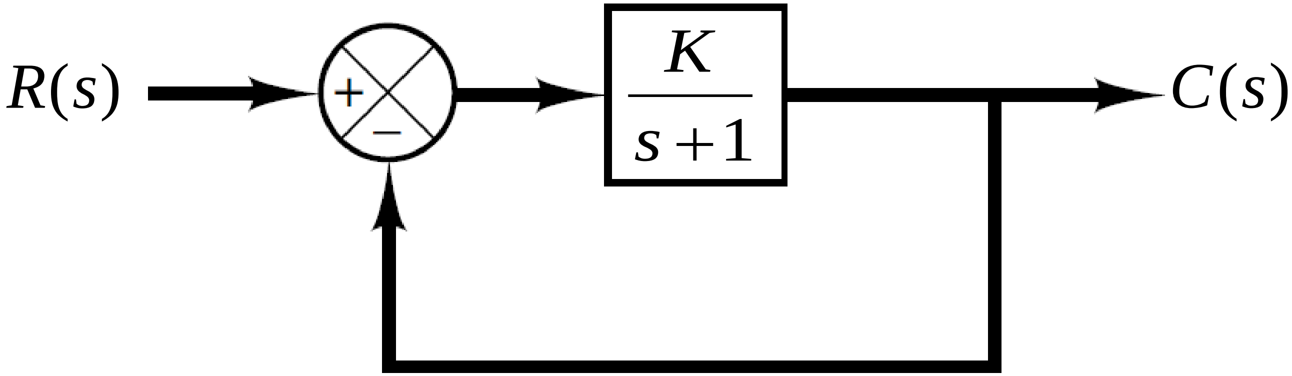
Construction of Root Locus

Lecture Outline

- Introduction
- The definition of a root locus
- Construction of root loci
- Closed loop stability via root locus

Introduction

- Consider a unity feedback control system shown below.



- The open loop transfer function *G(s)* of the system is $G(s)=\dfrac{K}{s+1}$

- And the closed transfer function is

$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)}=\frac{K}{s+1+K}$$

Introduction

- The open loop stability does not depend upon gain K.

$$G(s) = \frac{K}{s+1}$$

- Whereas, the location of closed loop poles vary with the variation in gain.

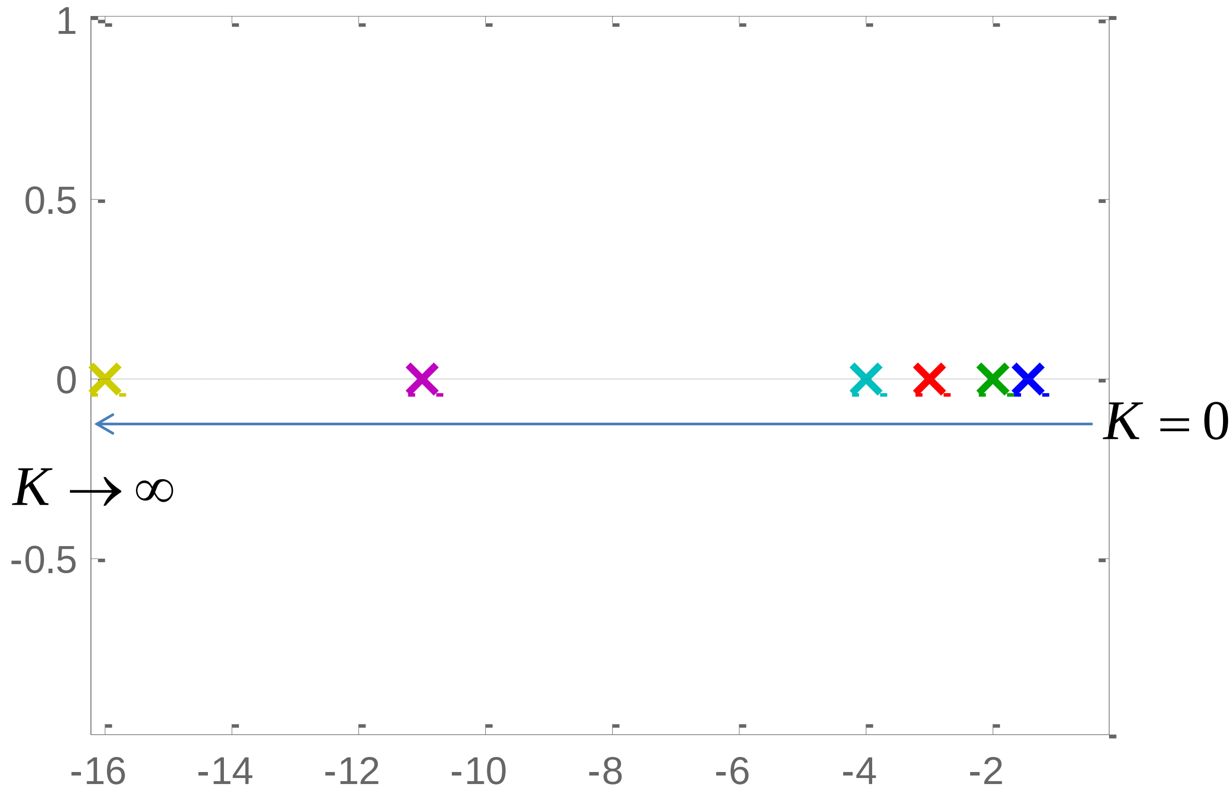
$$\frac{C(s)}{R(s)} = \frac{K}{s+1+K}$$

Introduction

- Location of closed loop Pole for different values of K (remember K>0).

$$\frac{C(s)}{R(s)} = \frac{K}{s+1+K}$$

K	Pole
0.5	-1.5
1	-2
2	-3
3	-4
5	-6
10	-11
15	-16



What is Root Locus?

- The root locus is the path of the roots of the characteristic equation traced out in the s-plane as a system parameter varies from zero to infinity.

How to Sketch root locus?

- One way is to compute the roots of the characteristic equation for all possible values of K.

$$\frac{C(s)}{R(s)} = \frac{K}{s+1+K}$$

K	Pole
0.5	-1.5
1	-2
2	-3
3	-4
5	-6
10	-11
15	-16

How to Sketch root locus?

- Computing the roots for all values of K might be tedious for higher order systems.

$$\frac{C(s)}{R(s)} = \frac{K}{s(s+1)(s+10)(s+20)+K}$$

K	Pole
0.5	?
1	?
2	?
3	?
5	?
10	?
15	?

Construction of Root Loci

- Finding the roots of the characteristic equation of degree higher than 3 is laborious and will need computer solution.
- A simple method for finding the roots of the characteristic equation has been developed by W. R. Evans and used extensively in control engineering.
- This method, called the *root-locus method,* is one in which the roots of the characteristic equation are plotted for all values of a system parameter.

Construction of Root Loci

- The roots corresponding to a particular value of this parameter can then be located on the resulting graph.

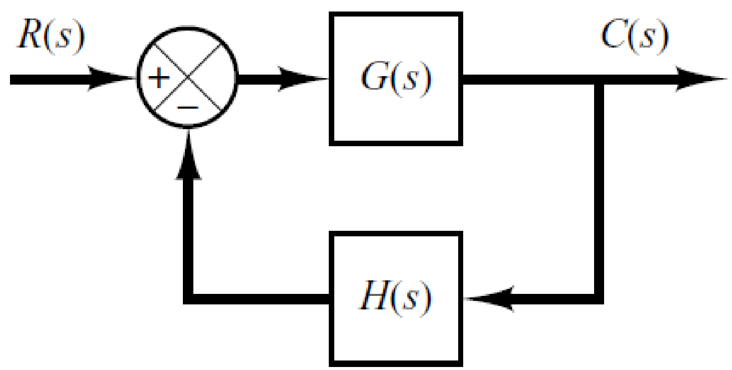
- Note that the parameter is usually the gain, but any other variable of the open-loop transfer function may be used.

- By using the root-locus method the designer can predict the effects on the location of the closed-loop poles of varying the gain value or adding open-loop poles and/or open-loop zeros.

Angle & Magnitude Conditions

- In constructing the root loci angle and magnitude conditions are important.

- Consider the system shown in following figure.



- The closed loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Construction of Root Loci

- The characteristic equation is obtained by setting the denominator polynomial equal to zero.

$$1 + G(s)H(s) = 0$$

- Or

$$G(s)H(s) = -1$$

- Where *G(s)H(s)* is a ratio of polynomial in s.

- Since *G(s)H(s)* is a complex quantity it can be split into angle and magnitude part.

Angle & Magnitude Conditions

- The angle of *G(s)H(s)=-1* is

$$\angle G(s)H(s) = \angle -1$$

$$\angle G(s)H(s) = \pm 180^{\circ}(2k+1)$$

- Where *k*=0,1,2,3…

- The magnitude of *G(s)H(s)=-1* is

$$\left|G(s)H(s)\right| = \left|-1\right|$$

$$\left|G(s)H(s)\right| = 1$$

Angle & Magnitude Conditions

- Angle Condition

$$\angle G(s)H(s) = \pm 180^\circ (2k+1) \qquad (k = 0,1,2,3\ldots)$$

- Magnitude Condition

$$|G(s)H(s)| = 1$$

- The values of s that fulfill both the angle and magnitude conditions are the roots of the characteristic equation, or the closed-loop poles.

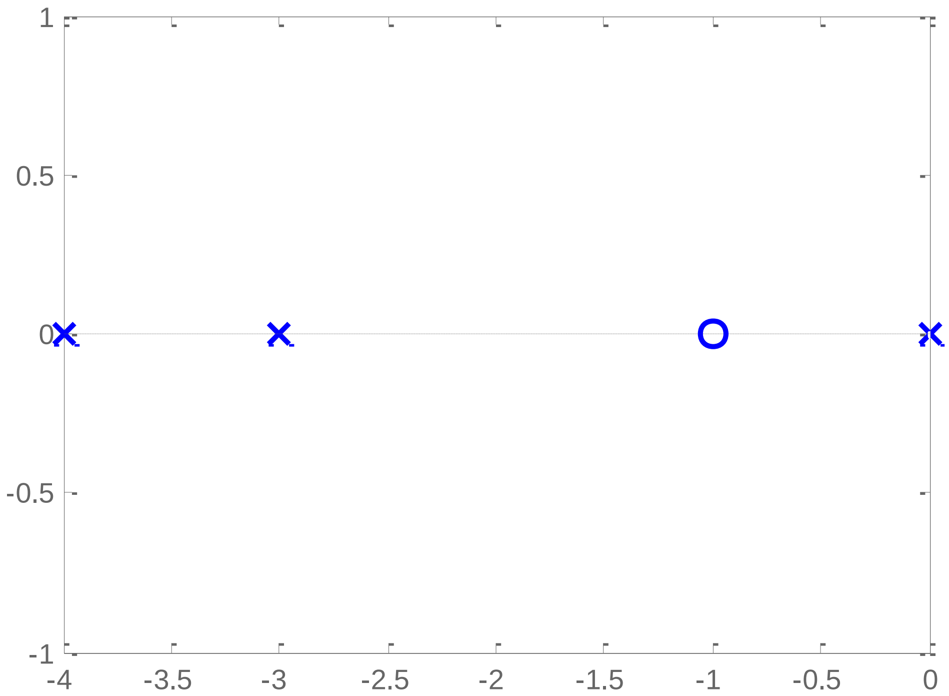
- A locus of the points in the complex plane satisfying the angle condition alone is the root locus.

Angle and Magnitude Conditions (Graphically)

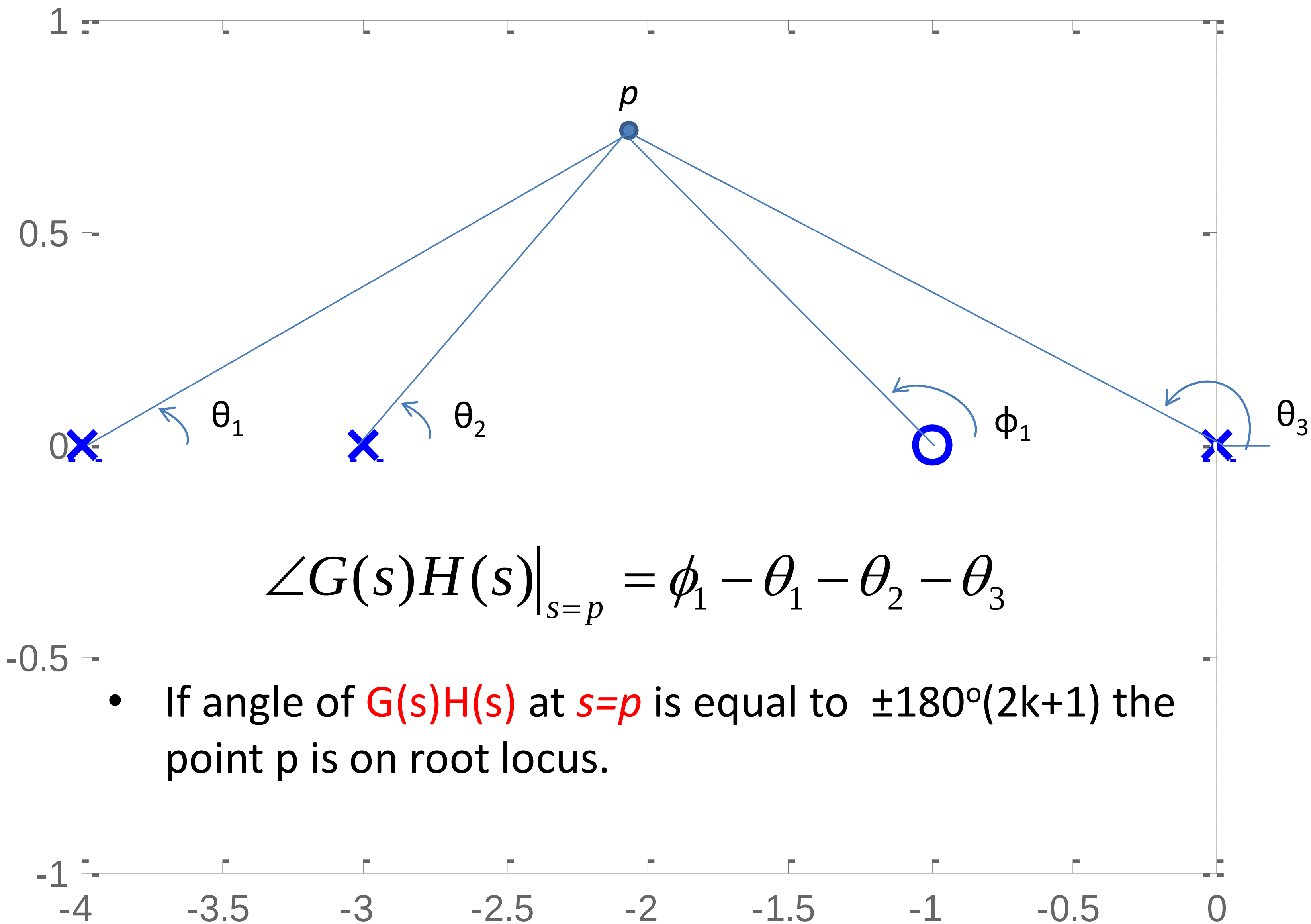
- To apply Angle and magnitude conditions graphically we must first draw the poles and zeros of G(s)H(s) in s-plane.

- For example if G(s)H(s) is given by

$$G(s)H(s) = \frac{s+1}{s(s+3)(s+4)}$$



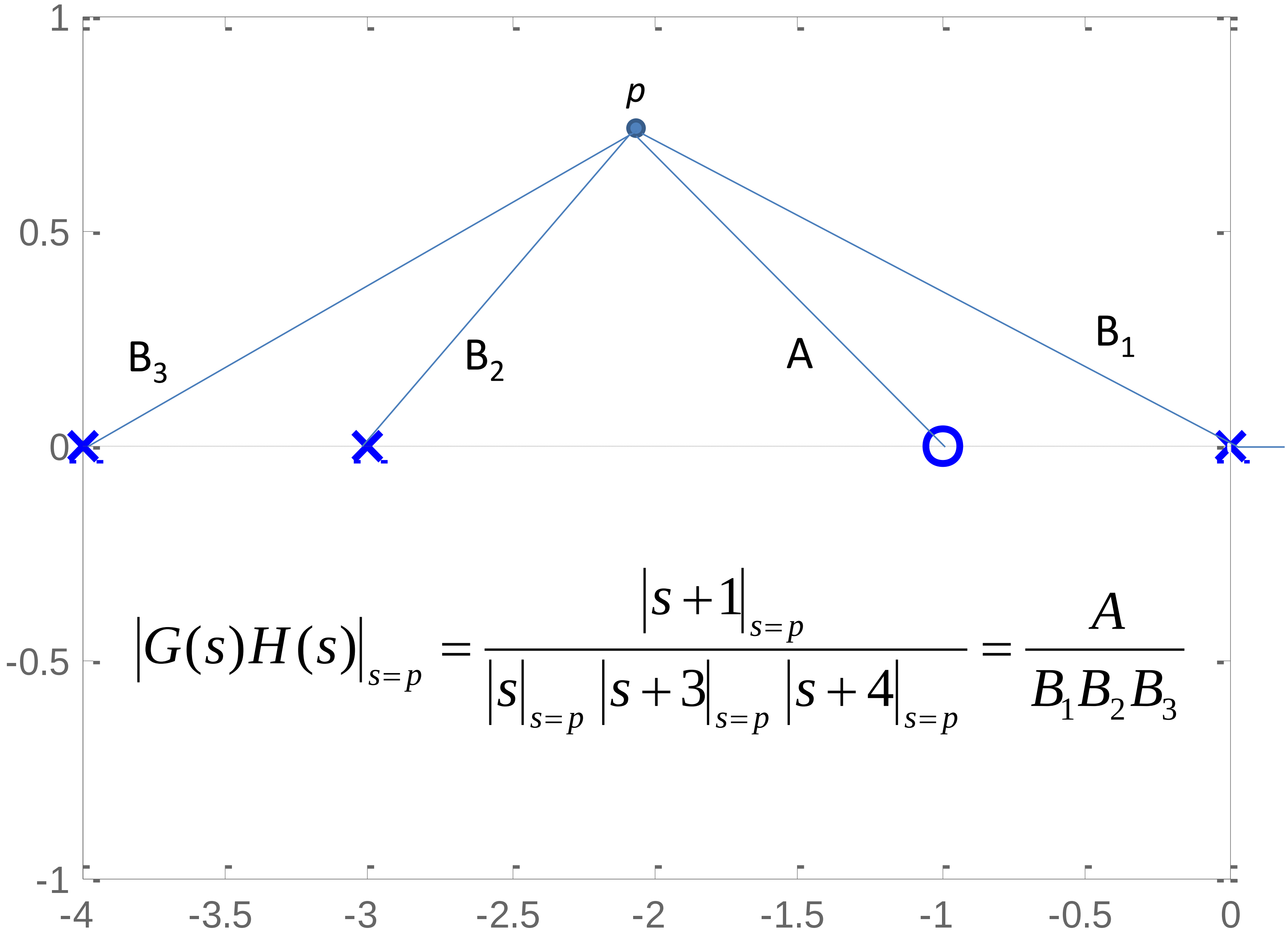
Angle and Magnitude Conditions (Graphically)



$$\angle G(s)H(s)\Big|_{s=p} = \phi_1 - \theta_1 - \theta_2 - \theta_3$$

- If angle of G(s)H(s) at *s=p* is equal to $\pm 180^o(2k+1)$ the point p is on root locus.

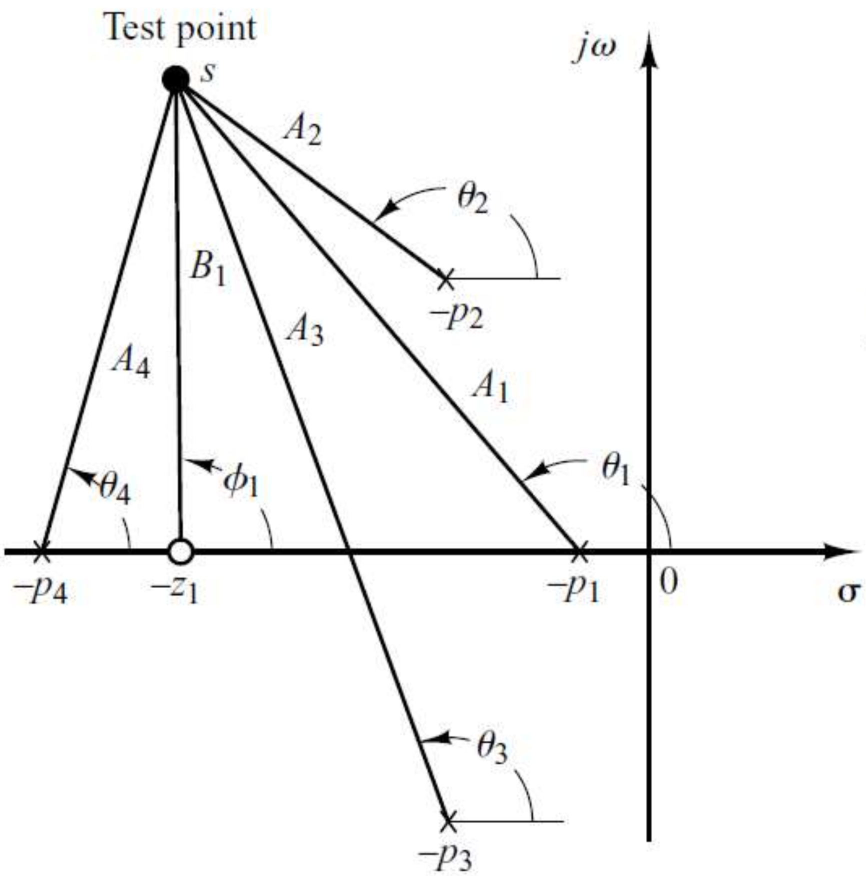
Angle and Magnitude Conditions graphically



$$\left|G(s)H(s)\right|_{s=p} = \frac{\left|s+1\right|_{s=p}}{\left|s\right|_{s=p}\left|s+3\right|_{s=p}\left|s+4\right|_{s=p}} = \frac{A}{B_1B_2B_3}$$

Angle and Magnitude Conditions (Graphically)

$$G(s)H(s) = \frac{K(s+z_1)}{(s+p_1)(s+p_2)(s+p_3)(s+p_4)}$$



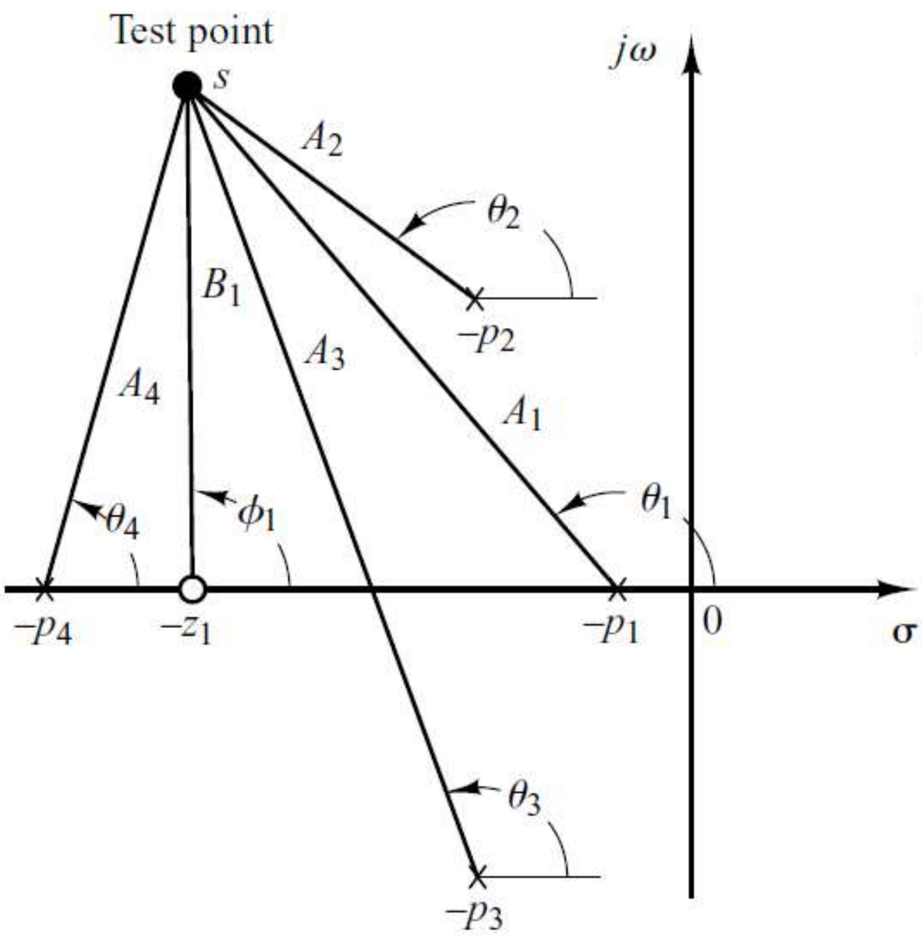
Angle condition

$$\underline{/G(s)H(s)} = \phi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4$$

- Where ϕ_1, θ_1, θ_2, θ_3, and θ_4 are measured in counter clockwise direction.

Angle and Magnitude Conditions (Graphically)

$$G(s)H(s) = \frac{K(s + z_1)}{(s + p_1)(s + p_2)(s + p_3)(s + p_4)}$$



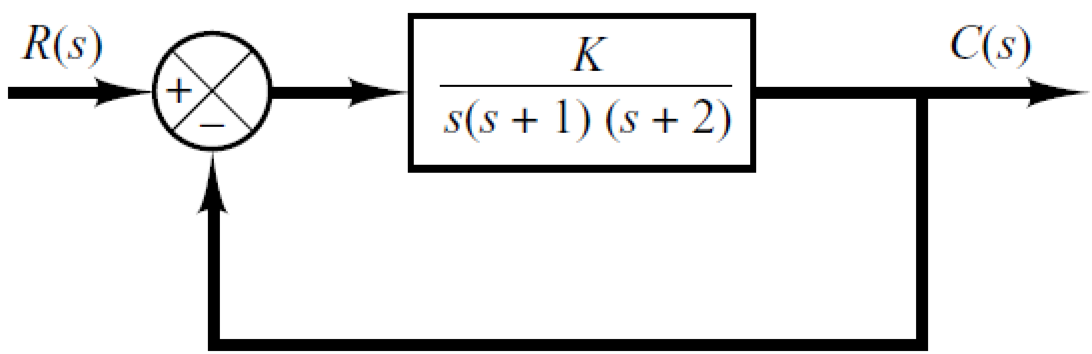
Magnitude Condition

$$|G(s)H(s)| = \frac{KB_1}{A_1 A_2 A_3 A_4}$$

- Where B_1, A_1, A_2, A_3, and A_4 are the magnitudes of the complex quantities $s+p_1$, $s+p_2$.....

Illustrative Example#1

- Apply angle and magnitude conditions (Analytically as well as graphically) on following unity feedback system.



Illustrative Example#1

- Here $$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

- For the given system the angle condition becomes

$$\angle G(s)H(s) = \angle \frac{K}{s(s+1)(s+2)}$$

$$\angle G(s)H(s) = \angle K - \angle s - \angle(s+1) - \angle(s+2)$$

$$\angle K - \angle s - \angle(s+1) - \angle(s+2) = \pm 180^\circ(2k+1)$$

Illustrative Example#1

- For example to check whether *s=-0.25* is on the root locus or not we can apply angle condition as follows.

$$\angle G(s)H(s)\big|_{s=-0.25} = \angle K\big|_{s=-0.25} - \angle s\big|_{s=-0.25} - \angle(s+1)\big|_{s=-0.25} - \angle(s+2)\big|_{s=-0.25}$$

$$\angle G(s)H(s)\big|_{s=-0.25} = -\angle(-0.25) - \angle(0.75) - \angle(1.75)$$

$$\angle G(s)H(s)\big|_{s=-0.25} = -180^\circ - 0^\circ - 0^\circ$$

$$\angle G(s)H(s)\big|_{s=-0.25} = \pm 180^\circ(2k+1)$$

Illustrative Example#1

- **Home work:**
  - check whether *$s=-0.2+j0.937$* is on the root locus or not (Graphically as well as analytically) ?
  - check whether *$s=-1+j2$* is on the root locus or not (Graphically as well as analytically) ?

Illustrative Example#1

- Here $$G(s)H(s)=\frac{K}{s(s+1)(s+2)}$$

- And the Magnitude condition becomes

$$\left|G(s)H(s)\right|=\left|\frac{K}{s(s+1)(s+2)}\right|=1$$

Illustrative Example#1

- Now we know from angle condition that the point s=-0.25 is on the rot locus. But we do not know the value of gain K at that specific point.

- We can use magnitude condition to determine the value of gain at any point on the root locus.

$$\left| \frac{K}{s(s+1)(s+2)} \right|_{s=-0.25} = 1$$

$$\left| \frac{K}{(-0.25)(-0.25+1)(-0.25+2)} \right|_{s=-0.25} = 1$$

Illustrative Example#1

$$\left|\frac{K}{(-0.25)(-0.25+1)(-0.25+2)}\right|_{s=-0.25} = 1$$

$$\left|\frac{K}{(-0.25)(0.75)(1.75)}\right| = 1$$

$$\left|\frac{K}{-0.3285}\right| = 1$$

$$\frac{K}{0.328} = 1$$

$$K = 0.328$$

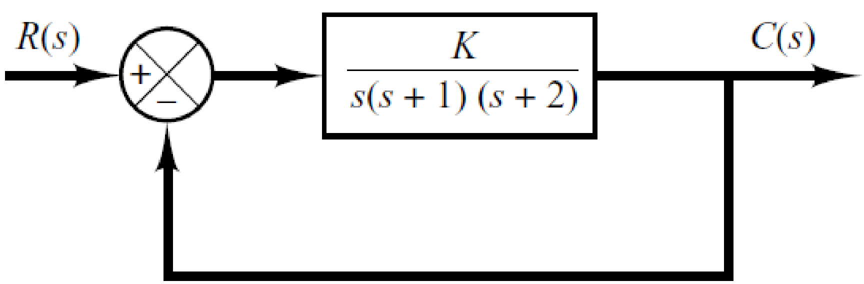
Illustrative Example#1

- **Home work:**
  - If $s=-0.2+j0.937$ is on the root locus determine the value of gain K at that point.
  - If $s=-1+j2$ is on the root locus determine the value of gain K at that point.

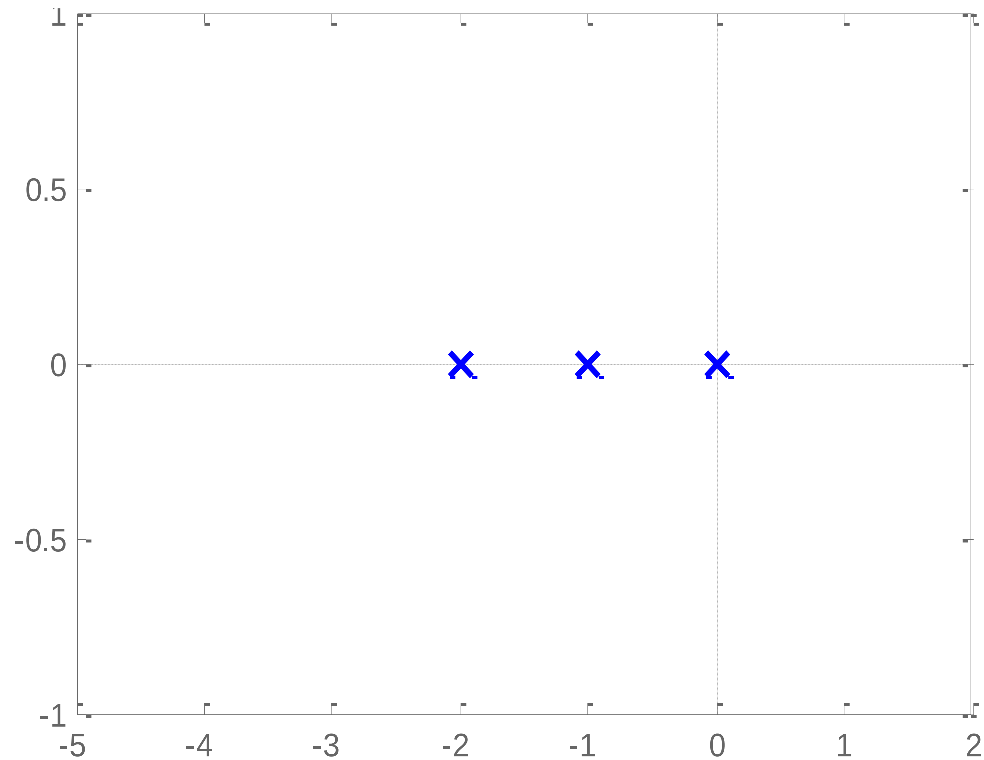
Construction of root locus

Construction of root loci

- Step-1: The first step in constructing a root-locus plot is to locate the open-loop poles and zeros in s-plane.



$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$



Construction of root loci

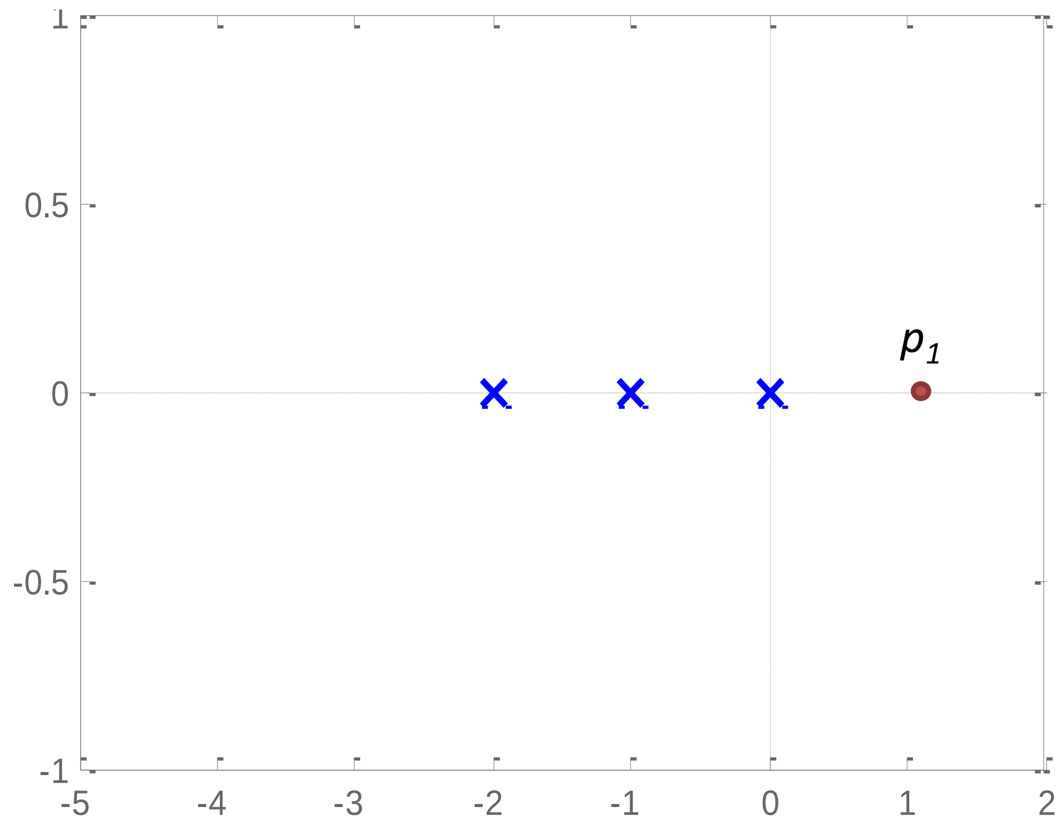
- Step-2: Determine the root loci on the real axis.

- To determine the root loci on real axis we select some test points.
- e.g: p_1 (on positive real axis).

$$\angle s = \angle s + 1 = \angle s + 2 = 0°$$

- The angle condition is not satisfied.

- Hence, there is no root locus on the positive real axis.



Construction of root loci

- Step-2: Determine the root loci on the real axis.

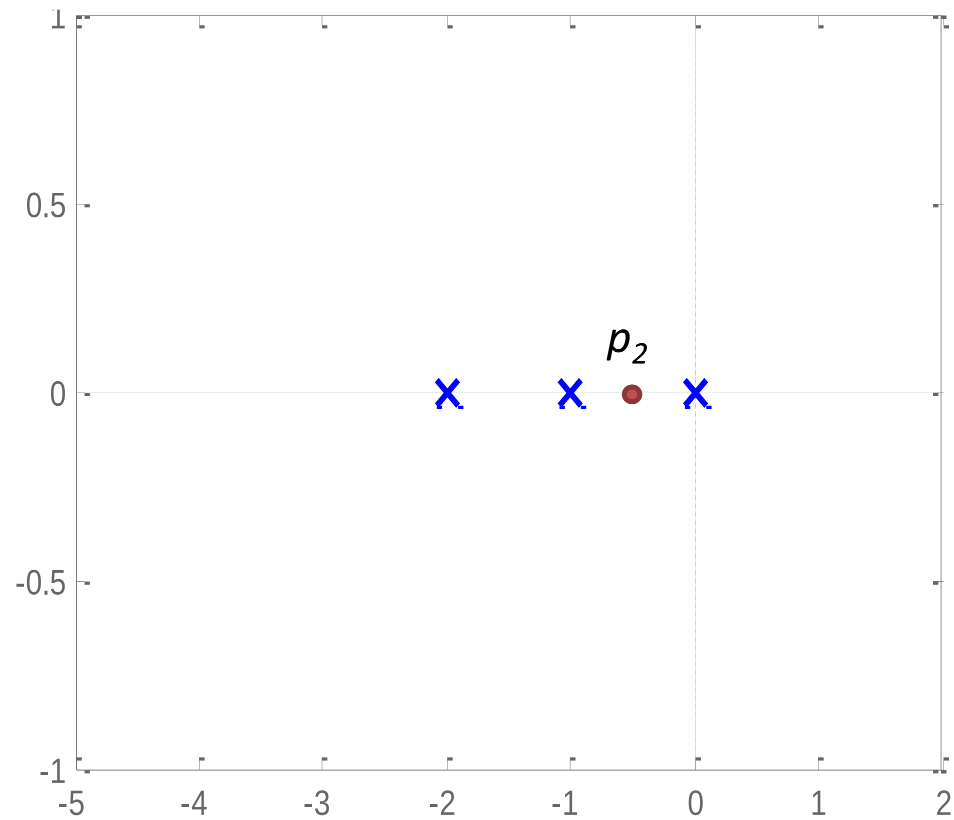
- Next, select a test point on the negative real axis between 0 and –1.
- Then

$$\underline{/s} = 180^\circ, \qquad \underline{/s+1} = \underline{/s+2} = 0^\circ$$

- Thus

$$-\underline{/s} - \underline{/s+1} - \underline{/s+2} = -180^\circ$$

- The angle condition is satisfied. Therefore, the portion of the negative real axis between 0 and –1 forms a portion of the root locus.



Construction of root loci

- Step-2: Determine the root loci on the real axis.

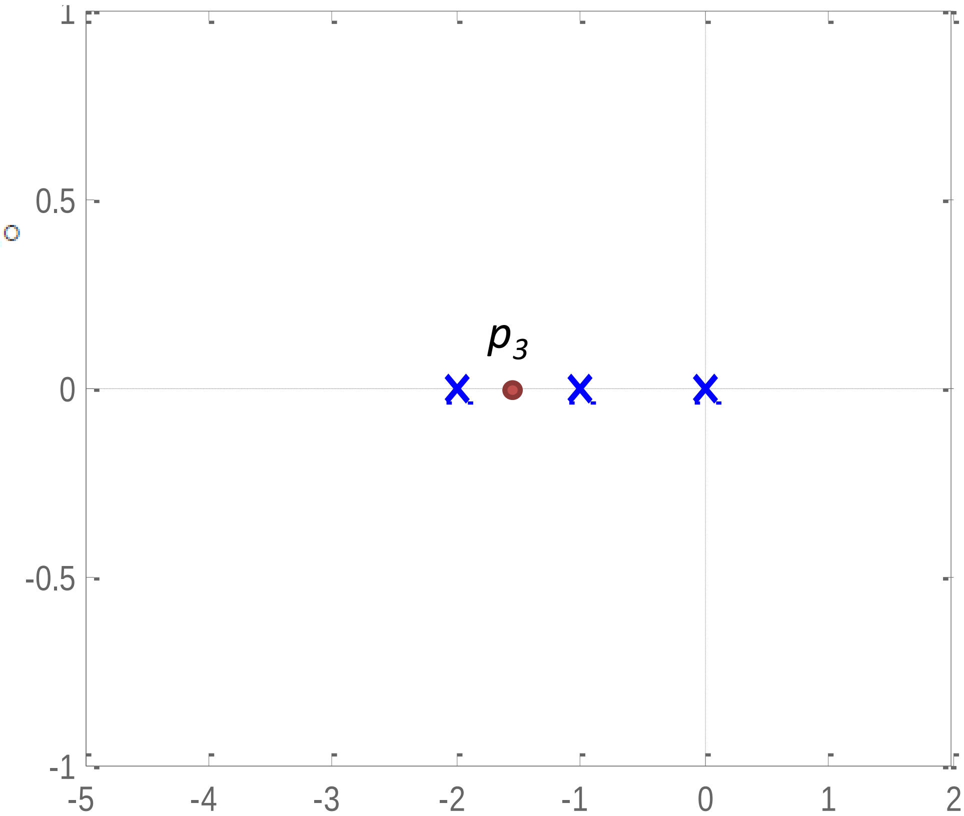
- Now, select a test point on the negative real axis between -1 and –2.
- Then

$$\underline{/s} = \underline{/s+1} = 180^\circ, \qquad \underline{/s+2} = 0^\circ$$

- Thus

$$-\underline{/s} - \underline{/s+1} - \underline{/s+2} = -360^\circ$$

- The angle condition is not satisfied. Therefore, the negative real axis between -1 and –2 is not a part of the root locus.

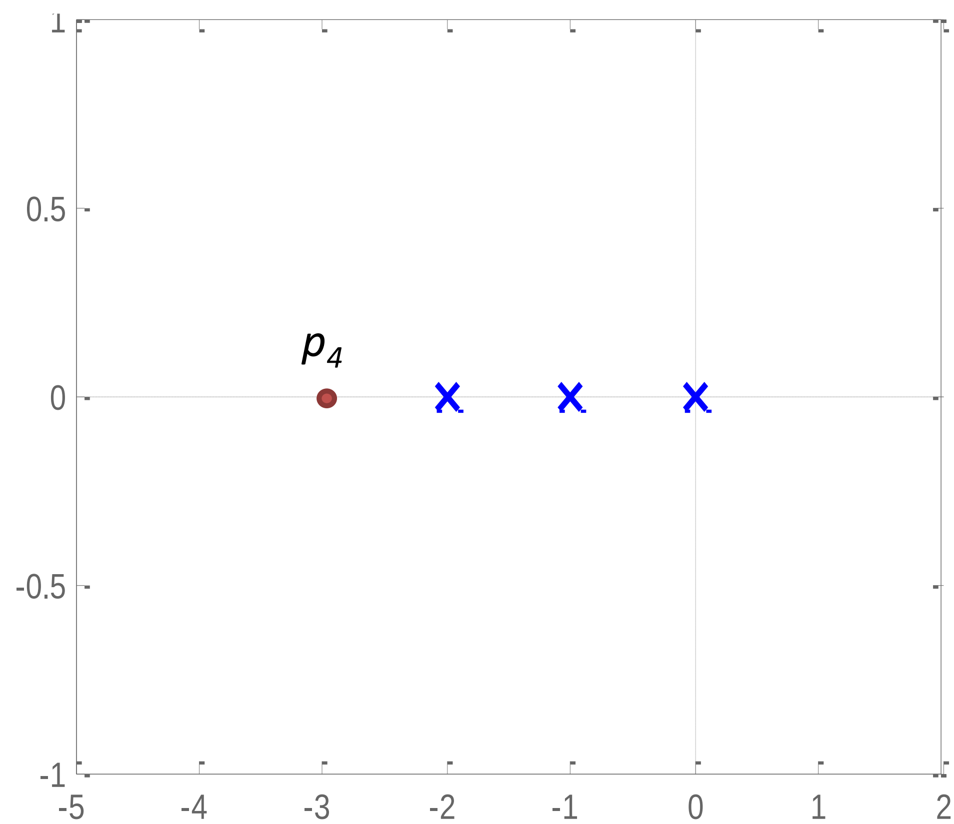


Construction of root loci

- Step-2: Determine the root loci on the real axis.

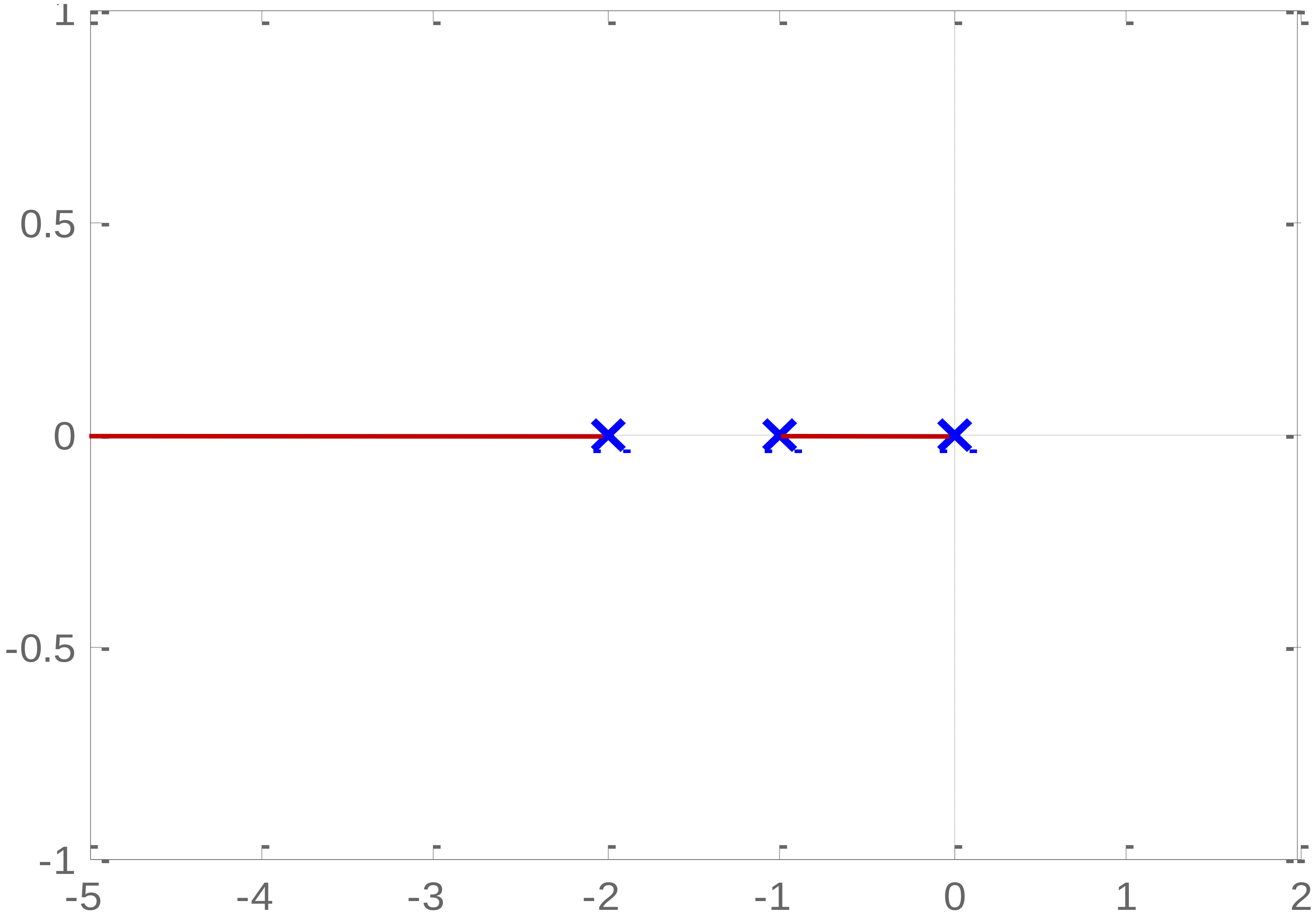
- Similarly, test point on the negative real axis between -2 and $-\infty$ satisfies the angle condition.

- Therefore, the negative real axis between -2 and $-\infty$ is part of the root locus.



Construction of root loci

- Step-2: Determine the root loci on the real axis.



Construction of root loci

- Step-3: Determine the *asymptotes* of the root loci.

$$Angle\ of\ \ asymptotes = \psi = \frac{\pm 180°(2k+1)}{n-m}$$

- where
- n-----> number of poles
- m-----> number of zeros
- For this Transfer Function $G(s)H(s) = \dfrac{K}{s(s+1)(s+2)}$

$$\psi = \frac{\pm 180°(2k+1)}{3-0}$$

Construction of root loci

- Step-3: Determine the *asymptotes* of the root loci.

$$
\begin{aligned}
\psi &= \pm 60^\circ && \text{when } k = 0 \\
&= \pm 180^\circ && \text{when } k = 1 \\
&= \pm 300^\circ && \text{when } k = 2 \\
&= \pm 420^\circ && \text{when } k = 3
\end{aligned}
$$

- Since the angle repeats itself as k is varied, the distinct angles for the asymptotes are determined as 60°, –60°, -180°and 180°.
- Thus, there are three asymptotes having angles 60°, –60°, 180°.

Construction of root loci

- Step-3: Determine the *asymptotes* of the root loci.

- Before we can draw these asymptotes in the complex plane, we must find the point where they intersect the real axis.

- Point of intersection of asymptotes on real axis (or centroid of asymptotes) can be find as out

$$\sigma = \frac{\sum poles - \sum zeros}{n-m}$$

Construction of root loci

- Step-3: Determine the *asymptotes* of the root loci.

- For $G(s)H(s) = \dfrac{K}{s(s+1)(s+2)}$

$$\sigma = \frac{(0-1-2)-0}{3-0}$$

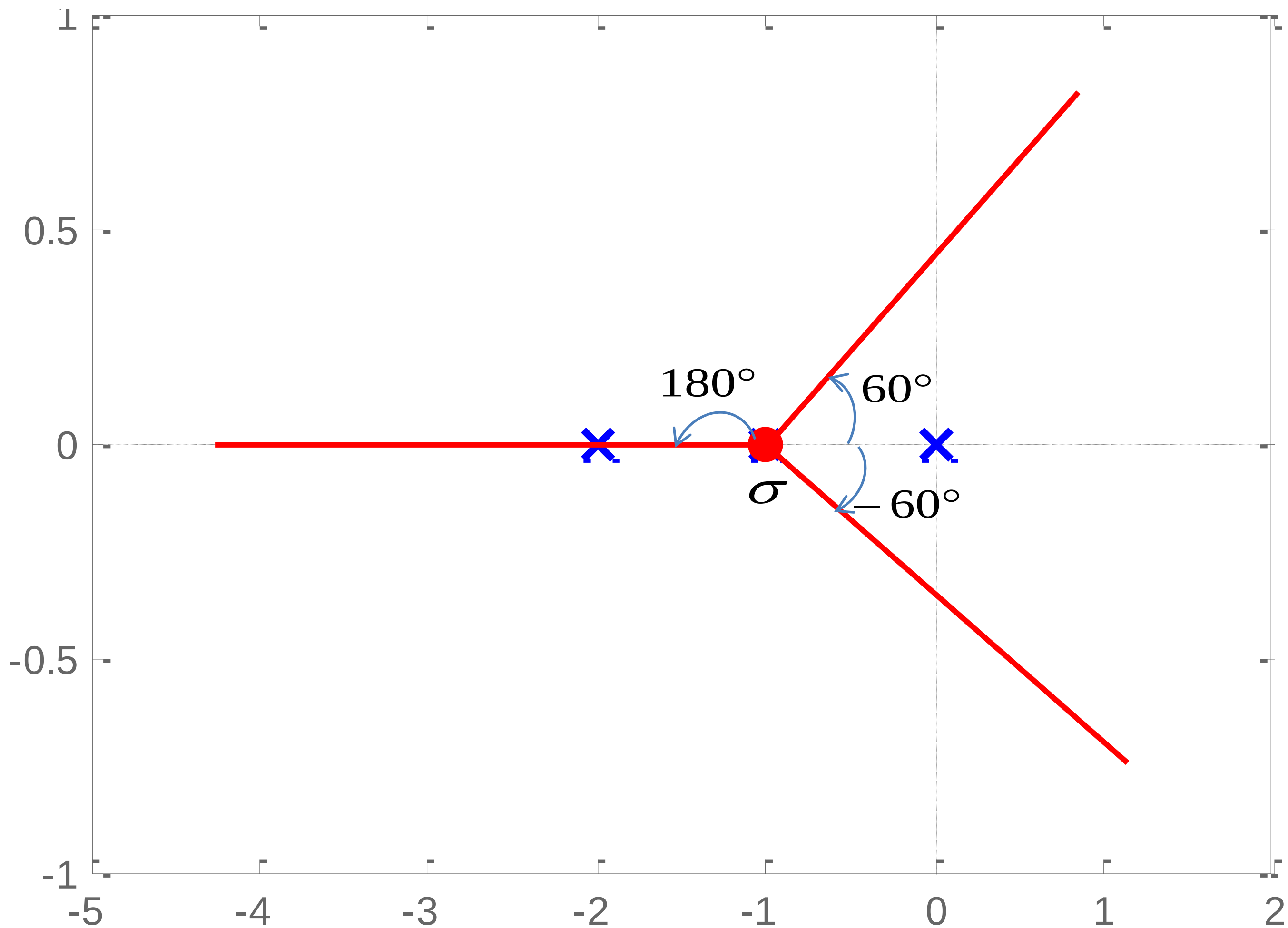
$$\sigma = \frac{-3}{3} = -1$$

Construction of root loci

- Step-3: Determine the *asymptotes* of the root loci.

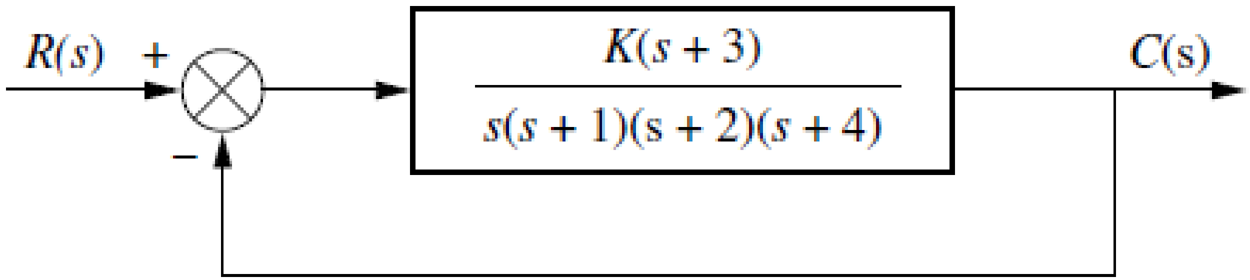
$$\psi = 60°, -60°, 180°$$

$$\sigma = -1$$



Home Work

- Consider following unity feedback system.



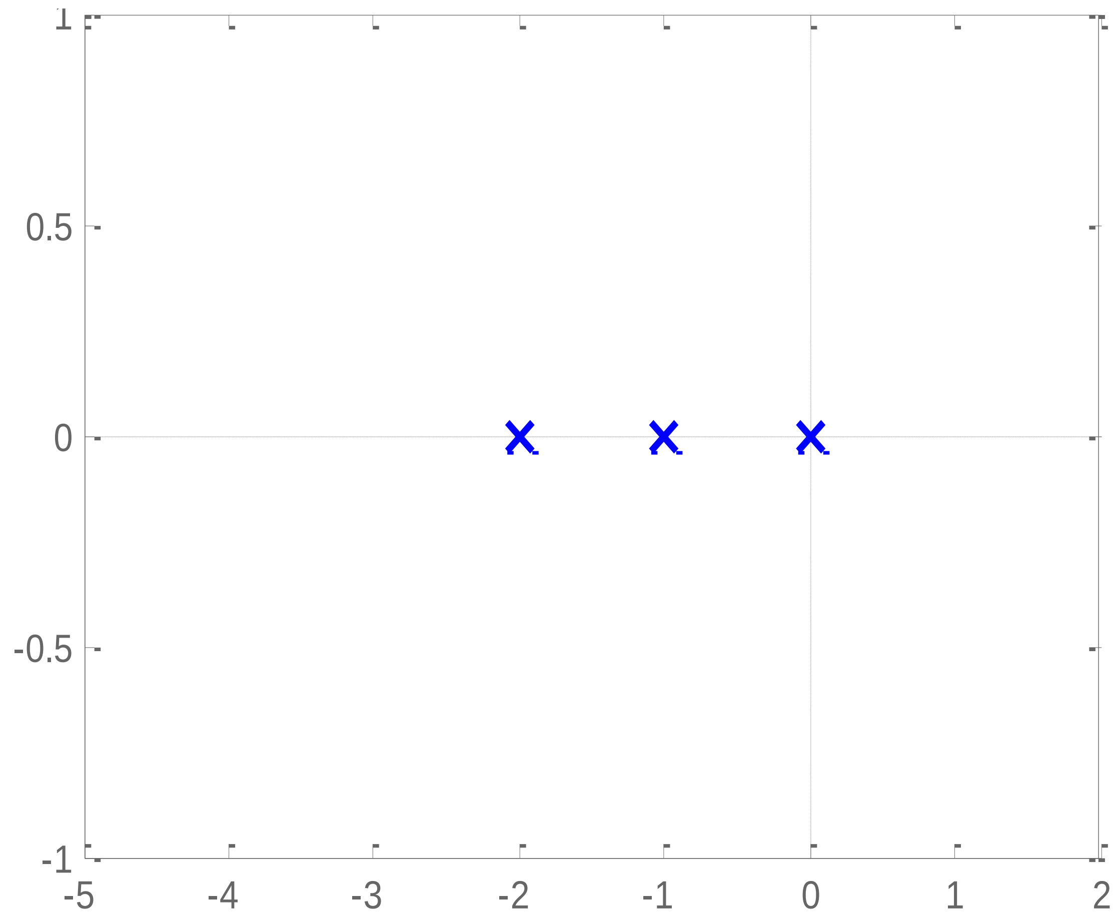
- Determine
  - Root loci on real axis
  - Angle of asymptotes
  - Centroid of asymptotes

Construction of root loci

- Step-4: Determine the *breakaway point*.

  - The breakaway point corresponds to a point in the s plane where multiple roots of the characteristic equation occur.

  - It is the point from which the root locus branches leaves real axis and enter in complex plane.

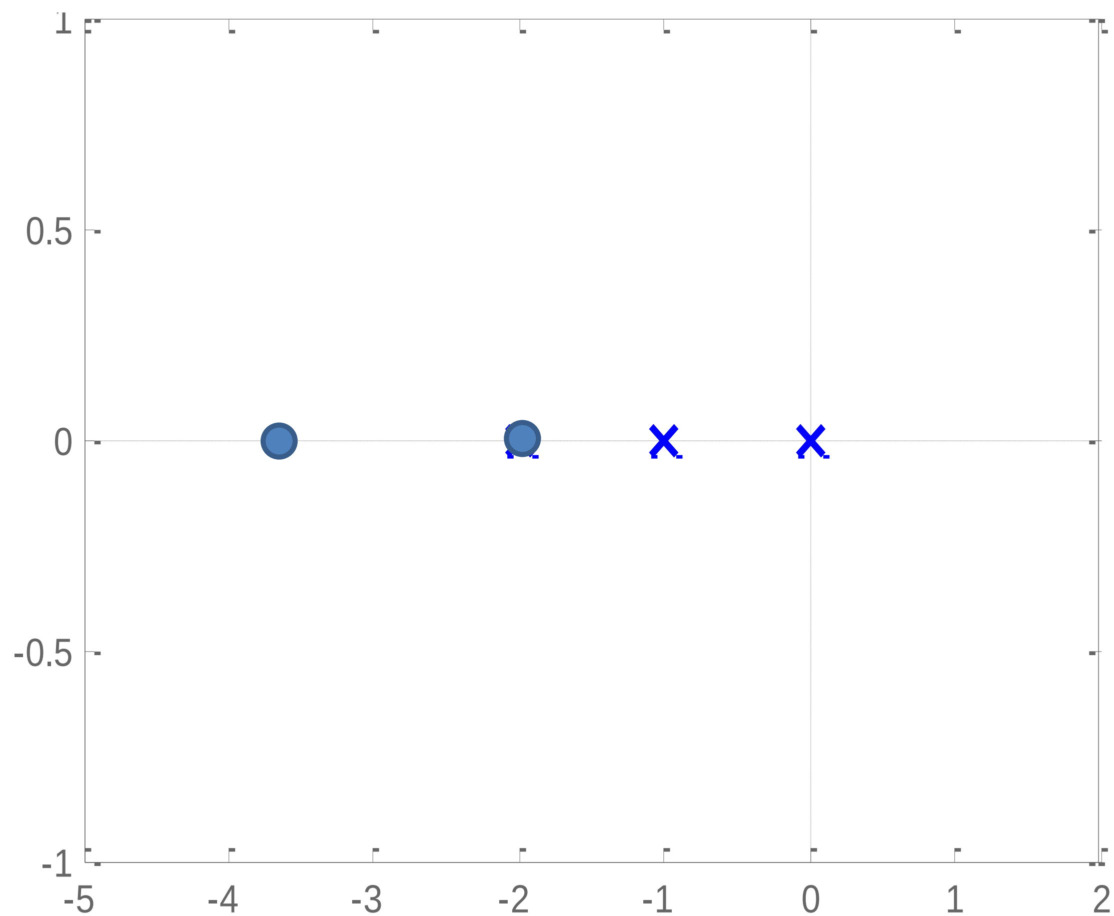


Construction of root loci

- Step-4: Determine the *break-in point*.

  - The break-in point corresponds to a point in the s plane where multiple roots of the characteristic equation occur.

  - It is the point where the root locus branches arrives at real axis.



Construction of root loci

- Step-4: Determine the *breakaway point* or *break-in point*.

  - The breakaway or break-in points can be determined from the roots of

$$\frac{dK}{ds} = 0$$

  - It should be noted that not all the solutions of dK/ds=0 correspond to actual breakaway points.

  - If a point at which dK/ds=0 is on a root locus, it is an actual breakaway or break-in point.

  - Stated differently, if at a point at which dK/ds=0 the value of K takes a real positive value, then that point is an actual breakaway or break-in point.

Construction of root loci

- Step-4: Determine the *breakaway point* or *break-in point*.

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

- The characteristic equation of the system is

$$1 + G(s)H(s) = 1 + \frac{K}{s(s+1)(s+2)} = 0$$

$$\frac{K}{s(s+1)(s+2)} = -1$$

$$K = -\left[s(s+1)(s+2)\right]$$

- The breakaway point can now be determined as

$$\frac{dK}{ds} = -\frac{d}{ds}\left[s(s+1)(s+2)\right]$$

Construction of root loci

- Step-4: Determine the *breakaway point* or *break-in point*.

$$\frac{dK}{ds} = -\frac{d}{ds}\left[s(s+1)(s+2)\right]$$

$$\frac{dK}{ds} = -\frac{d}{ds}\left[s^3 + 3s^2 + 2s\right]$$

$$\frac{dK}{ds} = -3s^2 - 6s - 2$$

- Set *dK/ds=0* in order to determine breakaway point.

$$-3s^2 - 6s - 2 = 0$$

$$3s^2 + 6s + 2 = 0$$

$$s = -0.4226$$

$$= -1.5774$$

Construction of root loci

- Step-4: Determine the *breakaway point* or *break-in point*.

$$s = -0.4226$$
$$= -1.5774$$

- Since the breakaway point must lie on a root locus between 0 and –1, it is clear that *s=–0.4226* corresponds to the actual breakaway point.

- Point s=–1.5774 is not on the root locus. Hence, this point is not an actual breakaway or break-in point.

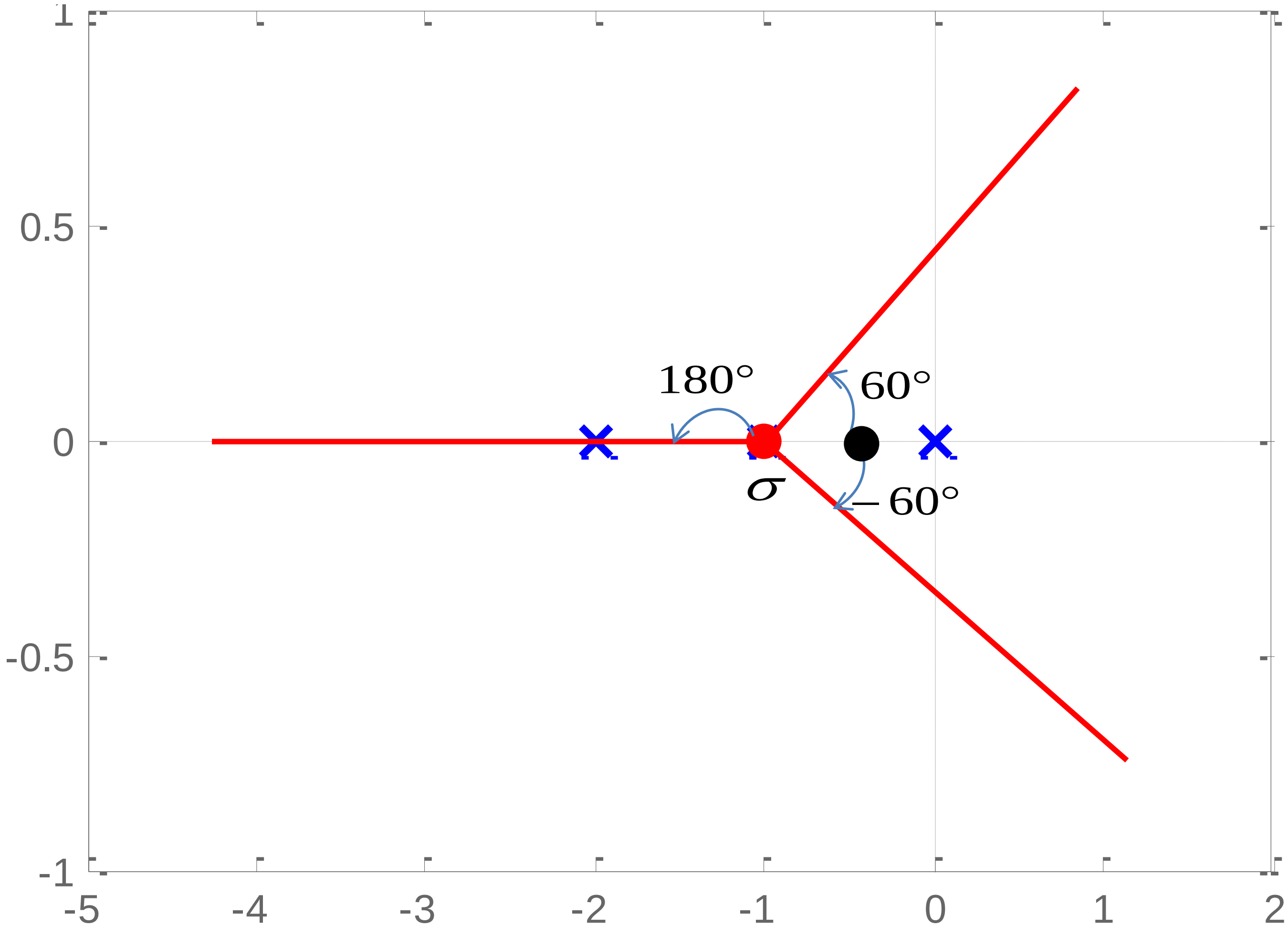
- In fact, evaluation of the values of K corresponding to s=–0.4226 and s=–1.5774 yields

$$K = 0.3849, \qquad \text{for } s = -0.4226$$
$$K = -0.3849, \qquad \text{for } s = -1.5774$$

Construction of root loci

- Step-4: Determine the *breakaway point*.

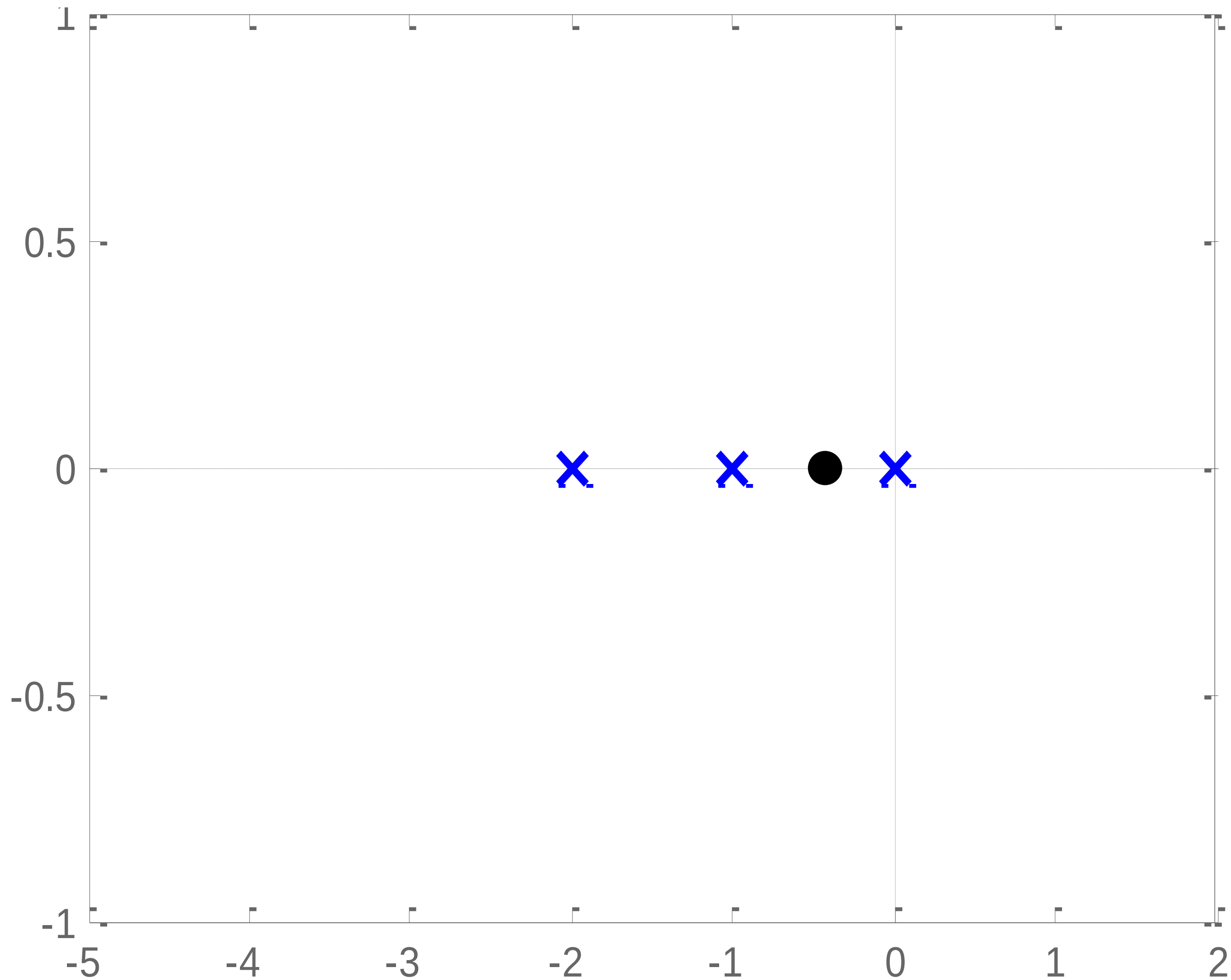
$$s = -0.4226$$



Construction of root loci

- Step-4: Determine the *breakaway point*.

$$s = -0.4226$$



Home Work

- Determine the Breakaway and break in points

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)}$$

Solution

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2-8s+15)}{(s^2+3s+2)}$$

$$\frac{K(s^2-8s+15)}{s^2+3s+2} = -1$$

$$K = -\frac{(s^2+3s+2)}{(s^2-8s+15)}$$

- Differentiating ***K*** with respect to **s** and setting the derivative equal to zero yields;

$$\frac{dK}{ds} = -\frac{[(s^2-8s+15)(2s+3)-(s^2+3s+2)(2s-8)]}{(s^2-8s+15)^2} = 0$$

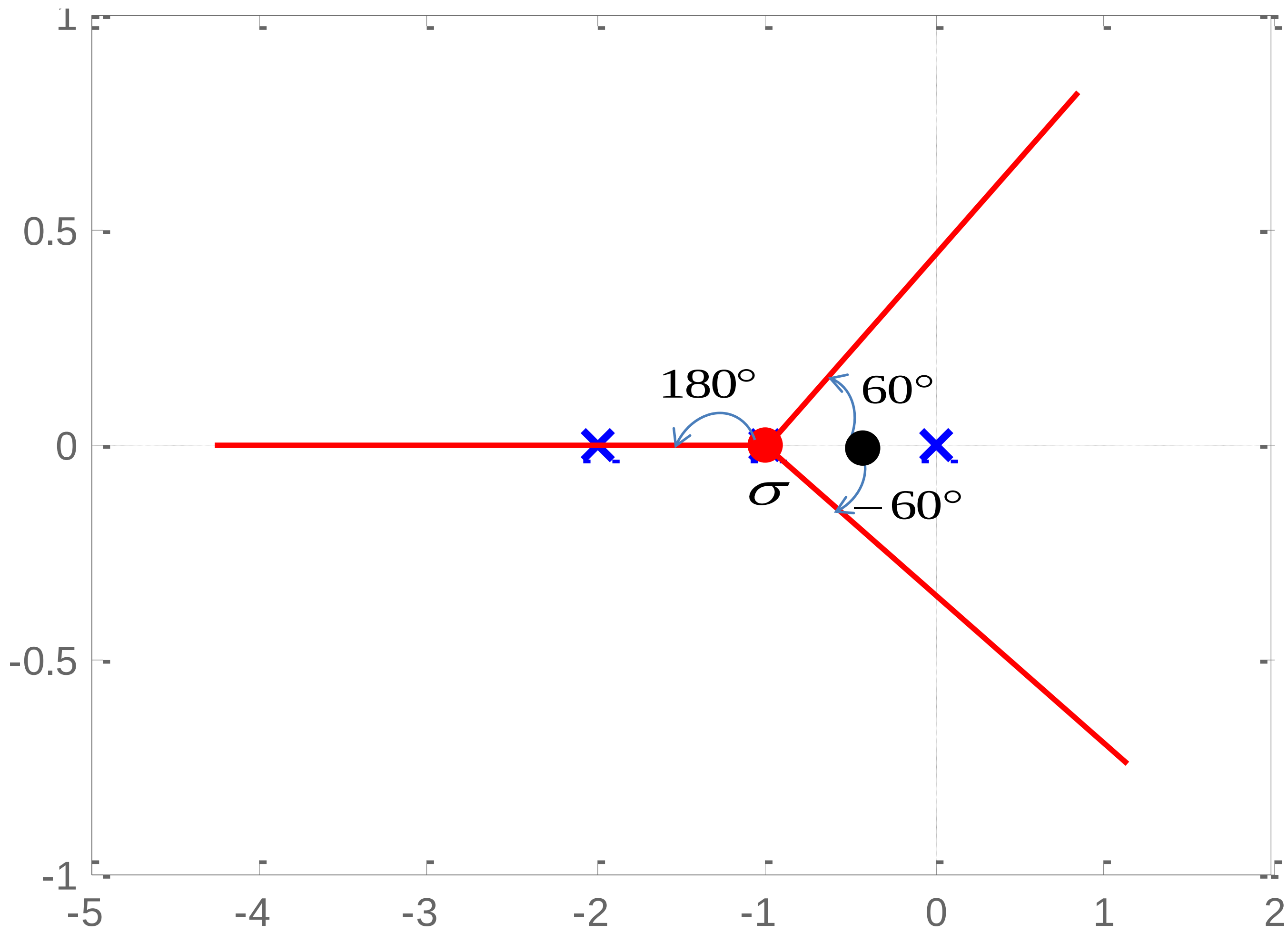
$$11s^2-26s-61=0$$

Hence, solving for s, we find the break-away and break-in points;

$$\boxed{s = -1.45 \text{ and } 3.82}$$

Construction of root loci

- Step-5: Determine the points where root loci cross the imaginary axis.



Construction of root loci

- Step-5: Determine the points where root loci cross the imaginary axis.
  - These points can be found by use of Routh's stability criterion.
  - Since the characteristic equation for the present system is

$$s^3 + 3s^2 + 2s + K = 0$$

  - The Routh Array Becomes

$$\begin{array}{lll} s^3 & 1 & 2 \\ s^2 & 3 & K \\ s^1 & \dfrac{6-K}{3} & \\ s^0 & K & \end{array}$$

Construction of root loci

- Step-5: Determine the points where root loci cross the imaginary axis.

- The value(s) of K that makes the system marginally stable is 6.

- The crossing points on the imaginary axis can then be found by solving the auxiliary equation obtained from the s^2 row, that is,

$$3s^2 + K = 3s^2 + 6 = 0$$

- Which yields

$$s = \pm j\sqrt{2}$$

s^3	1	2
s^2	3	K
s^1	$\dfrac{6-K}{3}$	
s^0	K	

Construction of root loci

- Step-5: Determine the points where root loci cross the imaginary axis.

- An alternative approach is to let $s=j\omega$ in the characteristic equation, equate both the real part and the imaginary part to zero, and then solve for ω and K.

- For present system the characteristic equation is

$$s^3 + 3s^2 + 2s + K = 0$$

$$(j\omega)^3 + 3(j\omega)^2 + 2j\omega + K = 0$$

$$(K - 3\omega^2) + j(2\omega - \omega^3) = 0$$

Construction of root loci

- Step-5: Determine the points where root loci cross the imaginary axis.

$$(K-3\omega^2)+j(2\omega-\omega^3)=0$$

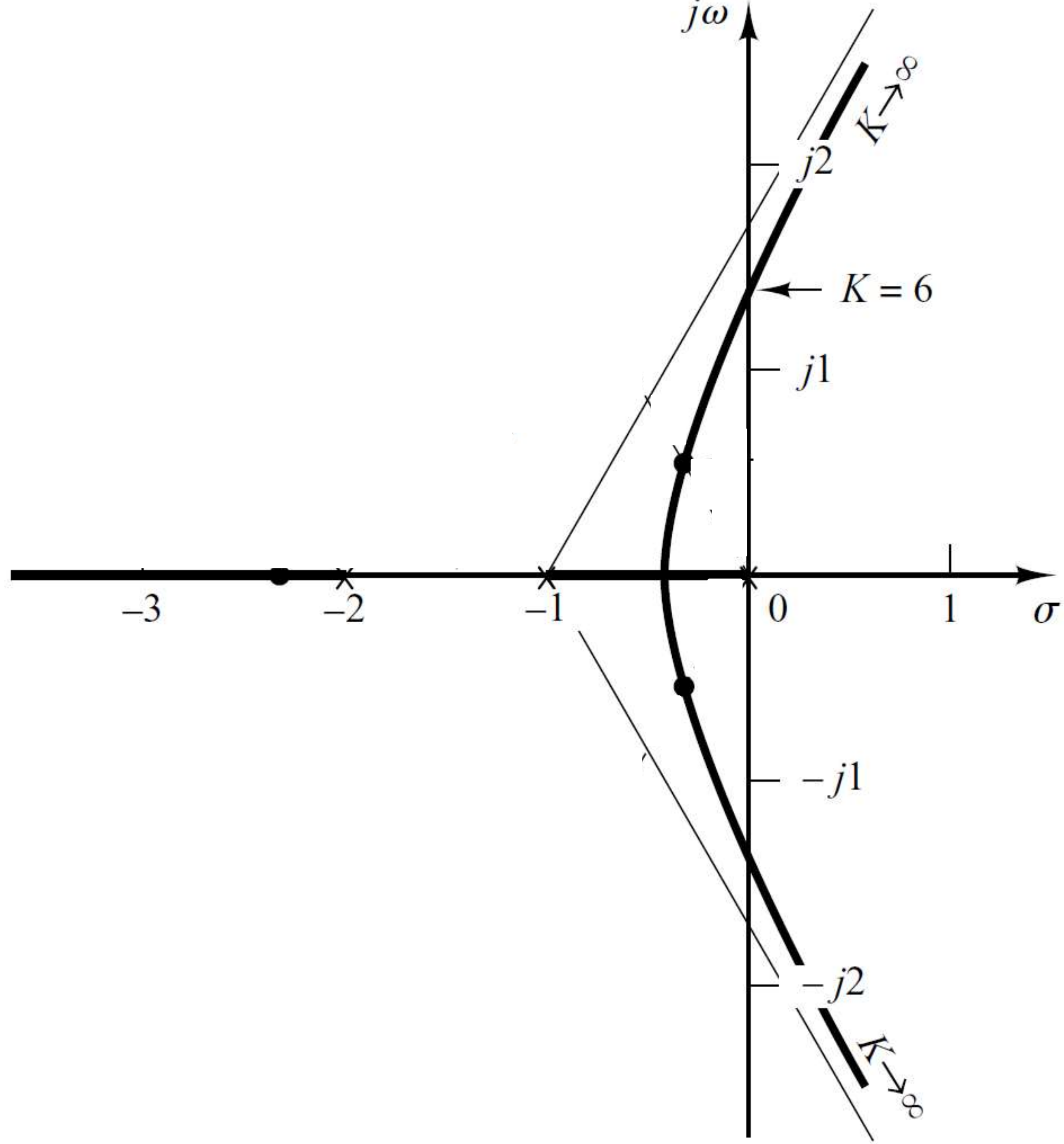
- Equating both real and imaginary parts of this equation to zero

$$(2\omega-\omega^3)=0$$

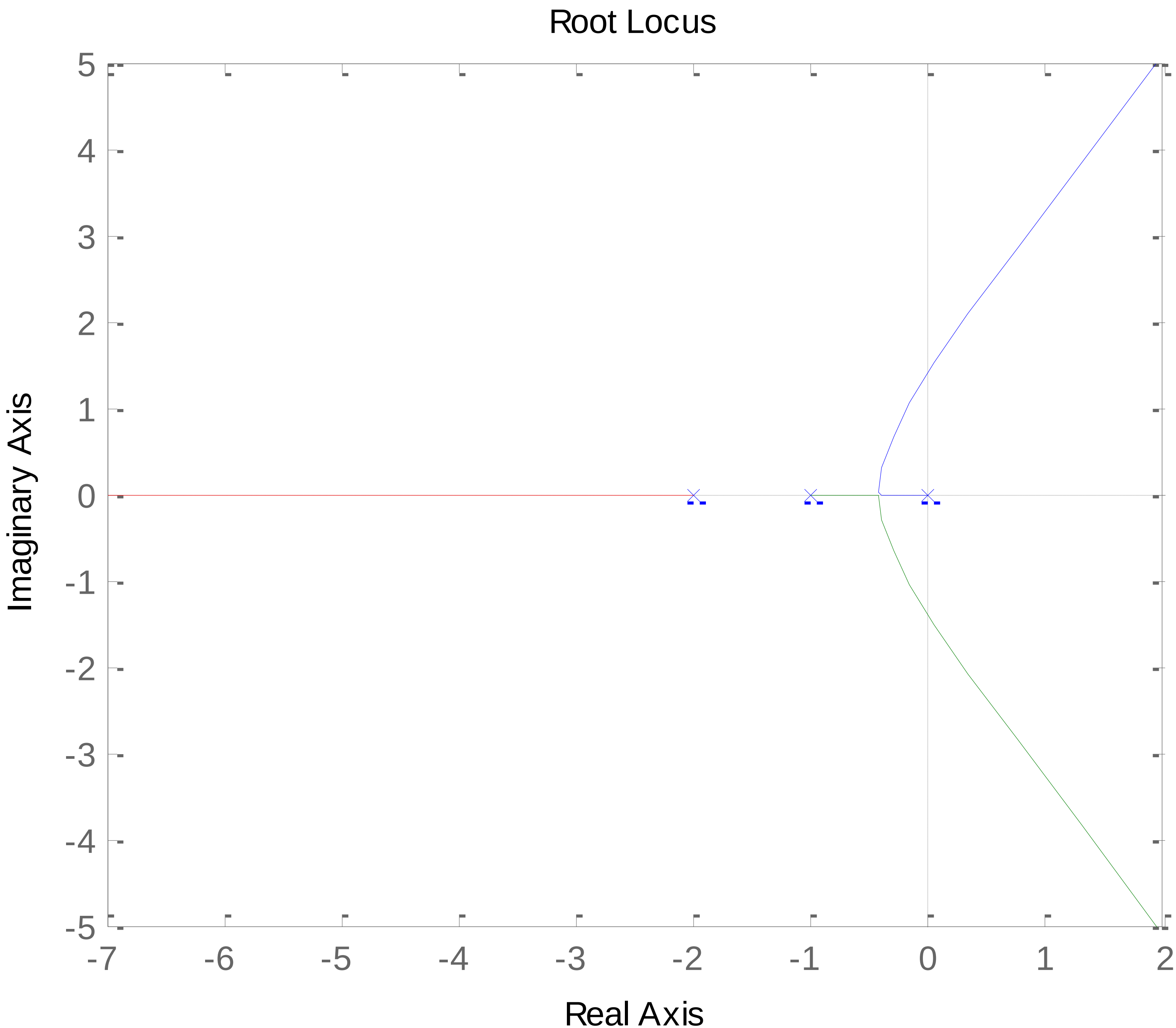
$$(K-3\omega^2)=0$$

- Which yields

$$\omega = \pm\sqrt{2}, \qquad K = 6 \qquad \text{or} \qquad \omega = 0, \qquad K = 0$$



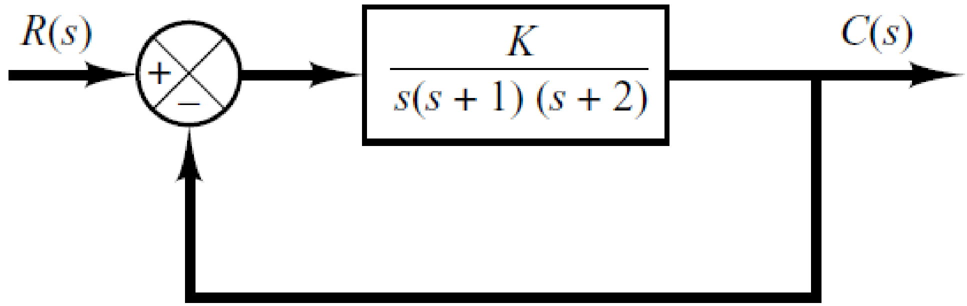
jω
K→∞
j2
K = 6
j1
−j1
−j2
K→∞
−3
−2
−1
0
1
σ



Root Locus
Imaginary Axis
Real Axis
5
4
3
2
1
0
-1
-2
-3
-4
-5
-7
-6
-5
-4
-3
-2
-1
0
1
2

Example#1

- Consider following unity feedback system.



- Determine the value of K such that the damping ratio of a pair of dominant complex-conjugate closed-loop poles is 0.5.

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$

Example#1

- The damping ratio of 0.5 corresponds to

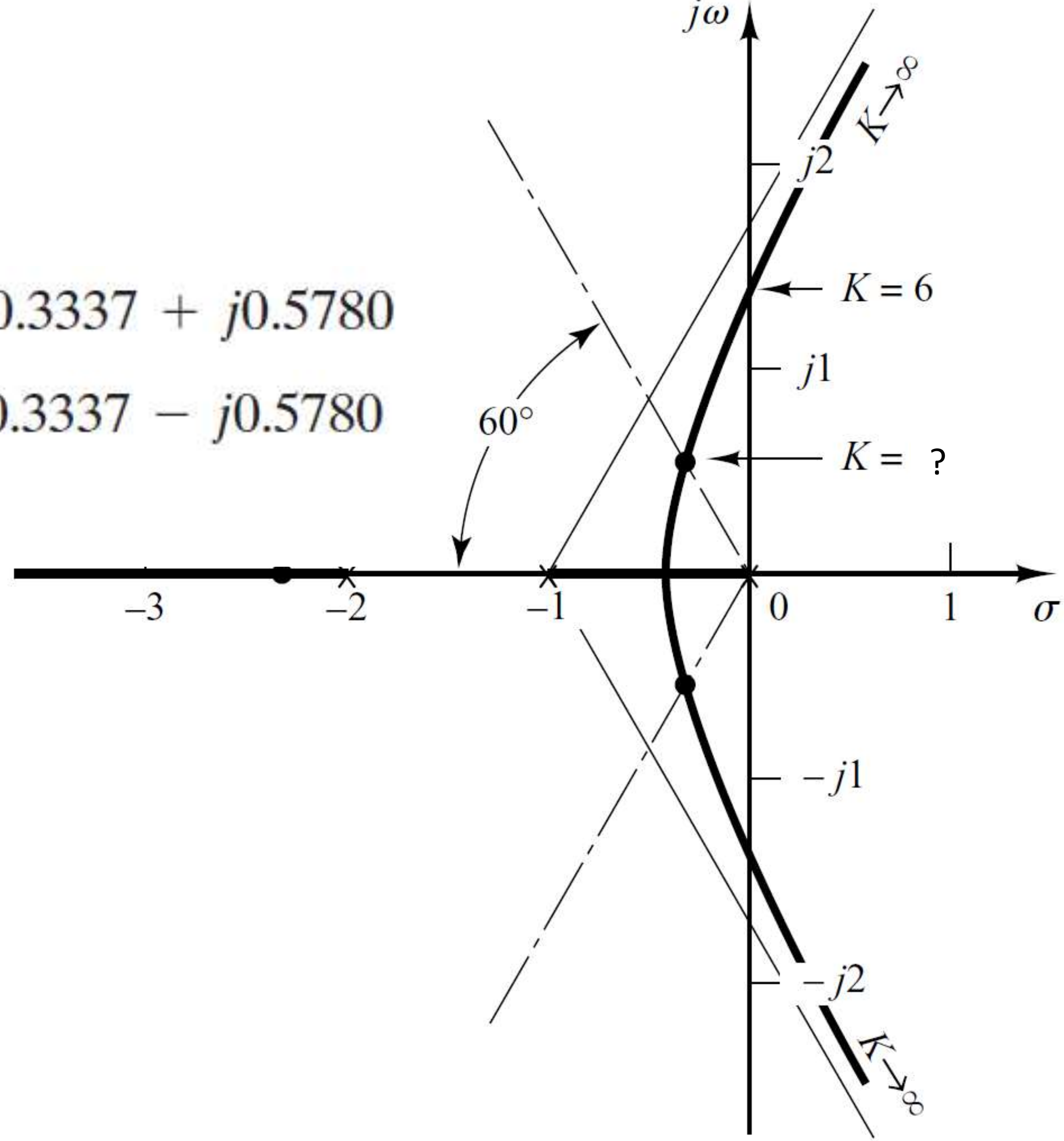
$$\zeta = \cos\theta$$

$$\theta = \cos^{-1}\zeta$$

$$\theta = \cos^{-1}(0.5) = 60^\circ$$

$$s_1 = -0.3337 + j0.5780$$

$$s_2 = -0.3337 - j0.5780$$

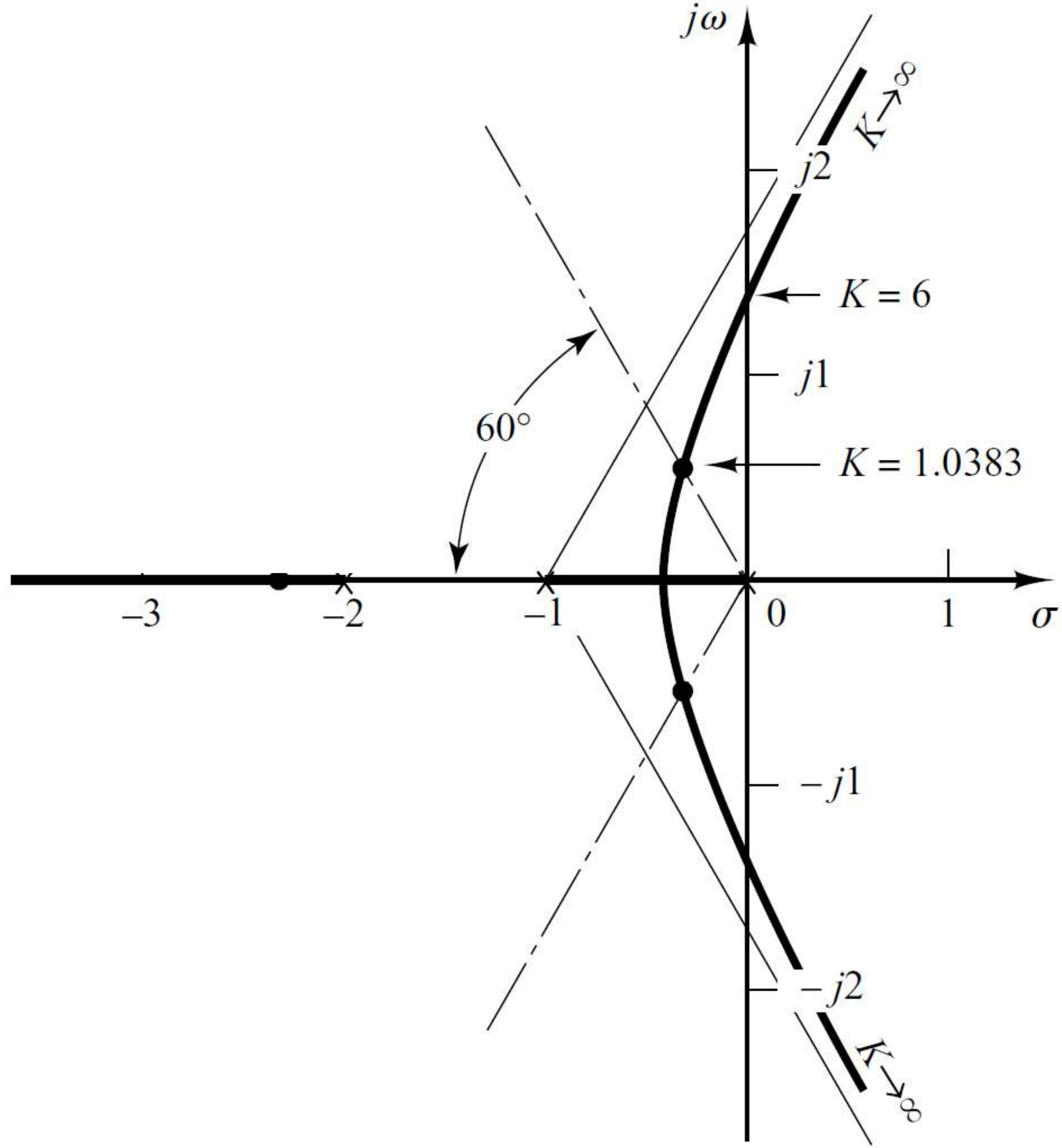


Example#1

- The value of K that yields such poles is found from the magnitude condition

$$\left| \frac{K}{s(s+1)(s+2)} \right|_{s=-0.3337+j0.5780} = 1$$

$$\begin{aligned} K &= |s(s+1)(s+2)|_{s=-0.3337+j0.5780} \\ &= 1.0383 \end{aligned}$$



$j\omega$
$K \to \infty$
$j2$
$K = 6$
$j1$
$60°$
$K = 1.0383$
-3
-2
-1
0
1
σ
$-j1$
$-j2$
$K \to \infty$

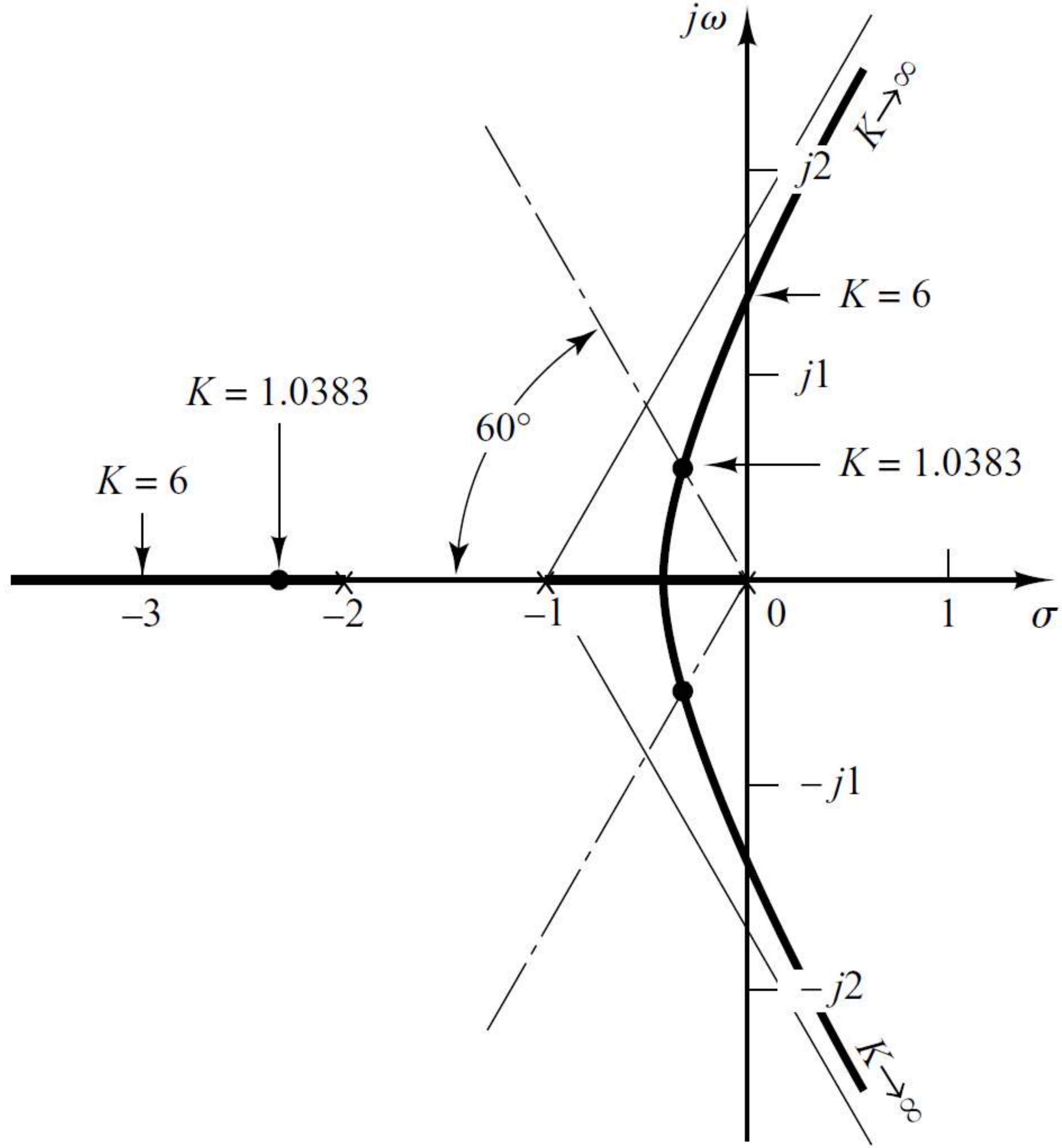
Example#1

- The third closed loop pole at K=1.0383 can be obtained as

$$1+G(s)H(s)=1+\frac{K}{s(s+1)(s+2)}=0$$

$$1+\frac{1.0383}{s(s+1)(s+2)}=0$$

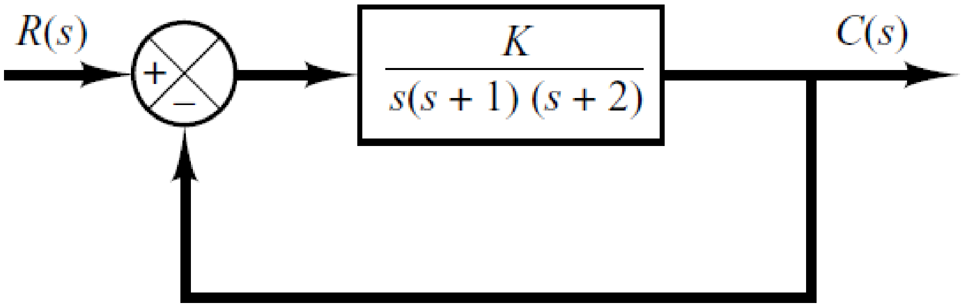
$$s(s+1)(s+2)+1.0383=0$$



jω
K→∞
j2
K = 6
j1
K = 1.0383
60°
K = 1.0383
K = 6
–3
–2
–1
0
1
σ
–j1
–j2
K→∞

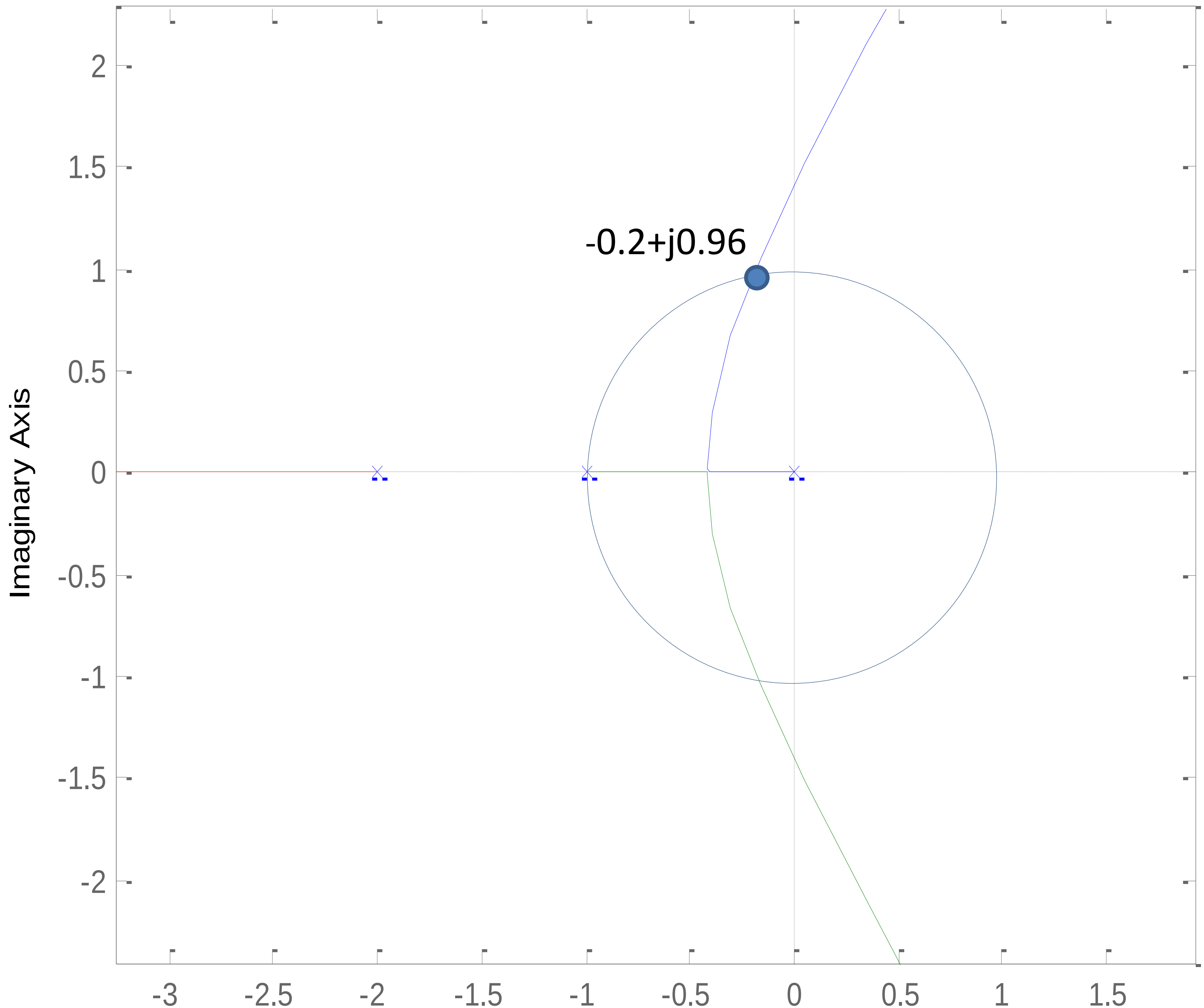
Home Work

- Consider following unity feedback system.



- Determine the value of K such that the natural undamped frequency of dominant complex-conjugate closed-loop poles is 1 rad/sec.

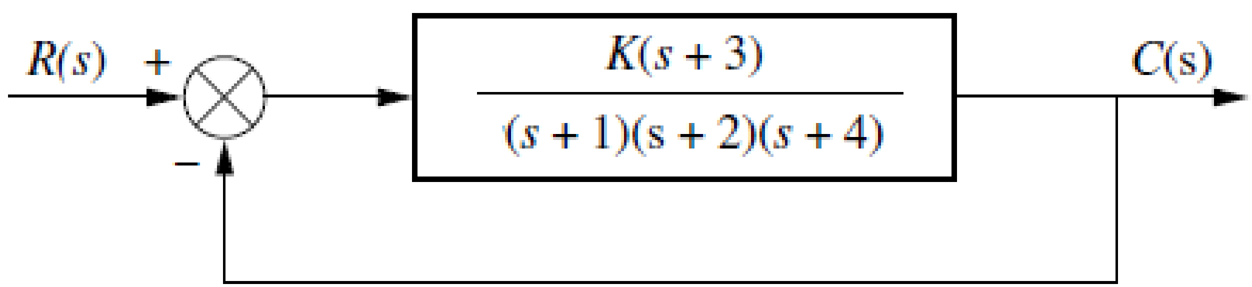
$$G(s)H(s) = \frac{K}{s(s+1)(s+2)}$$



-0.2+j0.96
Imaginary Axis
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5

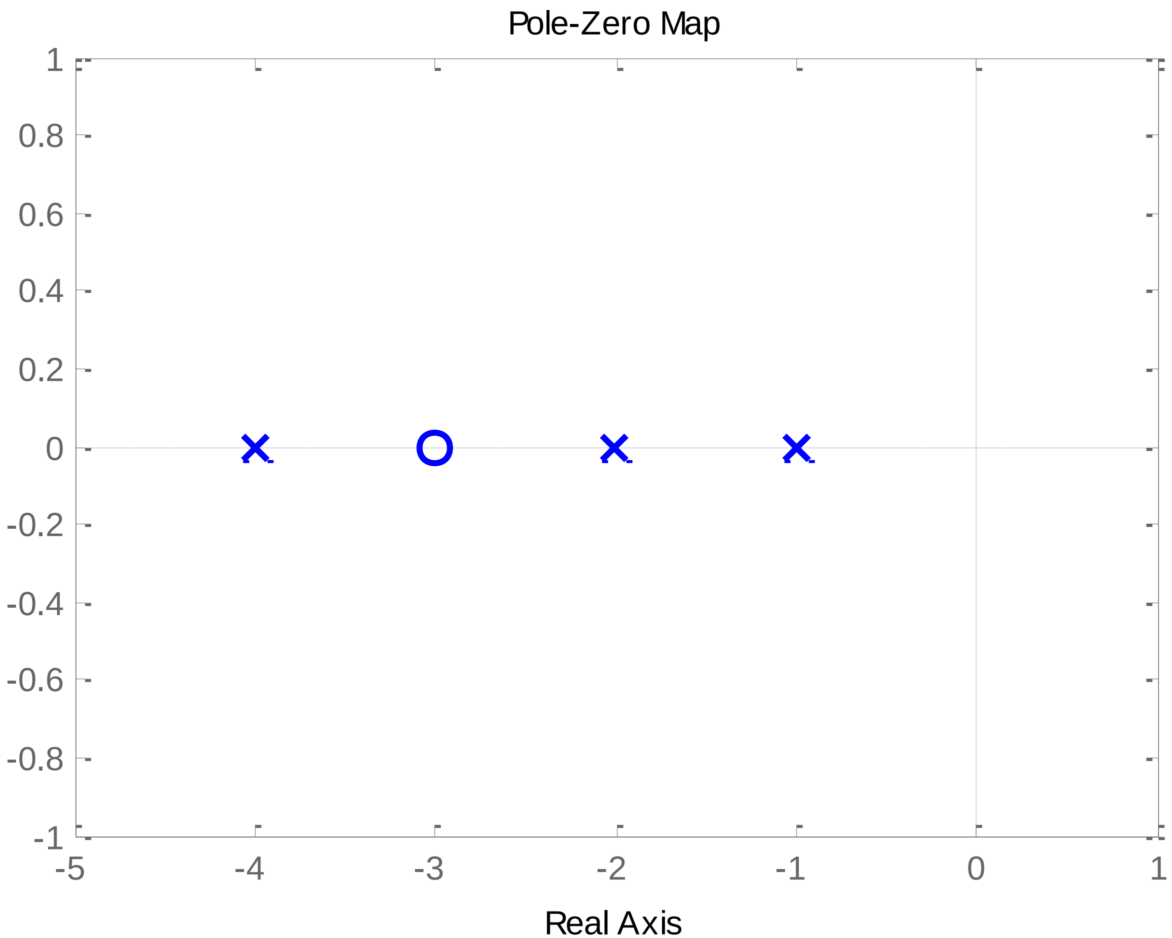
Example#2

- Sketch the root locus of following system and determine the location of dominant closed loop poles to yield maximum overshoot in the step response less than 30%.



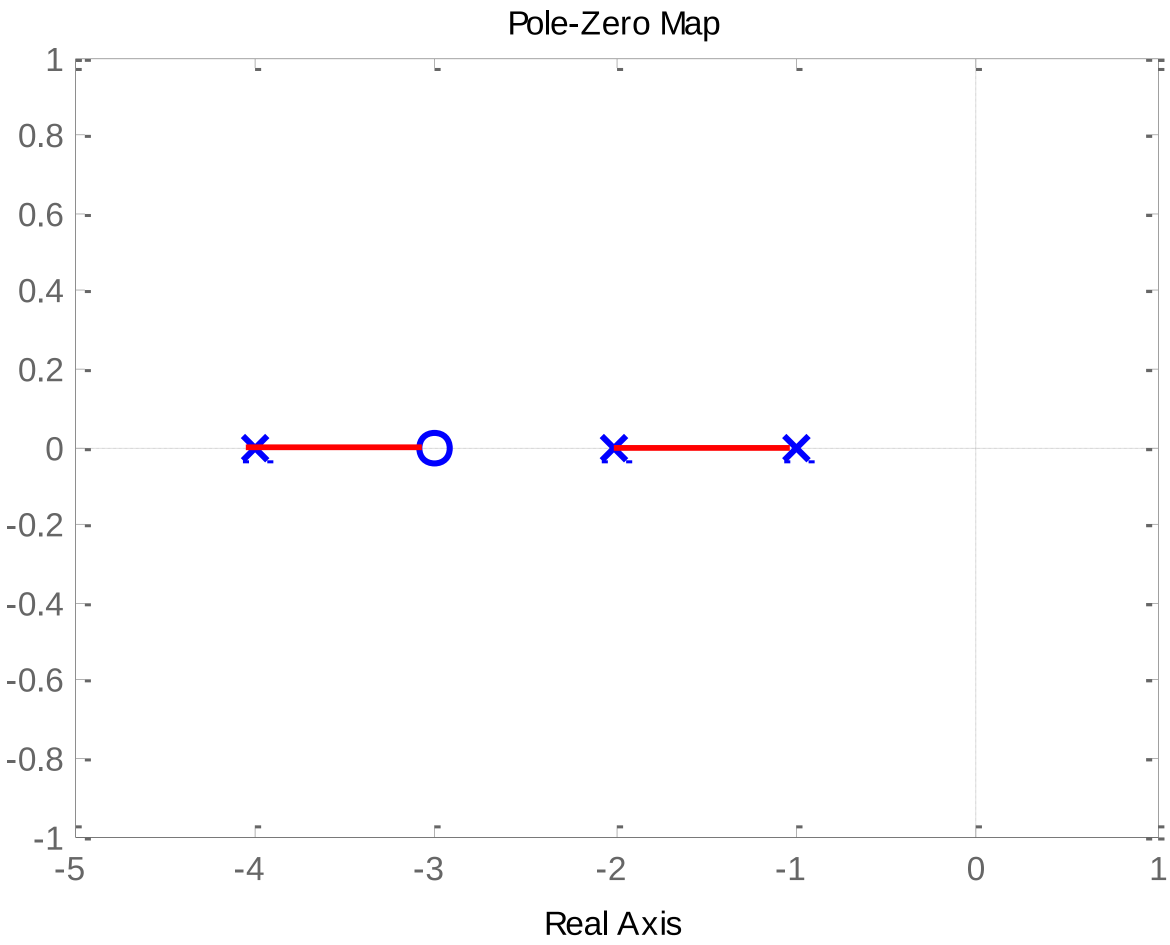
Example#2

- Step-1: Pole-Zero Map



Example#2

- Step-2: Root Loci on Real axis

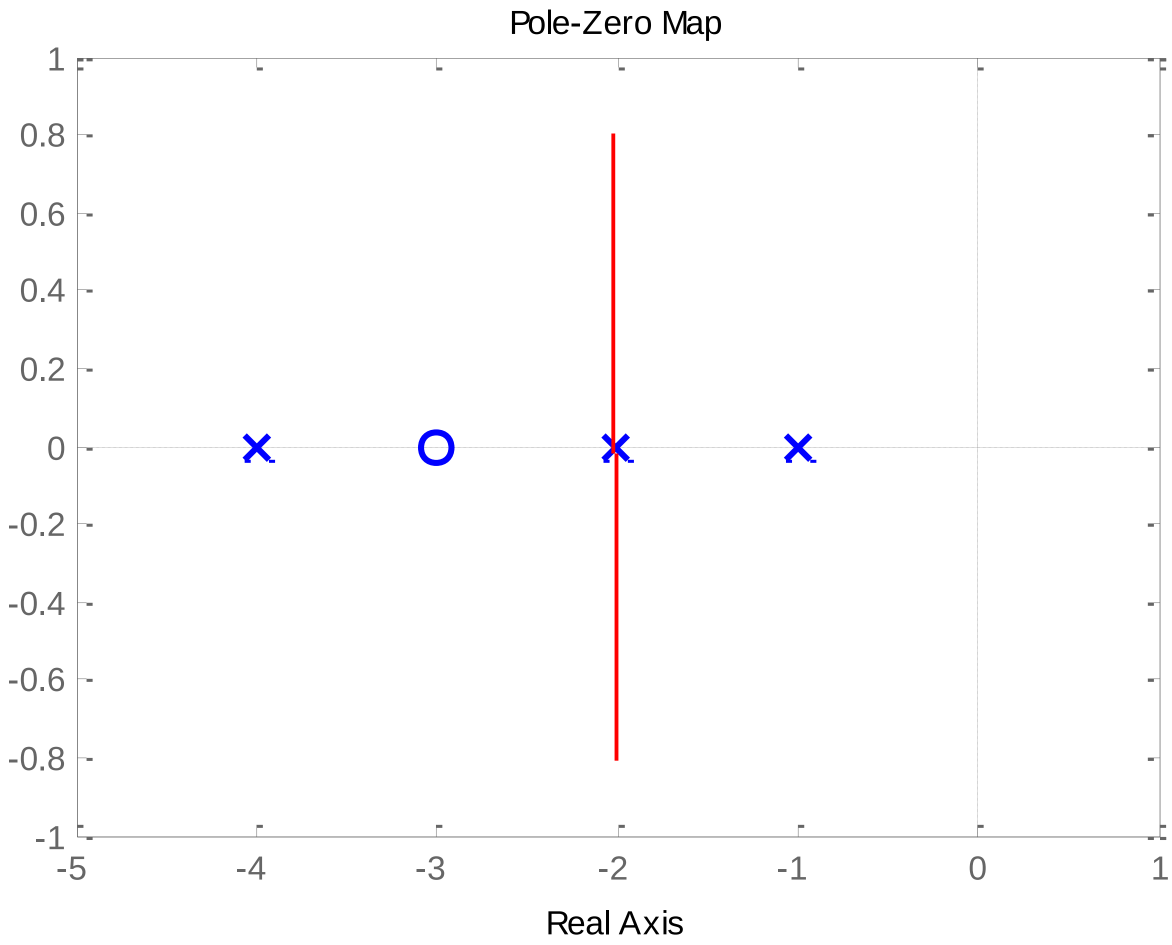


Example#2

- Step-3: Asymptotes

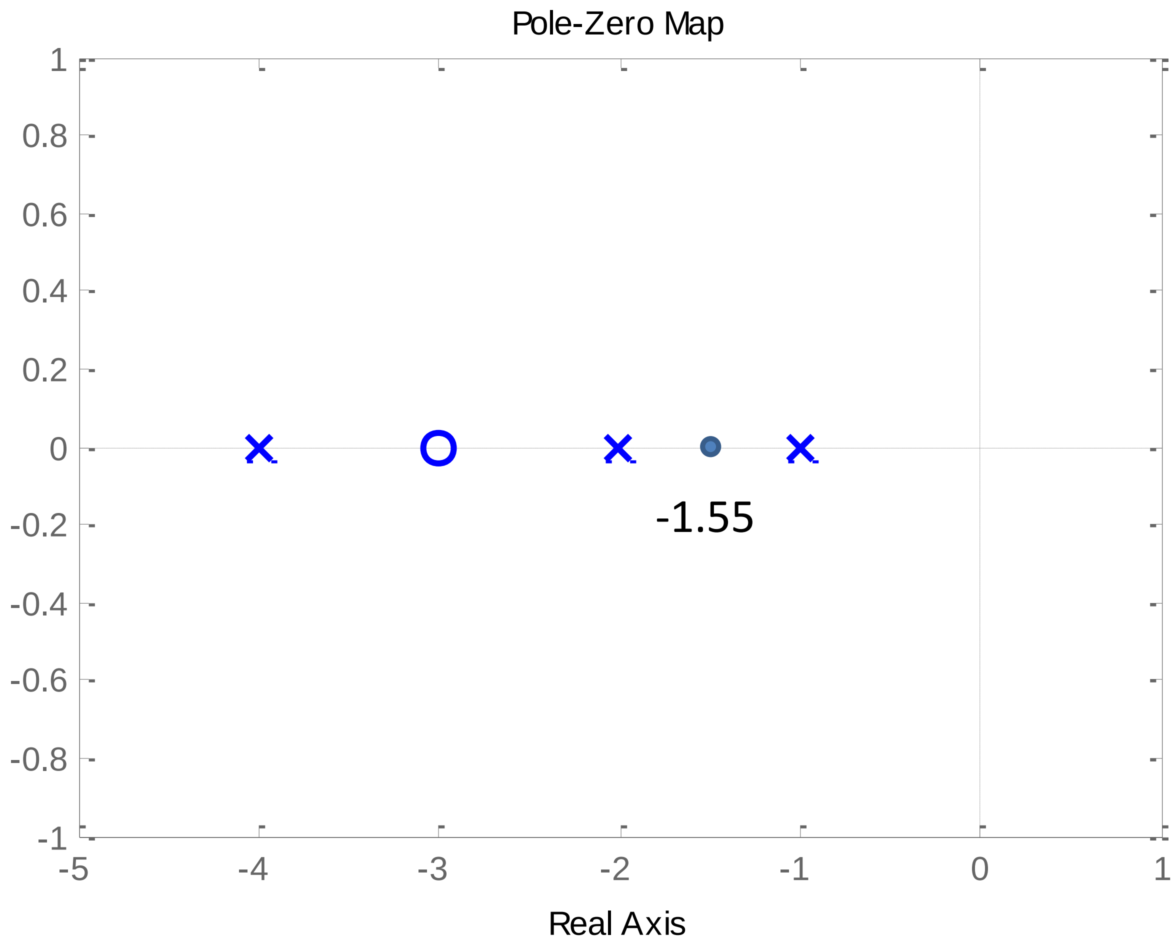
$$\psi = \pm 90°$$

$$\sigma = -2$$

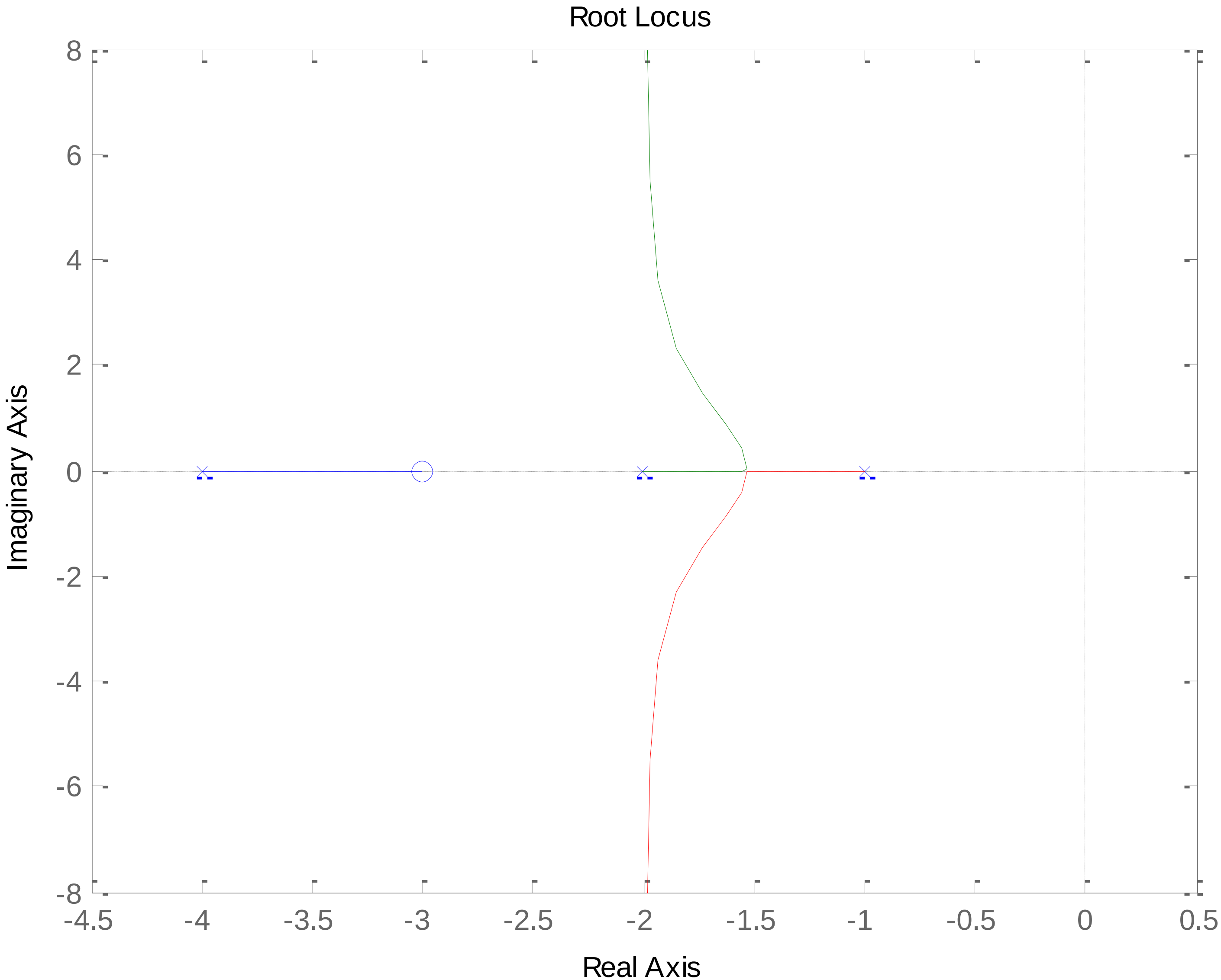


Example#2

- Step-4: breakaway point



Example#2



Root Locus
Imaginary Axis
Real Axis
8
6
4
2
0
-2
-4
-6
-8
-4.5
-4
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5

Example#2

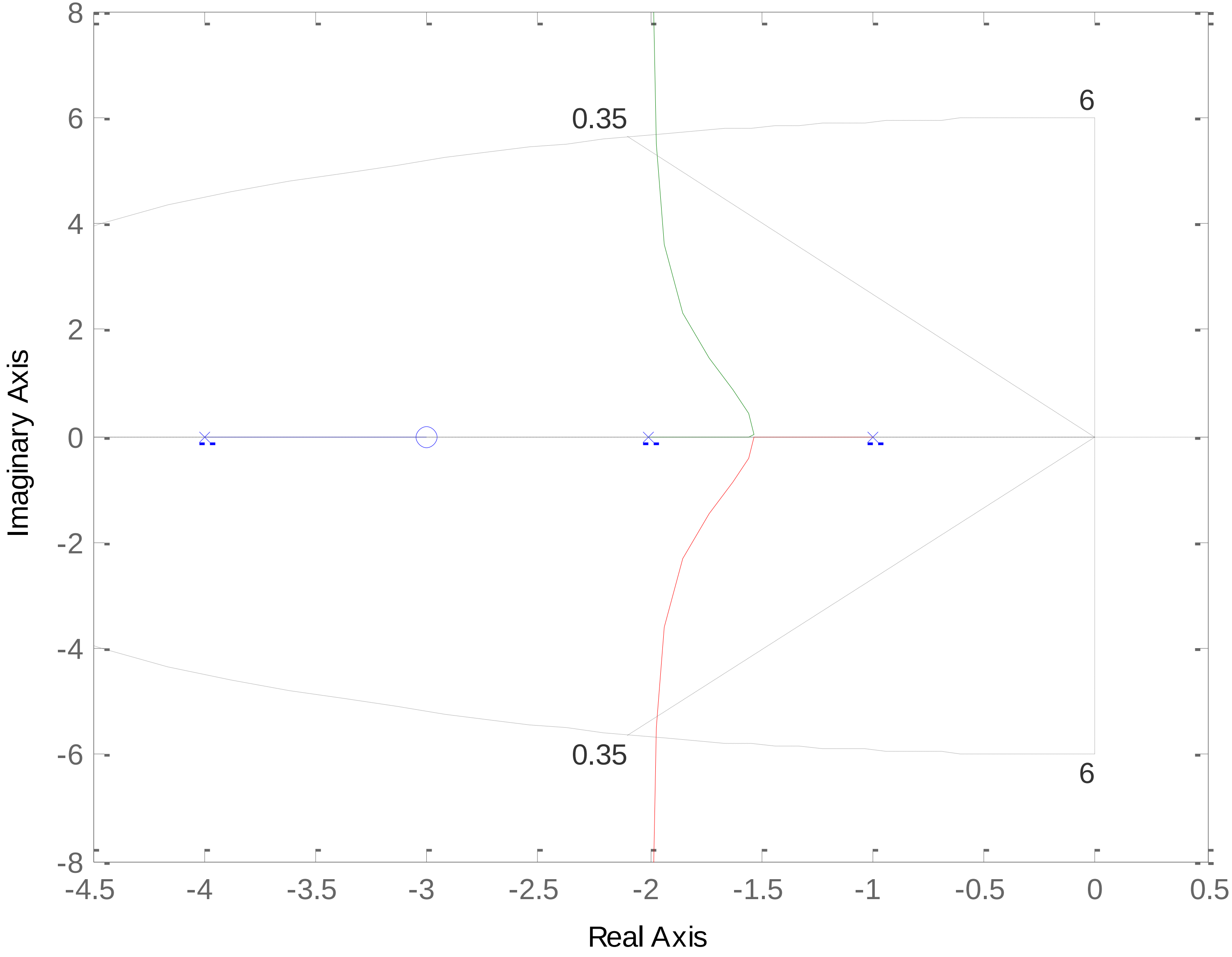
- Mp<30% corresponds to

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

$$30\% = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100$$

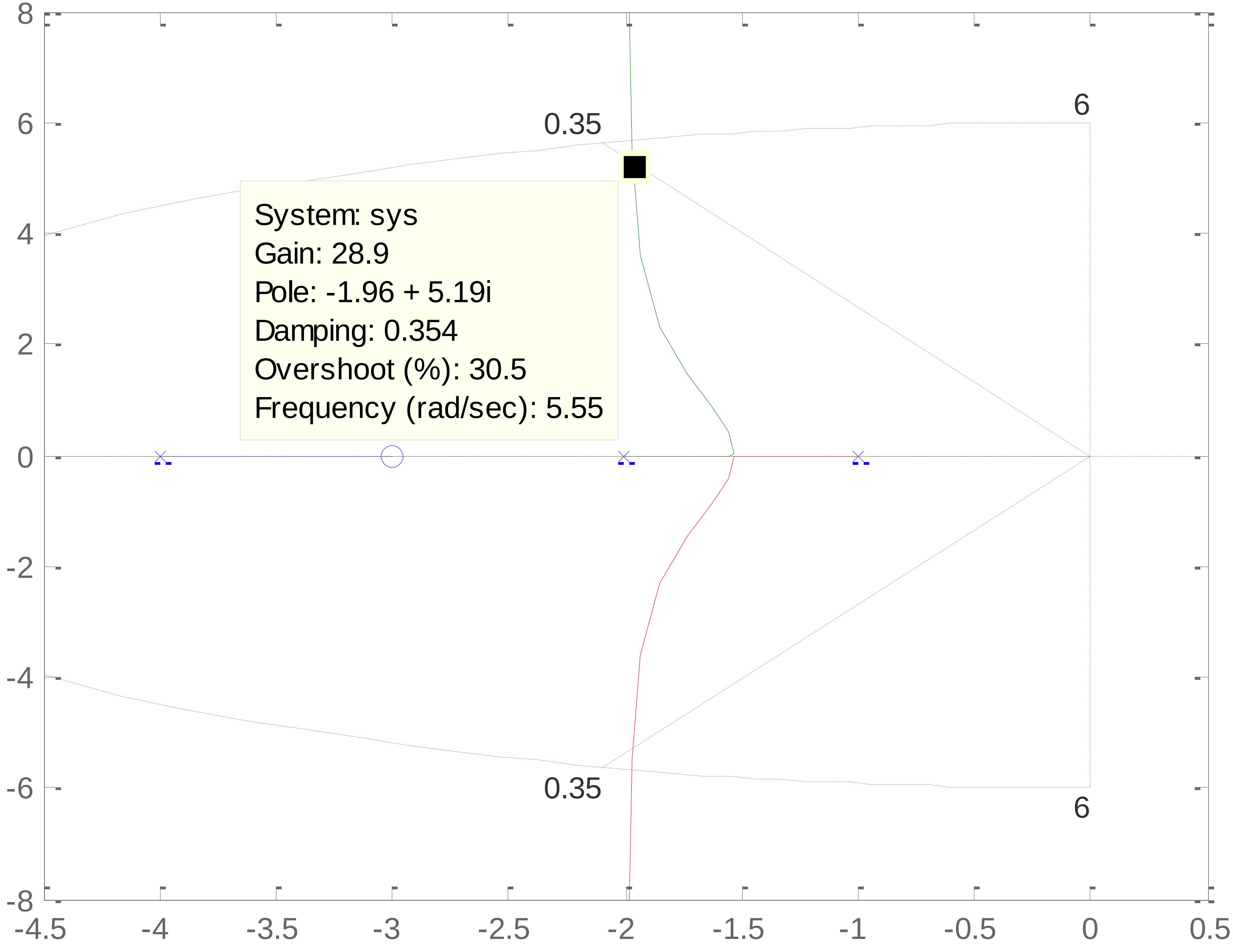
$$\zeta > 0.35$$

Example#2



8
6
4
2
0
-2
-4
-6
-8
Imaginary Axis
-4.5
-4
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
Real Axis
0.35
6
0.35
6

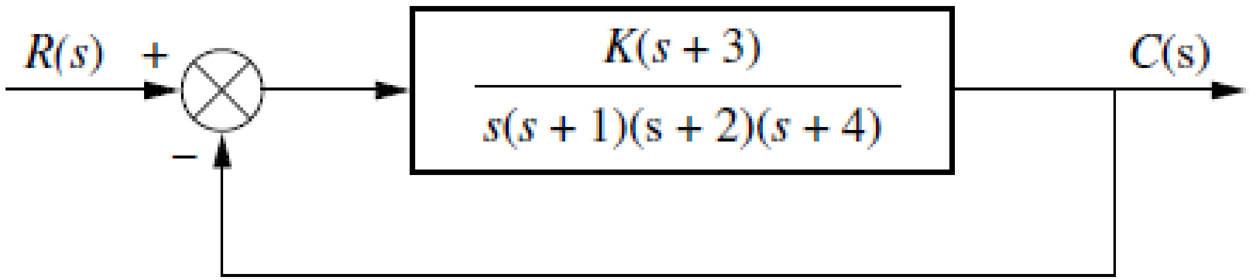
Example#2



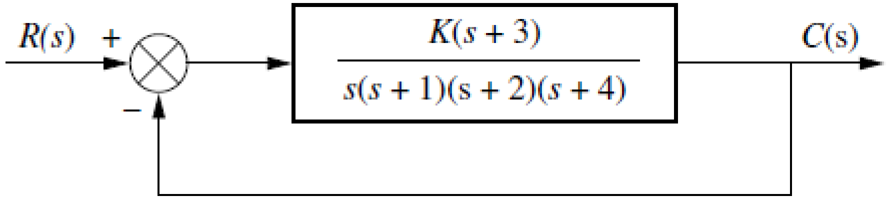
System: sys
Gain: 28.9
Pole: -1.96 + 5.19i
Damping: 0.354
Overshoot (%): 30.5
Frequency (rad/sec): 5.55
0.35
6
0.35
6
8
6
4
2
0
-2
-4
-6
-8
-4.5
-4
-3.5
-3
-2.5
-2
-1.5
-1
-0.5
0
0.5

Example#3

- Sketch the Root Loci of following unity feedback system



$$G(s)H(s) = \frac{K(s+3)}{s(s+1)(s+2)(s+4)}$$



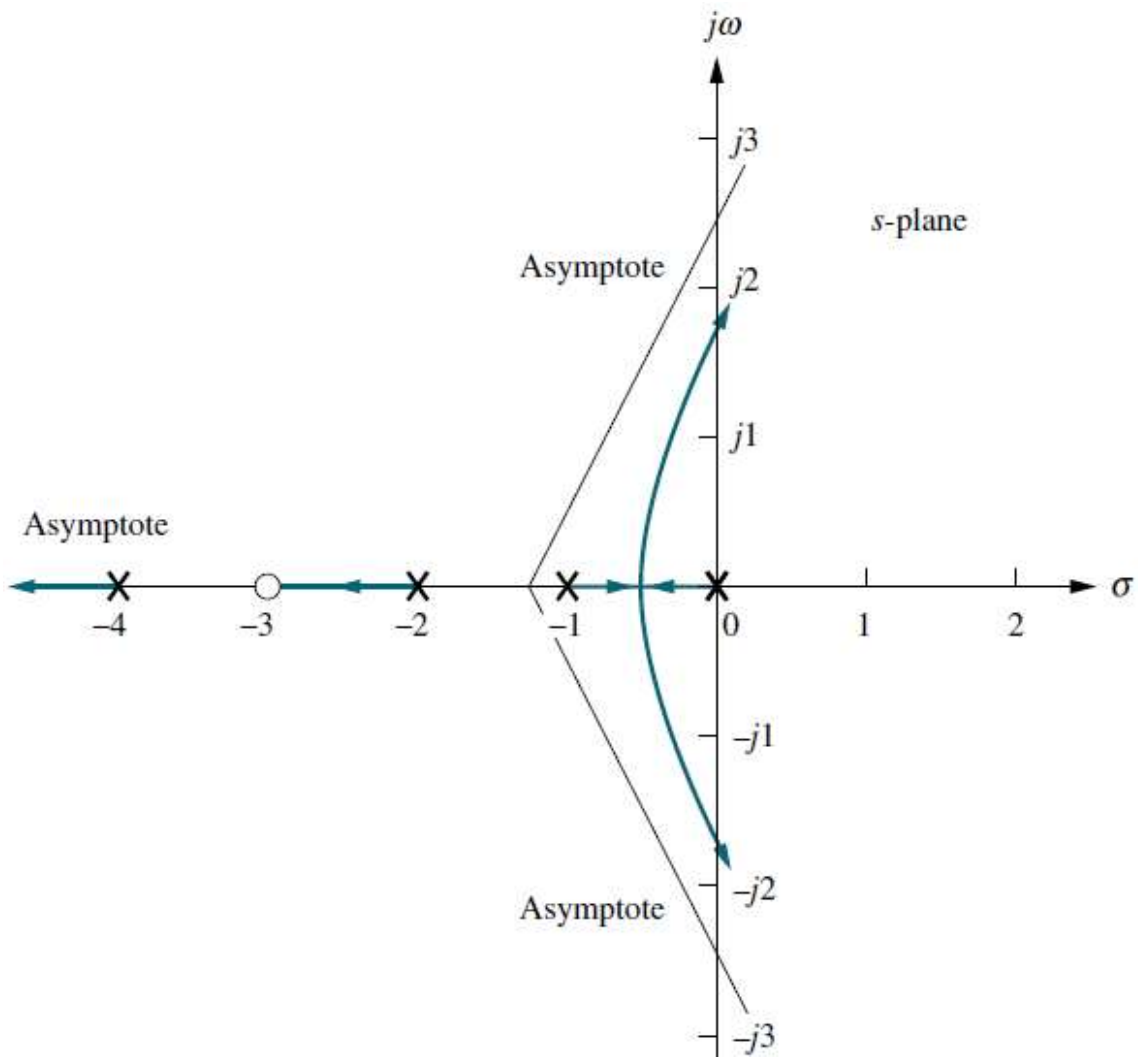
- Let us begin by calculating the asymptotes. The real-axis intercept is evaluated as;

$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3}$$

- The angles of the lines that intersect at - 4/3, given by

$$\begin{aligned}\theta_a &= \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}} \\ &= \pi/3 \qquad \text{for } k=0 \\ &= \pi \qquad \text{for } k=1 \\ &= 5\pi/3 \qquad \text{for } k=2\end{aligned}$$

- The Figure shows the complete root locus as well as the asymptotes that were just calculated.



Example: Sketch the root locus for the system with the characteristic equation of;

$$1 + GH(s) = 1 + \frac{K(s+1)}{s(s+2)(s+4)^2}$$

a) Number of finite poles = n = 4.
b) Number of finite zeros = m = 1.
c) Number of asymptotes = n - m = 3.
d) Number of branches or loci equals to the number of finite poles (n) = 4.
e) The portion of the real-axis between, 0 and -2, and between, -4 and -∞, lie on the root locus for $K > 0$.

- Using Eq. (v), the real-axis asymptotes intercept is evaluated as;

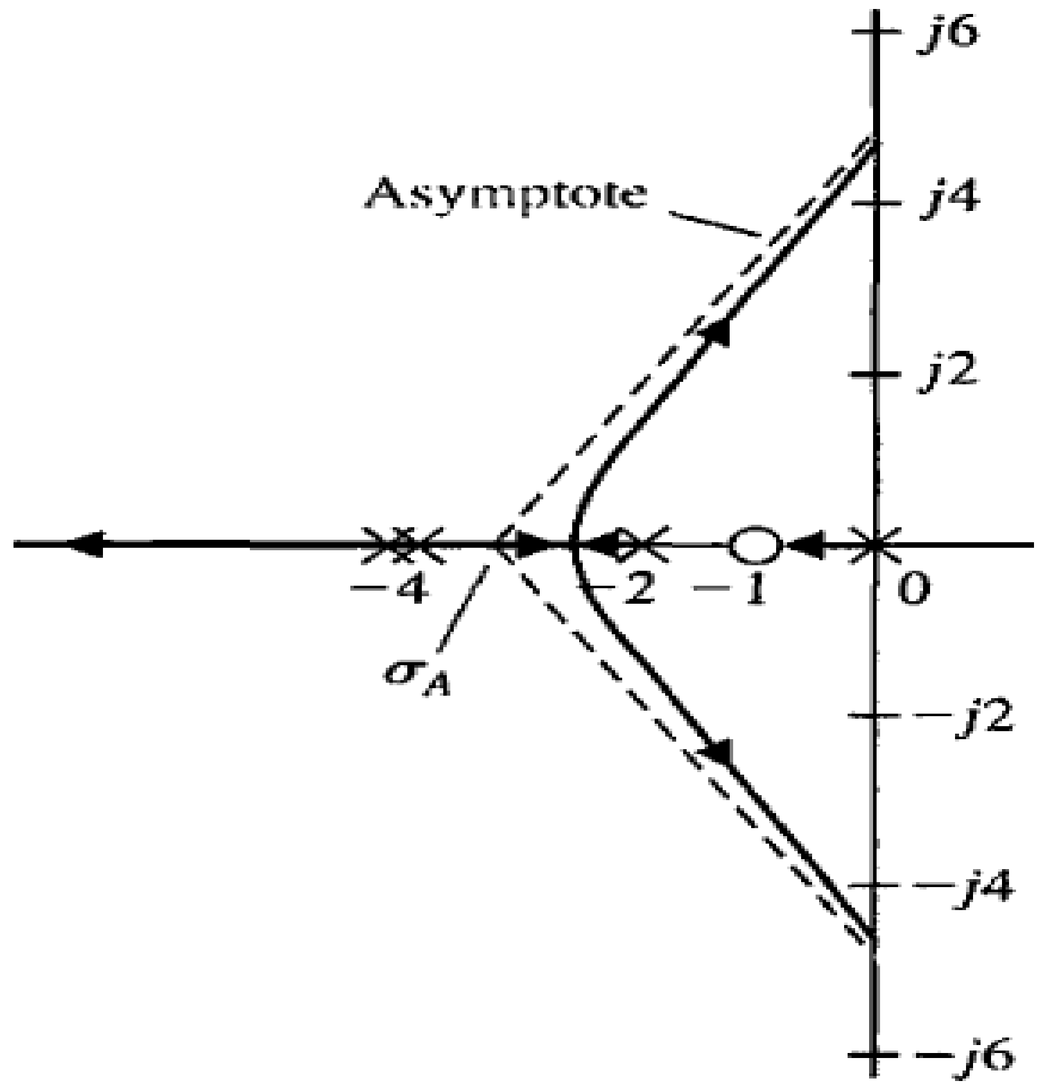
$$\sigma_a = \frac{(-2) + 2(-4) - (-1)}{n - m} = \frac{-10 + 1}{4 - 1} = -3$$

- The angles of the asymptotes that intersect at - 3, given by Eq. (vi), are;

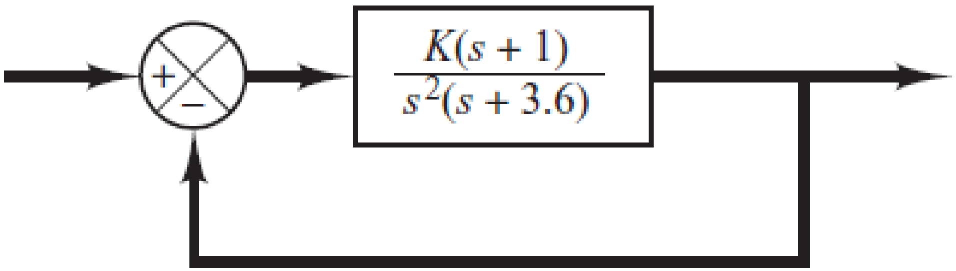
$$\theta_a = \frac{(2k+1)\pi}{n - m} = \frac{(2k+1)\pi}{4 - 1}$$

For $K = 0$, $\theta a = 60^o$
For $K = 1$, $\theta a = 180^o$
For $K = 2$, $\theta a = 300^o$

- The root-locus plot of the system is shown in the figure below.
- It is noted that there are three asymptotes. Since n – m = 3.
- The root loci must begin at the poles; two loci (or branches) must leave the double pole at s = -4.
- The breakaway point, σ, is -2.59



Example: Sketch the root loci for the system.



- A root locus exists on the real axis between points **$s = -1$** and **$s = -3.6$**.
- The intersection of the asymptotes and the real axis is determined as,

$$\sigma_a = \frac{0+0+3.6-1}{n-m} = \frac{2.6}{3-1} = -1.3$$

- The angles of the asymptotes that intersect at – 1.3, given by Eq. (vi), are;

$$\theta_a = \frac{(2k+1)\pi}{n-m} = \frac{(2k+1)\pi}{3-1}$$

For $K = 0$, $\theta a = 90^o$
For $K = 1$, $\theta a = -90^o$ or 270^o

- Since the characteristic equation is $s^3 + 3.6s^2 + K(s+1) = 0$
- We have $K = -\frac{s^3 + 3.6s^2}{s+1}$ $\longrightarrow$ (a)

- The breakaway and break-in points are found from Eq. (a) as,

$$\frac{dK}{ds} = -\frac{(3s^2 + 7.2s)(s + 1) - (s^3 + 3.6s^2)}{(s + 1)^2} = 0$$

or $$s^3 + 3.3s^2 + 3.6s = 0$$

From which we get,

$$s = 0, \qquad s = -1.65 + j0.9367, \qquad s = -1.65 - j0.9367$$

- Point s = 0 corresponds to the actual breakaway point. But points $s = 1.65 \pm j0.9367$ neither breakaway nor break-in points, because the corresponding gain values K become complex quantities.

- To check the points where root-locus branches may cross the imaginary axis, substitute $s = j\omega$ into the characteristic equation, yielding.

$$(j\omega)^3 + 3.6(j\omega)^2 + Kj\omega + K = 0$$

or

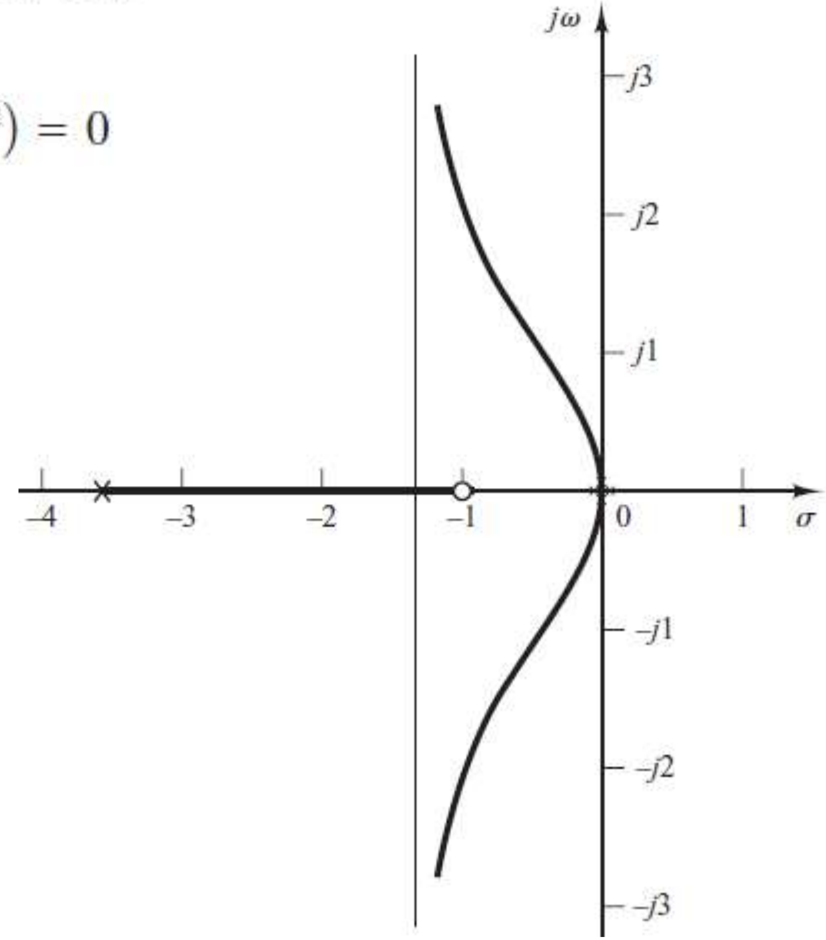
$$\left(K - 3.6\omega^2\right) + j\omega\left(K - \omega^2\right) = 0$$

- Notice that this equation can be satisfied only if $\omega = 0, K = 0$.

- Because of the presence of a double pole at the origin, the root locus is tangent to the $j\omega$axis at $k = 0$.

- The root-locus branches do not cross the $j\omega$axis.

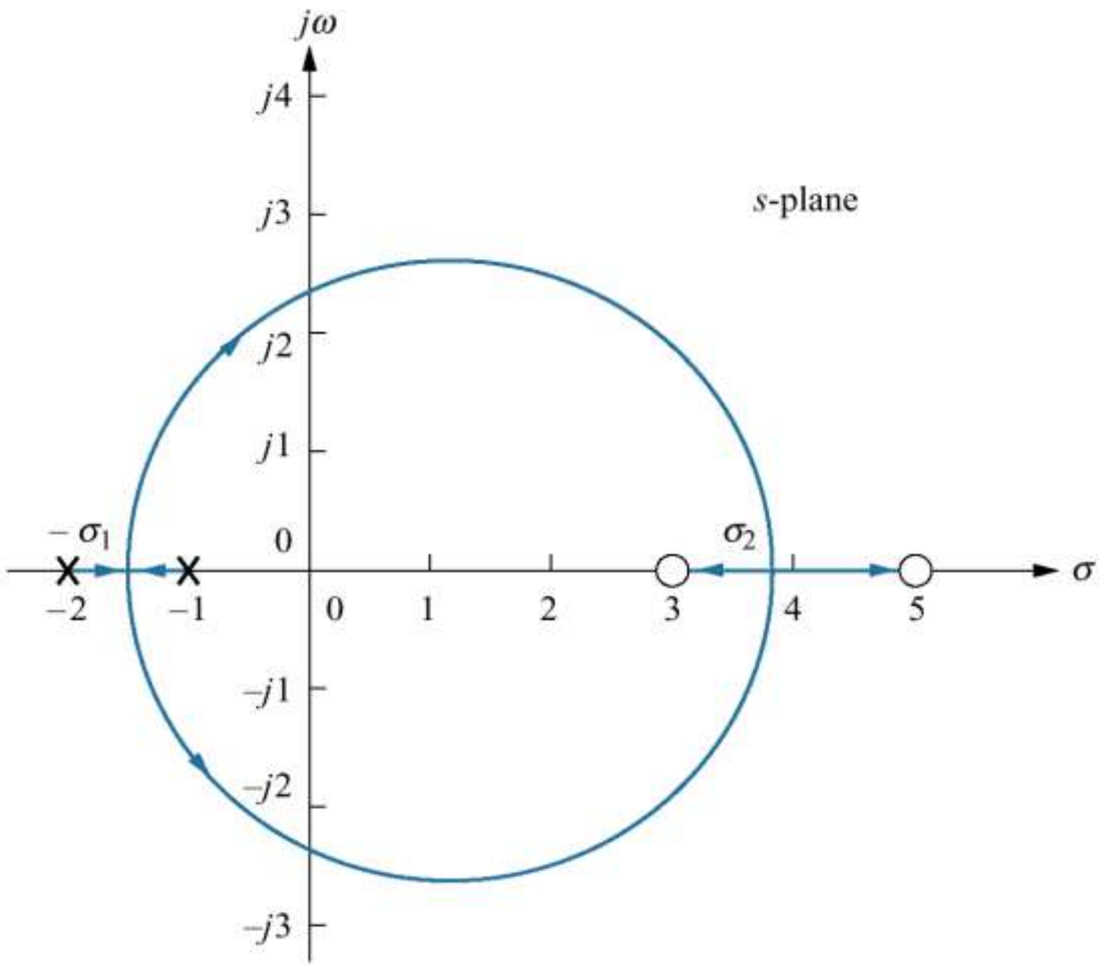
- The root loci of this system is shown in the Figure.



Real-Axis breakaway and Break-in Points via differentiation

Problem Find the breakaway and break-in points for the root locus of Shown Figure.

Solution Using the open-loop poles and zeros, we have



Real-Axis breakaway and Break-in Points via differentiation

Problem Find the breakaway and break-in points for the root locus of Shown Figure.

Solution Using the open-loop poles and zeros, we have

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2-8s+15)}{(s^2+3s+2)}$$

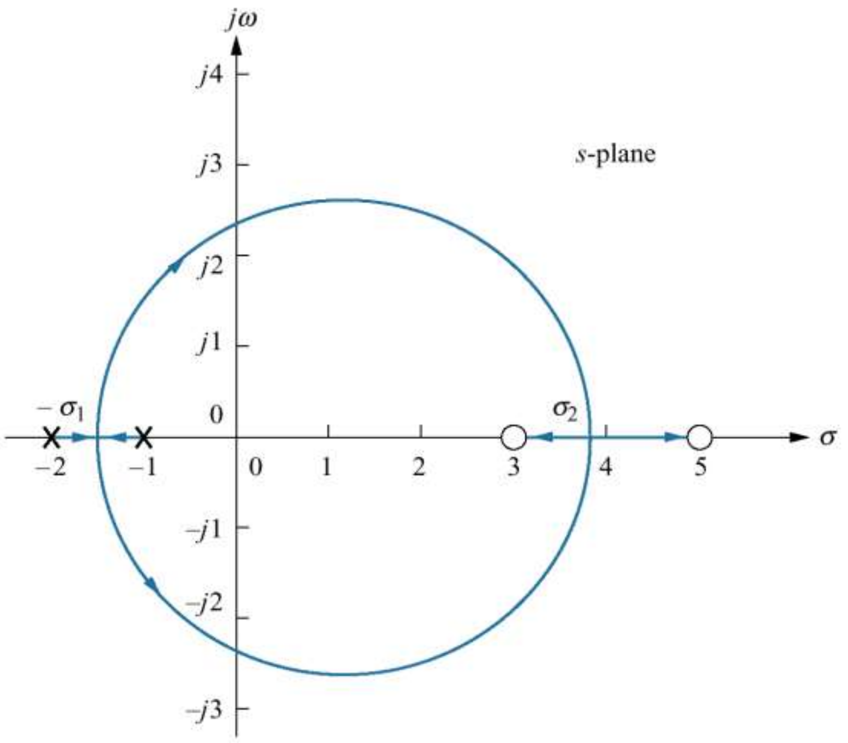
But for all points along the root locus, KG(s)H(s) = -1, and along the real axis $s = \sigma$

Hence, $\dfrac{K(\sigma^2-8\sigma+15)}{(\sigma^2+3\sigma+2)} = -1$

Solving for K, and then differentiating, we have

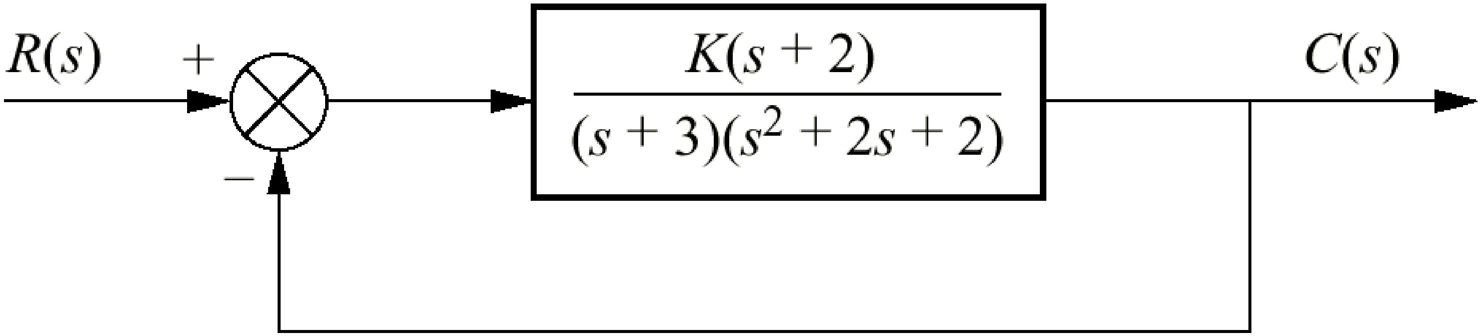
$$K = \frac{-(\sigma^2+3\sigma+2)}{(\sigma^2-8\sigma+15)} \quad \frac{dK}{d\sigma} = \frac{(11\sigma^2-26\sigma+-61)}{(\sigma^2-8\sigma+15)^2} = 0$$

Solving for σ we find σ = -1.45 and 3.82



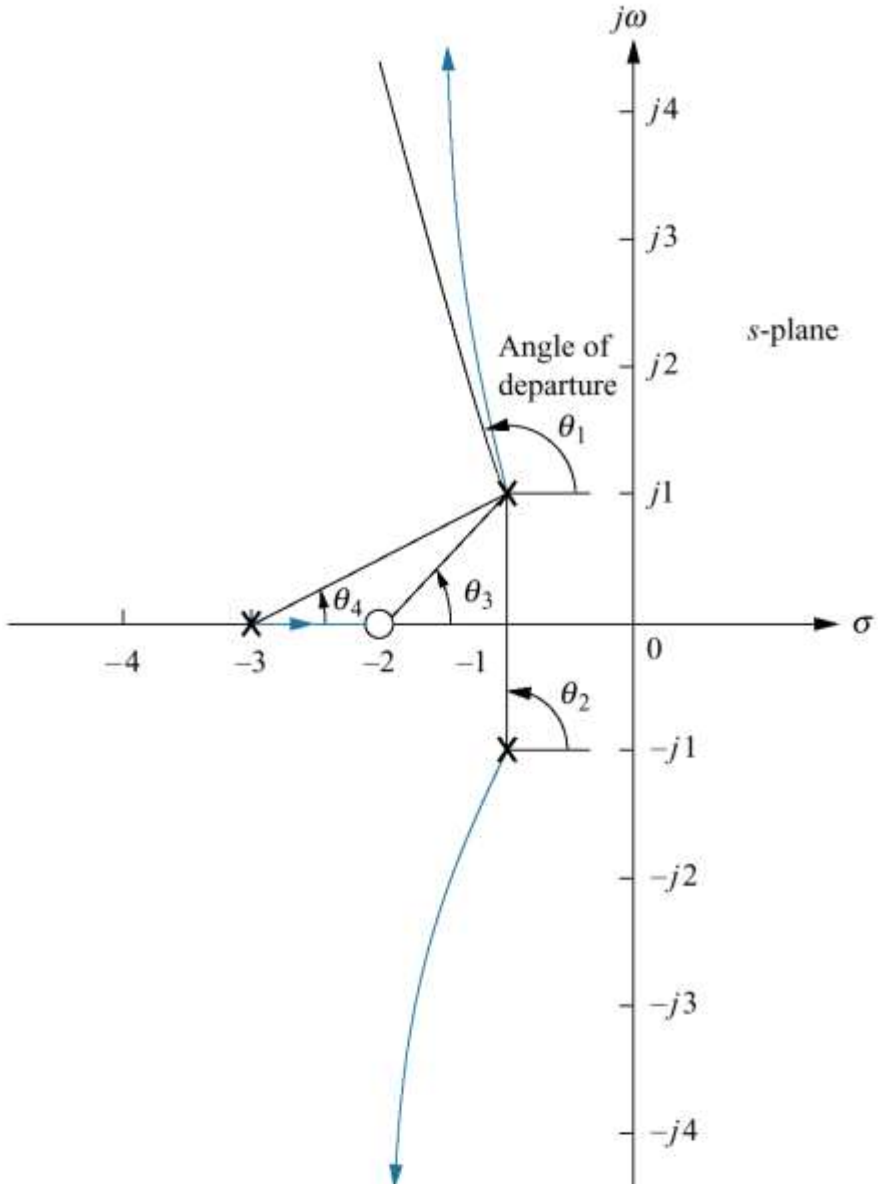
Angle of Departure from a complex pole Example

Problem sketch the root locus



pole, -3, -1+j1

Root locus for previous system showing angle of departure



Problem Sketch the root locus for the system and find the following:

- The point and gain where the locus crosses the imaginary axis.

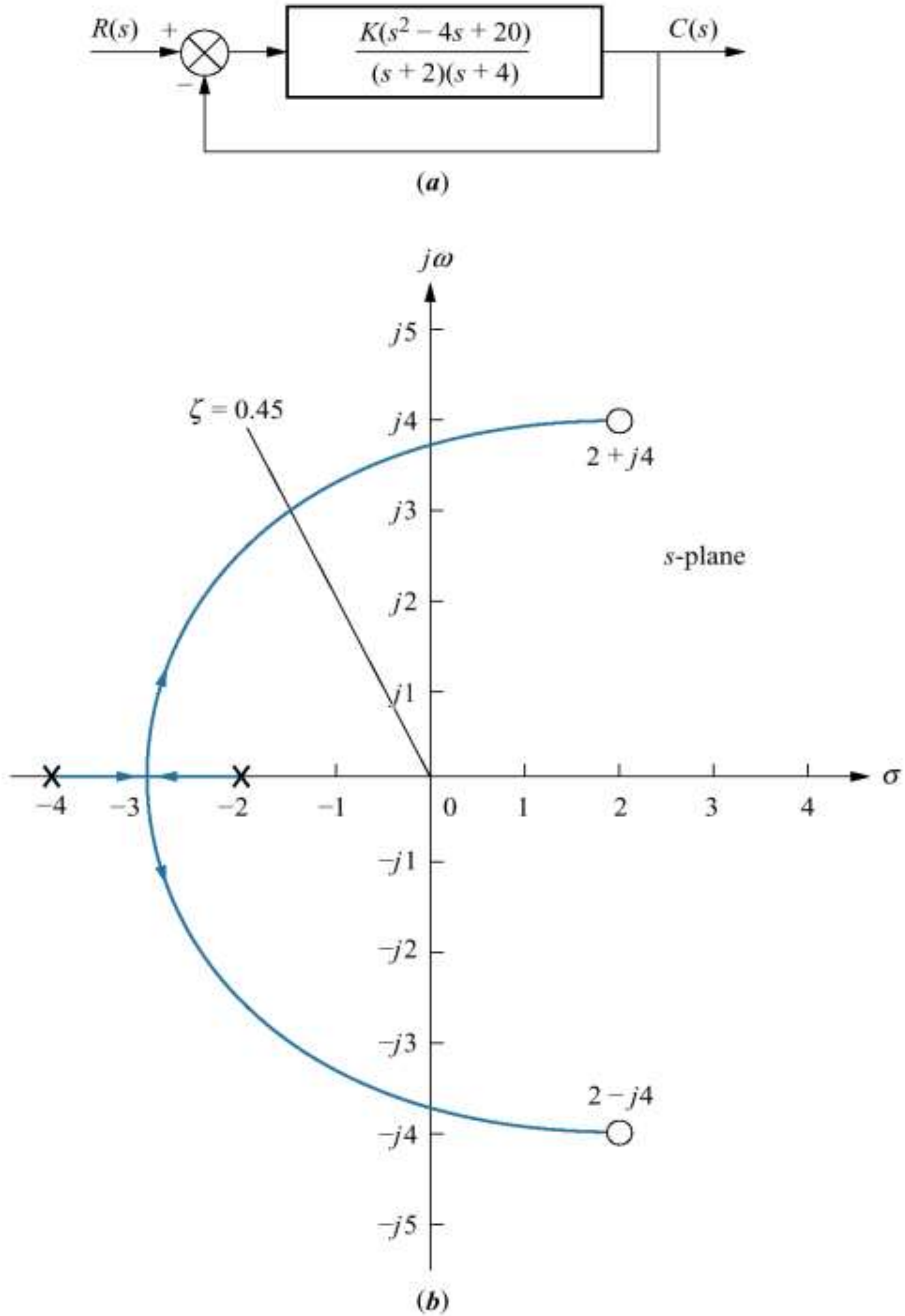
Crossing the jw axis at +/-j3.9 with a gain K=1.5

- The breakaway points on the real axis

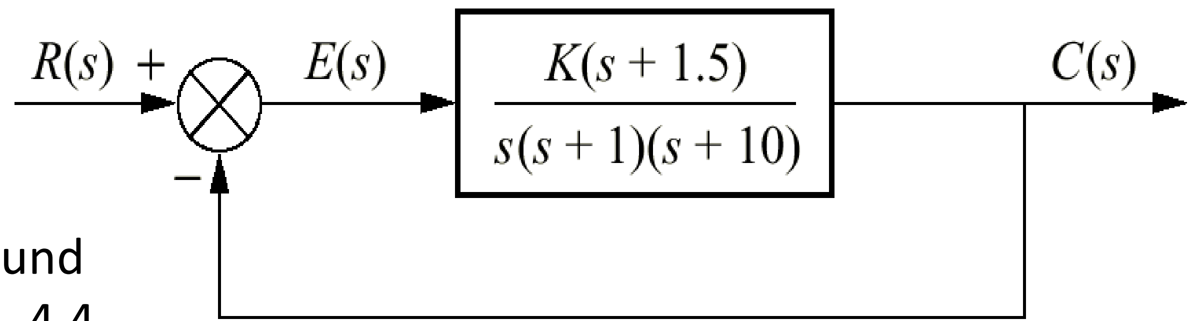
Breakaway point at -2.88

- The range of K for system to be stable

Stable for K between 0 and 1.5

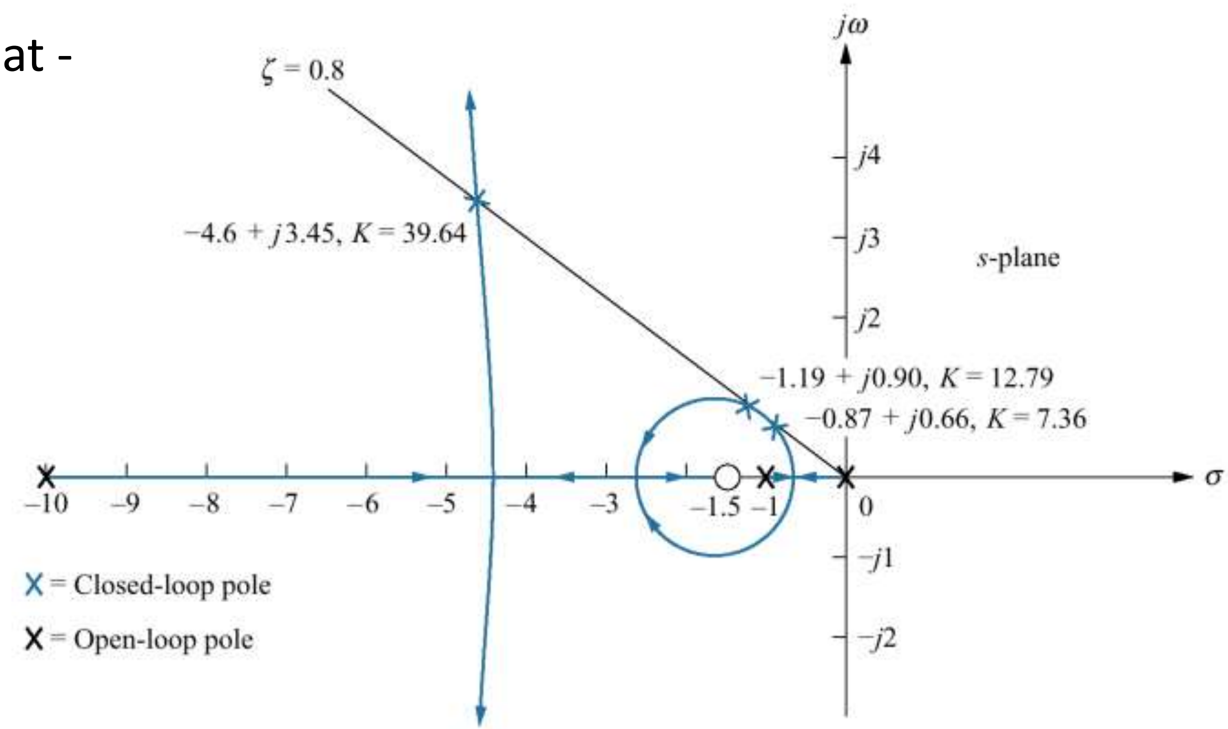


Problem Design the value of K to yield 1.52% OS. Estimate Ts,Tp and ss error.
Solution 1.52% corresponds to 0.8 damping ratio.

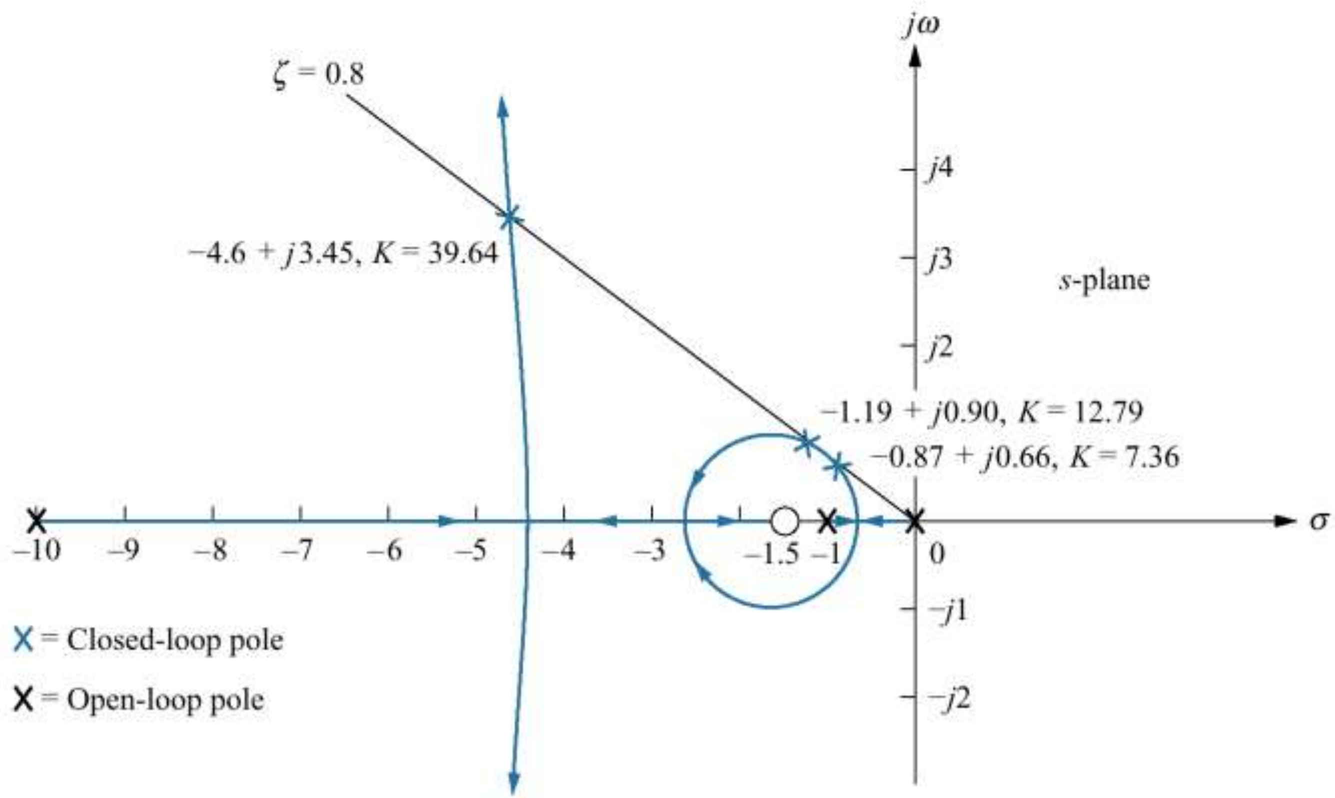


** Breakaway points are found at -0.62 with K=2.51 and at -4.4 with k=28.89

** Break-in point is found at -2.8 with k = 27.91



Characteristics of the system of Example 8.8



Case	Closed-loop poles	Closed-loop zero	Gain	Third closed-loop pole	Settling time	Peak time	K_v
1	$-0.87 \pm j0.66$	$-1.5 + j0$	7.36	-9.25	4.60	4.76	1.1
2	$-1.19 \pm j0.90$	$-1.5 + j0$	12.79	-8.61	3.36	3.49	1.9
3	$-4.60 \pm j3.45$	$-1.5 + j0$	39.64	-1.80	0.87	0.91	5.9

Home Work

- Draw the Root Locus of the following systems.

1) $G(s)H(s) = \dfrac{K}{s(s+2)}$

2) $G(s)H(s) = \dfrac{K}{s^2}$

3) $G(s)H(s) = \dfrac{K}{(s+1)(s-3)}$

4) $G(s)H(s) = \dfrac{K}{s^2+3s+10}$

5) $G(s)H(s) = \dfrac{K(s+1)}{s(s+2)}$

6) $G(s)H(s) = \dfrac{K(s-2)}{s^2}$

7) $G(s)H(s) = \dfrac{K(s+5)}{(s+1)(s-3)}$

END