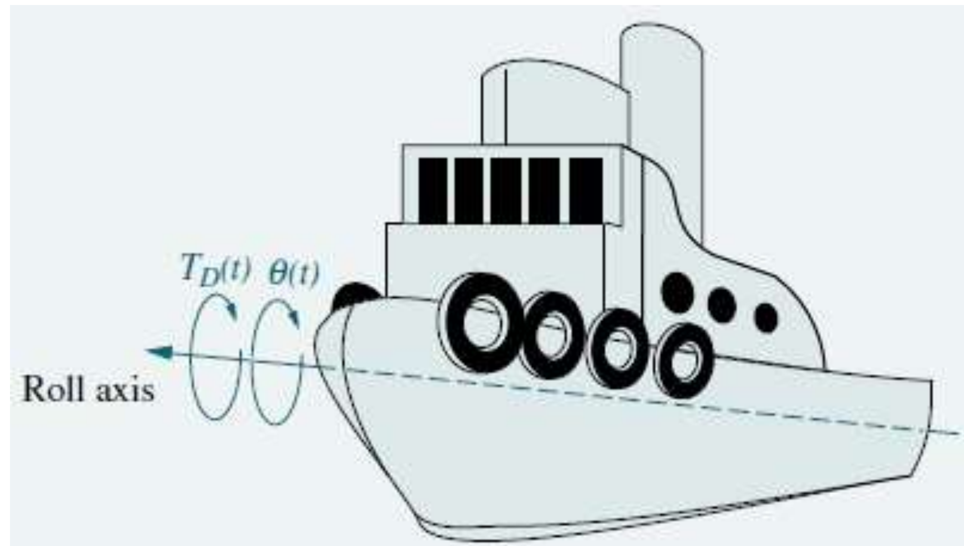
- Ships at sea undergo motion about their roll axis, as shown in Figure. Fins called stabilizers are used to reduce this rolling motion. The stabilizers can be positioned by a closed-loop roll control system that consists of components, such as fin actuators and sensors, as well as the ship's roll dynamics.



Example # 10

- Assume the roll dynamics, which relates the roll-angle output, $\theta(s)$, to a disturbance-torque input, $T_D(s)$, is

$$\frac{\theta(s)}{T_D(s)} = \frac{2.25}{(s^2 + 0.5s + 2.25)}$$



- Do the following:

  a)Find the natural frequency, damping ratio, peak time, settling time, rise time, and percent overshoot.

  b)Find the analytical expression for the output response to a unit step input.

Step Response of critically damped System ($\zeta = 1$)

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\omega_n)^2} \xrightarrow{\text{Step Response}} C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

- The partial fraction expansion of above equation is given as

$$\frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

$$c(t) = 1 - e^{-\omega_n t} - \omega_n e^{-\omega_n t} t$$

$$c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t)$$

Second – Order System

Example 11: Describe the nature of the second-order system response via the value of the damping ratio for the systems with transfer function

1. $G(s) = \dfrac{12}{s^2 + 8s + 12}$

2. $G(s) = \dfrac{16}{s^2 + 8s + 16}$

3. $G(s) = \dfrac{20}{s^2 + 8s + 20}$

Do them as your own revision

Example-12

- For each of the transfer functions find the locations of the poles and zeros, plot them on the s-plane, and then write an expression for the general form of the step response without solving for the inverse Laplace transform. State the nature of each response (overdamped, underdamped, and so on).

a. $T(s) = \dfrac{2}{s+2}$

b. $T(s) = \dfrac{5}{(s+3)(s+6)}$

c. $T(s) = \dfrac{10(s+7)}{(s+10)(s+20)}$

d. $T(s) = \dfrac{20}{s^2+6s+144}$

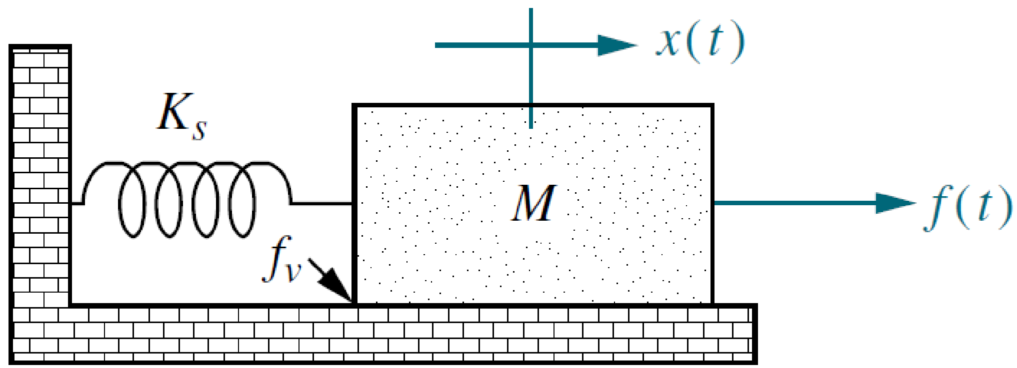
e. $T(s) = \dfrac{s+2}{s^2+9}$

f. $T(s) = \dfrac{(s+5)}{(s+10)^2}$

Example-13

- Solve for x(t) in the system shown in Figure if f(t) is a unit step.

$M = 1$ kg
$K_s = 5$ N/m
$f_v = 1$ N-s/m
$f(t) = u(t)$ N



END