Lecture 6.1 Applications of Determinants

Minors and cofactors of a Matrix

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Definition 1:

Given a matrix A, the Minor of $a_{ij} \equiv M_{ij}$, is determinant obtained from A by removing i^{th} row and j^{th} column.

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$ **is determinant obtained by deleting 1st row and 1st column**

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},\ M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},\ M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix},\ M_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix},\ M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix},\ M_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix},\ M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$\boxed{\textbf{Cofactor of } a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}}$$

Signs of Cofactors

$$\text{For 2x2 – matrix } \begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

$$\text{For 3x3 – matrix } \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\text{For 4x4 – matrix } \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Definition 2:

Given a matrix **A**, the **cofactor** of the element $\mathbf{a_{ij}}$ is a scalar obtained by multiplying together the term $\mathbf{(-1)^{i+j}}$ and the minor obtained from **A** by removing the i^{th} row and the j^{th} column.

Example:1.

Find all minors and cofactors of the matrix

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$

Solution:

$$\mathrm{M}_{11} = \begin{vmatrix} 0 & 3 \\ 5 & -4 \end{vmatrix} = -15, \quad \mathrm{M}_{12} = \begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} = -10, \quad \mathrm{M}_{13} = \begin{vmatrix} 1 & 0 \\ 2 & -5 \end{vmatrix} = 5$$

$$\mathrm{M}_{21} = \begin{vmatrix} 4 & -1 \\ 5 & -4 \end{vmatrix} = -11, \quad \mathrm{M}_{22} = \begin{vmatrix} 3 & -1 \\ 2 & -4 \end{vmatrix} = -10, \quad \mathrm{M}_{23} = \begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} = 7$$

$$\mathrm{M}_{31} = \begin{vmatrix} 4 & -1 \\ 0 & 3 \end{vmatrix} = 12, \quad \mathrm{M}_{32} = \begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} = 10, \quad \mathrm{M}_{33} = \begin{vmatrix} 3 & 4 \\ 1 & 0 \end{vmatrix} = -4$$

$$\boxed{\textbf{Cofactor of } a_{ij} = C_{ij} = (-1)^{i+j} M_{ij}}$$

$$\begin{array}{lll} C_{11} = -15, & C_{12} = 10, & C_{13} = 5 \\ C_{21} = 11, & C_{22} = -10, & C_{23} = -7 \\ C_{31} = 12, & C_{32} = -10, & C_{33} = -4 \end{array}$$

NOTE: Matrix of cofactors , $C = \begin{bmatrix} -15 & 10 & 5 \\ 11 & -10 & -7 \\ 12 & -10 & -4 \end{bmatrix}$

<u>NOTE: Determinant of matrix of Cofactors by the method of cofactors</u>

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$\det(A) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$
$$\det(A) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

The above equations can be used to check that the cofactors are found correctly as the values of determinants found must be equal, we open matrix from any row or column.

Example: 2 .

Find the determinant of the matrix A by method of cofactors,

$$A = \begin{bmatrix} 3 & 4 & -1 \\ 1 & 0 & 3 \\ 2 & 5 & -4 \end{bmatrix}$$

Solution:

Using the cofactors found in the last example.

Expanding from First row

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$= 3(-15)+4(10)+(-1)(5)$$
$$=-45 + 40 - 5 = -10$$

NOTE: 3. We can find determinant by opening matrix from second or third row or first column, the value of the determinant will be same

$$\det(A) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$
$$=(1)(11)+0(-10)+3(-7)=11 - 21 = -10$$

$$\det(A) = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$
$$= 2(12) +5(-10)+(-4)(-4) =24 - 50 +16 = -10$$

NOTE : 4. Determinant of A can be obtained by multiplying any row or any column of matrix A with the corresponding cofactors of the matrix.

NOTE: 5. Determinant of matrix A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

det A = $a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$.

Lecture 6.2 : Inverse by method of Cofactors

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \det \mathbf{A} \neq 0.$$

Step:1. **Find Matrix of cofactors**

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Step : 2. **Find Adjoint of matrix A , adj(A)**

$$\mathbf{Adj(A)} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

Step: 3.

If A is an invertible matrix, det(A) $\neq$ 0, then

$$A^{-1} = \frac{1}{\det A}[adj(A)]$$

Example: 3 . Find A^{-1} of matrix A

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ -2 & 0 & -4 \end{bmatrix} \text{ by the method of cofactors.}$$

Solution: Cofactors of the matrix A are

$$C_{11} = \begin{vmatrix} 3 & 2 \\ 0 & -4 \end{vmatrix} = -12, C_{12} = -\begin{vmatrix} 0 & 2 \\ -2 & -4 \end{vmatrix} = -4, C_{13} = \begin{vmatrix} 0 & 3 \\ -2 & 0 \end{vmatrix} = 6$$

$$C_{21} = -\begin{vmatrix} 0 & 3 \\ 0 & 4 \end{vmatrix} = 0, \quad C_{22} = \begin{vmatrix} 2 & 3 \\ -2 & -4 \end{vmatrix} = -2, \; C_{23} = -\begin{vmatrix} 2 & 0 \\ -2 & 0 \end{vmatrix} = 0,$$

$$C_{31} = \begin{vmatrix} 0 & 3 \\ 3 & 2 \end{vmatrix} = -9, \quad C_{32} = -\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = -4, \quad C_{33} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

$$\text{Matrix of cofactors,} \quad C = \begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix}$$

$$\text{Adjoint of matrix A,} \; \text{adj}(A) = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$
$$= 2(-12)+0(-4)+3(6)$$
$$= -24 + 18 = -6 \neq 0$$

Inverse of the matrix A is

$$A^{-1} = \frac{1}{\det A}[adj(A)] = \frac{1}{-6}\begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}$$

NOTE :

If we can find A^{-1} , then solution of linear system AX = B is $\quad X = A^{-1}B$

Lecture 6.3 : Cramer's Rule

Using determinants to solve a system of linear equations.

Theorem:

If A is $n \times n$ matrix with det(A)$\neq$ 0, then the linear system AX = B has a <u>unique</u> solution X = (x_j) given by

$$x_j = \frac{\det(A_j)}{\det(A)} \quad , j = 1,2,\ldots,n$$

Where A_j is the matrix obtained by replacing the jth column of A by B.

<u>NOTE:</u> If A is 3x3 matrix , then the solution of the system AX = B is

$$\mathrm{x} = \frac{\det(\mathrm{A}_1)}{\det(\mathrm{A})}, \qquad \mathrm{y} = \frac{\det(\mathrm{A}_2)}{\det(\mathrm{A})}, \qquad \mathrm{z} = \frac{\det(\mathrm{A}_3)}{\det(\mathrm{A})}$$

Example 4.

Use Cramer's Rule to solve

$$\begin{aligned} 4x + 5y \qquad &= 2 \\ 11x + y + 2z &= 3 \\ x + 5y + 2z &= 1 \end{aligned}$$

Solution:

$$\mathbf{A} = \begin{bmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}, \quad \mathbf{A_1} = \begin{bmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix}, \quad \mathbf{A_2} = \begin{bmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}, \quad \mathbf{A_3} = \begin{bmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{bmatrix}$$

det(A) = -132, det($\mathbf{A_1}$) =-36, det($\mathbf{A_2}$) = -24, det($\mathbf{A_3}$) = 12

$$x = \frac{\det(A_1)}{\det(A)} = \frac{-36}{-132} = \frac{3}{11},$$

$$y = \frac{\det(A_2)}{\det(A)} = \frac{-24}{-132} = \frac{2}{11},$$

$$z = \frac{\det(A_{31})}{\det(A)} = \frac{12}{-132} = \frac{-1}{11}$$

NOTE: If det(A) = 0, then there does not exist any solution of the system.