Control and Instrumentation

Unite code : 308EAC

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Outline

- Introduction
- Types of Models
- Mathematical Modeling of
  - Electrical Systems
  - Electronic Systems
  - Mechanical Systems
  - Electromechanical Systems

Types of Systems

- **Static System:** If a system does not change with time, it is called a static system.

- **Dynamic System:** If a system changes with time, it is called a dynamic system.

Dynamic Systems

- A system is said to be dynamic if its current output may depend on the past history as well as the present values of the input variables.
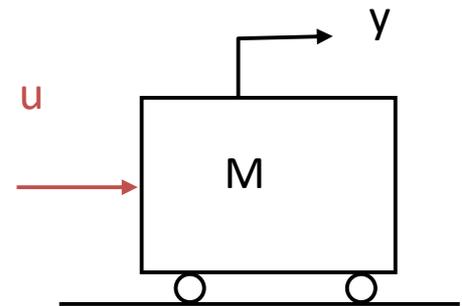
- Mathematically,

$$y(t) = \varphi[u(\tau), 0 \leq \tau \leq t]$$
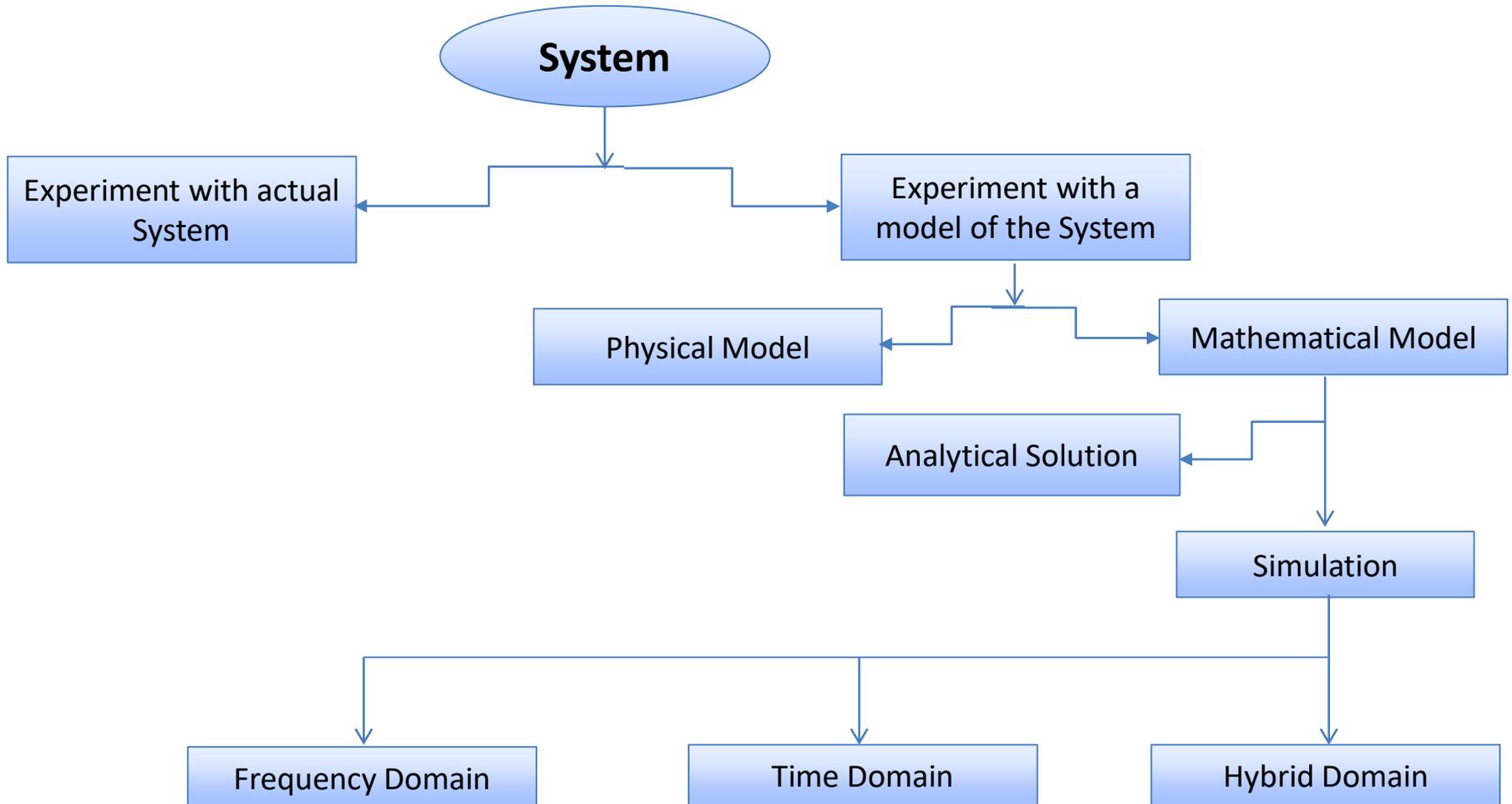
$$u : \text{Input, } t : \text{Time}$$

Example: A moving mass

Model: Force=Mass x Acceleration

$$M\ddot{y} = u$$

Ways to Study a System

Model

- A *model* is a simplified representation or abstraction of reality.

- Reality is generally too complex to copy exactly.

- Much of the complexity is actually *irrelevant* in problem solving.

What is Mathematical Model?

A set of mathematical equations (e.g., differential eqs.) that describes the input-output behavior of a system.

What is a model used for?

- Simulation

- Prediction/Forecasting

- Diagnostics

- Design/Performance Evaluation

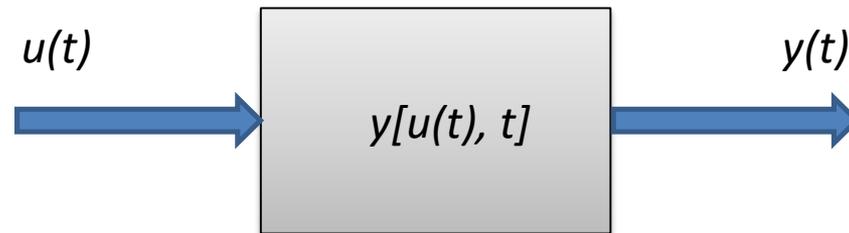- Control System Design

Black Box Model

- When only input and output are known.
- Internal dynamics are either too complex or unknown.

Input →■→ Output
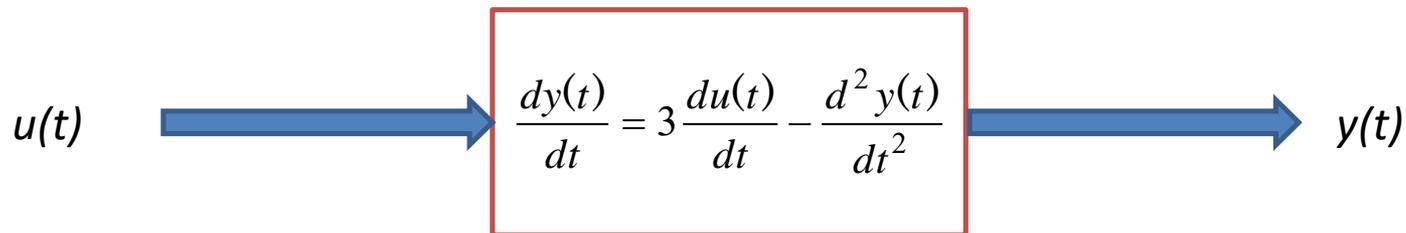
- Easy to Model

Grey Box Model

- When input and output and some information about the internal dynamics of the system is known.
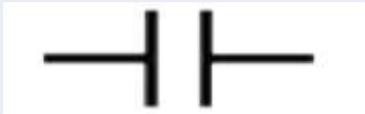

- Easier than white box Modelling.

White Box Model

- When input and output and internal dynamics of the system is known.

$$u(t) \rightarrow \boxed{\frac{dy(t)}{dt} = 3\frac{du(t)}{dt} - \frac{d^2 y(t)}{dt^2}} \rightarrow y(t)$$

- One should have complete knowledge of the system to derive a white box model.

# Modeling of Electrical Systems

Basic Elements of Electrical Systems

Component	Symbol	V-I Relation (Time Domain)	V-I Relation (Frequency Domain)
Resistor		$v_R(t) = i_R(t)R$	$V_R(s) = I_R(s)R$
Capacitor		$v_c(t) = \dfrac{1}{C}\int i_c(t)dt$	$V_c(s) = \dfrac{1}{Cs}I_c(s)$
Inductor		$v_L(t) = L\dfrac{di_L(t)}{dt}$	$V_L(s) = LsI_L(s)$

Transfer function for Electrical Systems

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
Capacitor	$v(t) = \dfrac{1}{C}\displaystyle\int_0^1 i(\tau)d\tau$	$i(t) = C\dfrac{dv(t)}{dt}$	$v(t) = \dfrac{1}{C}q(t)$	$\dfrac{1}{Cs}$	Cs
Resistor	$v(t) = Ri(t)$	$i(t) = \dfrac{1}{R}v(t)$	$v(t) = R\dfrac{dq(t)}{dt}$	R	$\dfrac{1}{R} = G$
Inductor	$v(t) = L\dfrac{di(t)}{dt}$	$i(t) = \dfrac{1}{L}\displaystyle\int_0^1 v(\tau)d\tau$	$v(t) = L\dfrac{d^2q(t)}{dt^2}$	Ls	$\dfrac{1}{Ls}$

Example#1

- The two-port network shown in the following figure has $v_i(t)$ as the input voltage and $v_o(t)$ as the output voltage. Find the transfer function $V_o(s)/V_i(s)$ of the network.
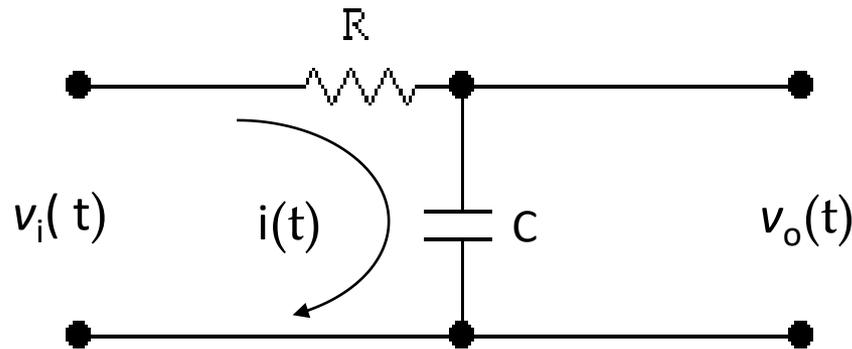
$$v_i(t) = i(t)R + \frac{1}{C}\int i(t)dt$$

$$v_o(t) = \frac{1}{C}\int i(t)dt$$

Example#1

$$v_i(t) = i(t)R + \frac{1}{C}\int i(t)dt \qquad\qquad v_o(t) = \frac{1}{C}\int i(t)dt$$

- Taking Laplace transform of both equations, considering initial conditions to zero.

$$V_i(s) = I(s)R + \frac{1}{Cs}I(s) \qquad\qquad V_o(s) = \frac{1}{Cs}I(s)$$

- Re-arrange both equations as:

$$V_i(s) = I(s)(R + \frac{1}{Cs}) \qquad\qquad CsV_o(s) = I(s)$$

Example#1

$$V_i(s) = I(s)(R + \frac{1}{Cs})$$

$$CsV_o(s) = I(s)$$

- Substitute *I(s)* in equation on left

$$V_i(s) = CsV_o(s)(R + \frac{1}{Cs})$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{Cs(R + \frac{1}{Cs})}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{1 + RCs}$$

Example#1

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{1 + RCs}$$

- The system has one pole at

$$1 + RCs = 0 \qquad \Rightarrow s = -\frac{1}{RC}$$
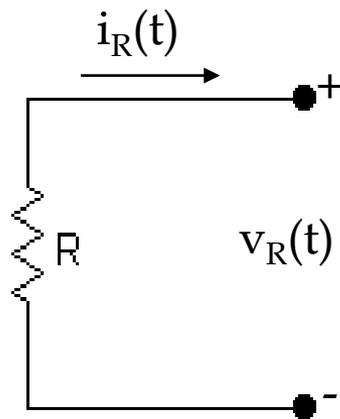
Example#2

- Find the transfer function G(S) of the following two port network.
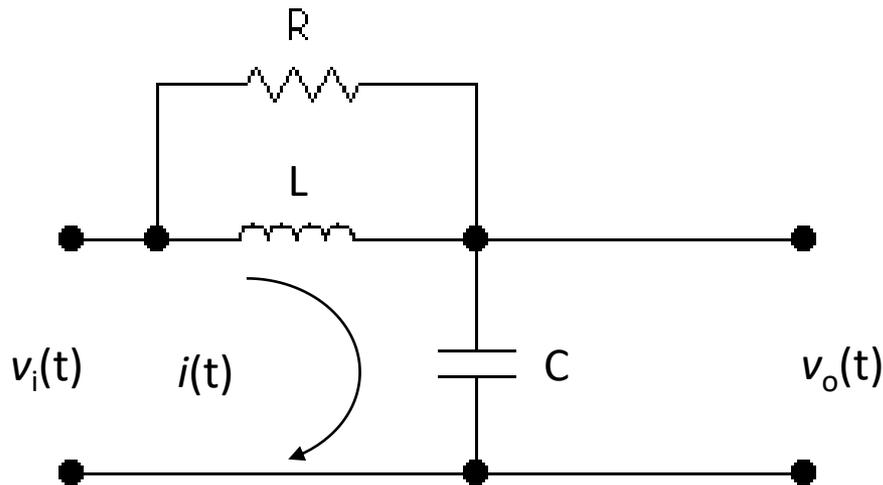
Example#2

- Simplify network by replacing multiple components with their equivalent transform impedance.
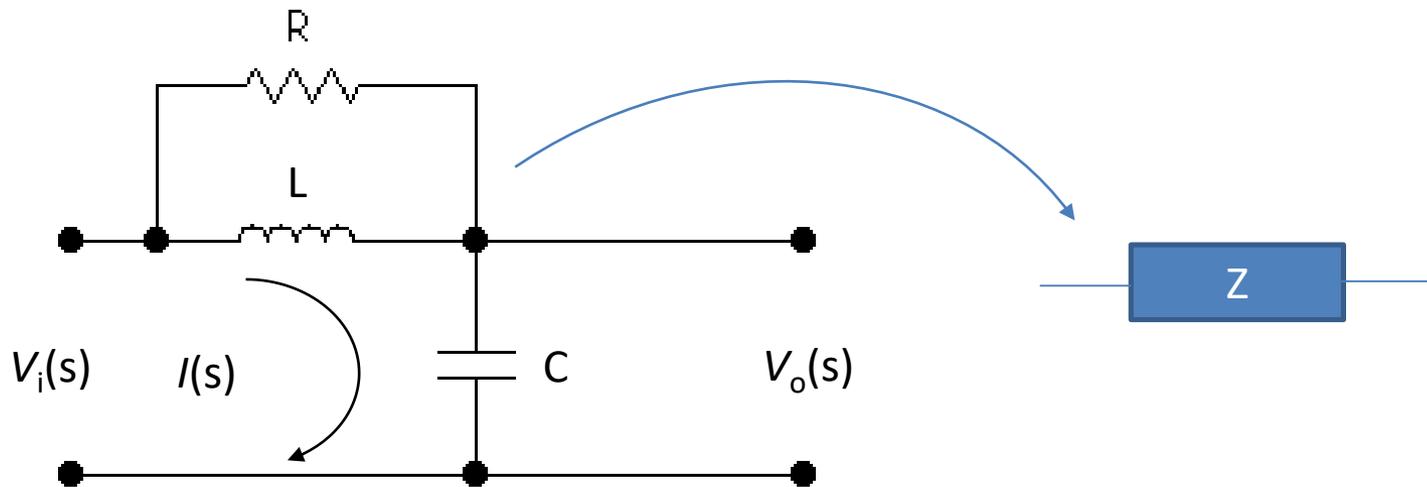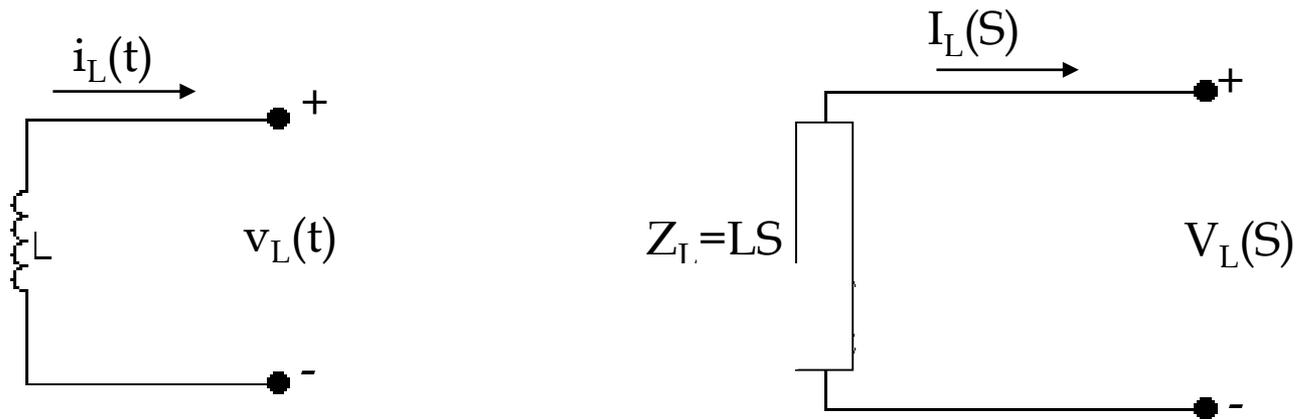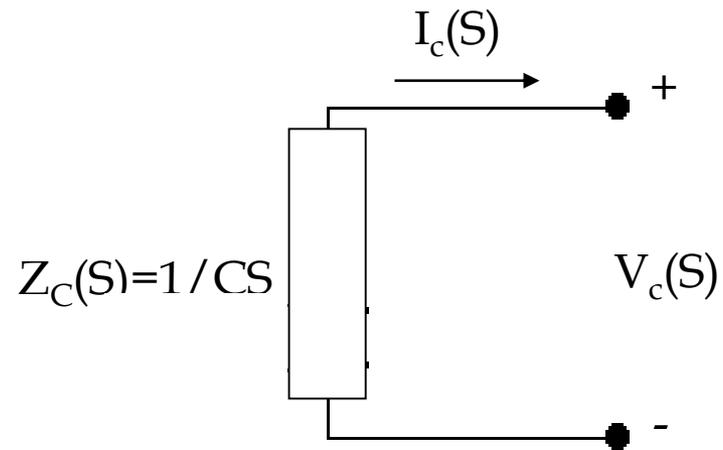
Transform Impedance (Resistor)

$i_R(t)$

$+$

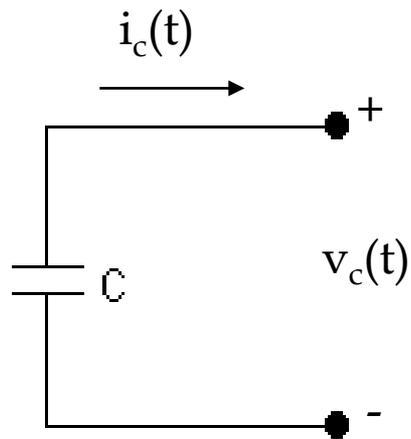R

$v_R(t)$

$-$

Transformation

$I_R(S)$

$+$

$Z_R = R$

$V_R(S)$

$-$
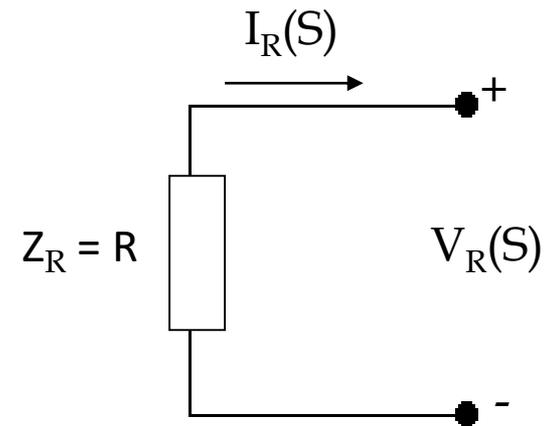
Transform Impedance (Inductor)

$i_L(t)$

$+$

$v_L(t)$

L

$-$

$I_L(S)$

$+$

$Z_L=LS$

$V_L(S)$

$-$

Transform Impedance (Capacitor)

$i_c(t)$

$v_c(t)$

$+$

$-$

C

$I_c(S)$

$Z_C(S) = 1/CS$

$V_c(S)$

$+$

$-$

Equivalent Transform Impedance (Series)

- Consider following arrangement, find out equivalent transform impedance.

$$Z_T = Z_R + Z_L + Z_C$$

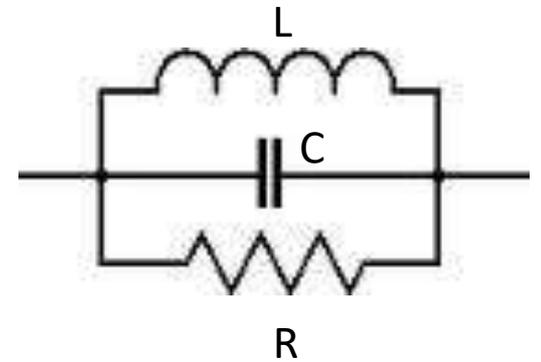$$Z_T = R + Ls + \frac{1}{Cs}$$
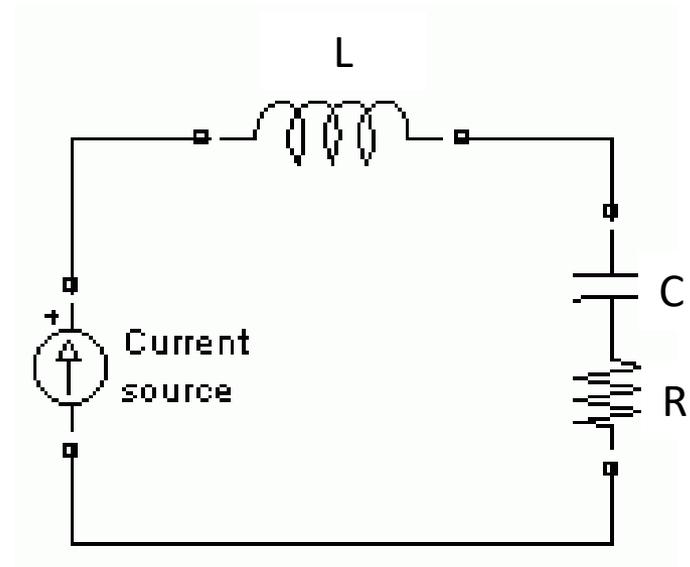
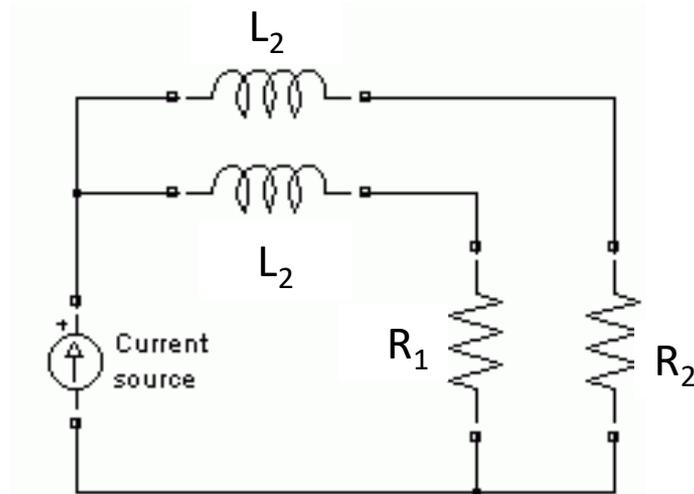# Equivalent Transform Impedance (Parallel)

$$\frac{1}{Z_T} = \frac{1}{Z_R} + \frac{1}{Z_L} + \frac{1}{Z_C}$$

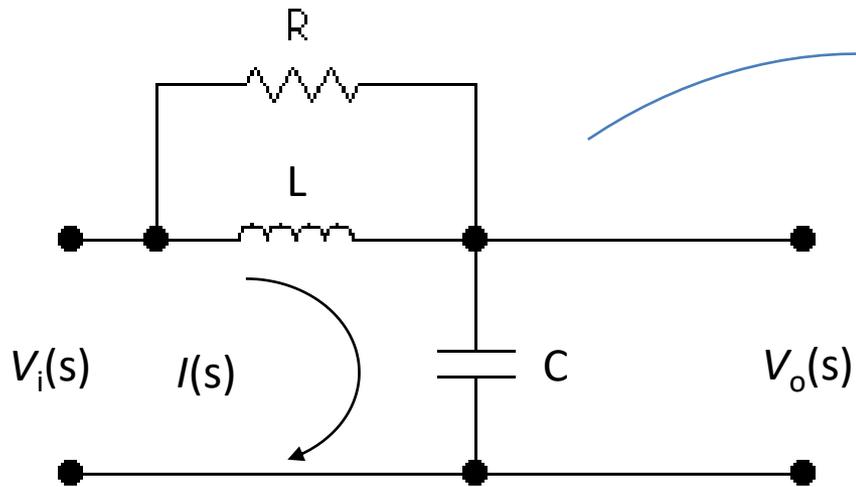$$\frac{1}{Z_T} = \frac{1}{R} + \frac{1}{Ls} + \frac{1}{\dfrac{1}{Cs}}$$

Equivalent Transform Impedance

- Find out equivalent transform impedance of following arrangement.
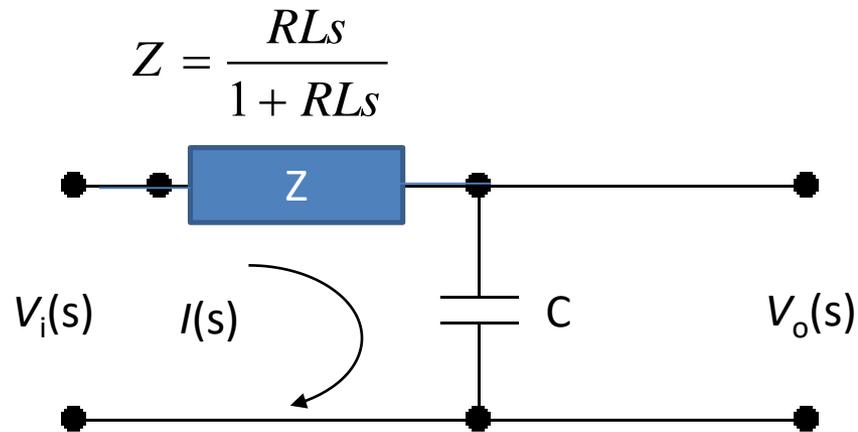
Back to Example#2

$$\frac{1}{Z} = \frac{1}{Z_R} + \frac{1}{Z_L}$$

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{Ls}$$
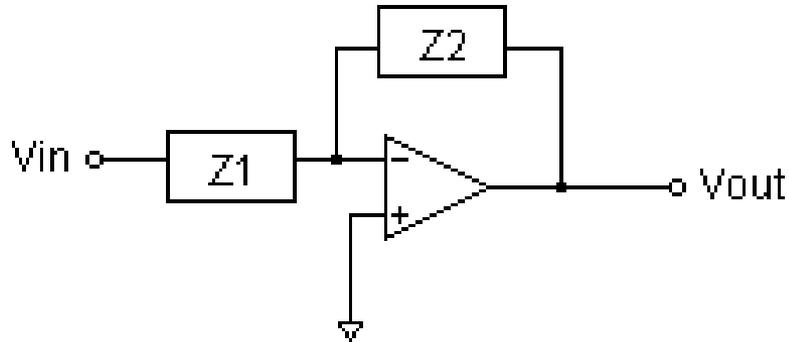
$$Z = \frac{RLs}{1 + RLs}$$

Example#2

$$Z = \frac{RLs}{1 + RLs}$$

Z

$V_i(s)$ $I(s)$ C $V_o(s)$
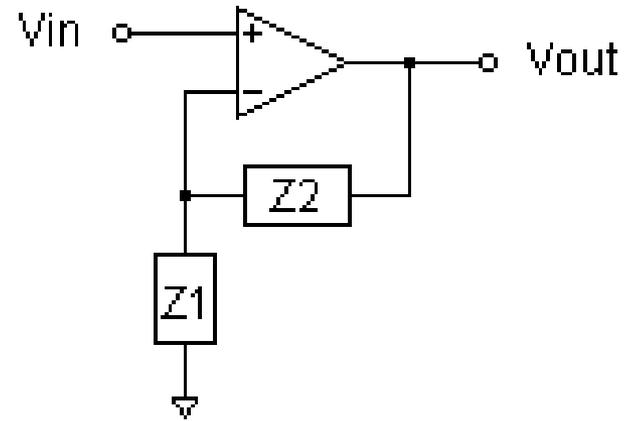
$$V_i(s) = I(s)Z + \frac{1}{Cs}I(s)$$

$$V_o(s) = \frac{1}{Cs}I(s)$$

Modeling of Electronic Systems
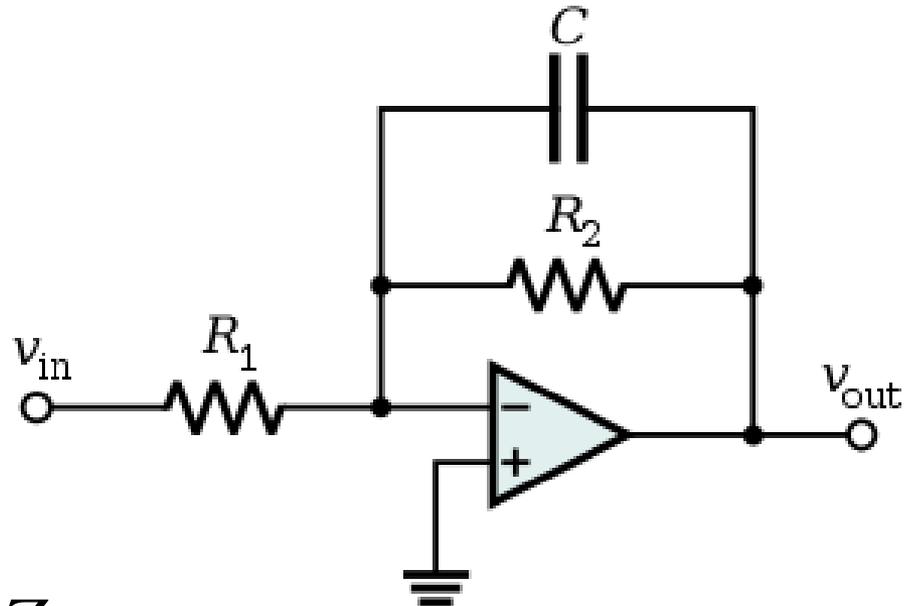
Operational Amplifiers

$$\frac{V_{out}}{V_{in}} = -\frac{Z_2}{Z_1}$$

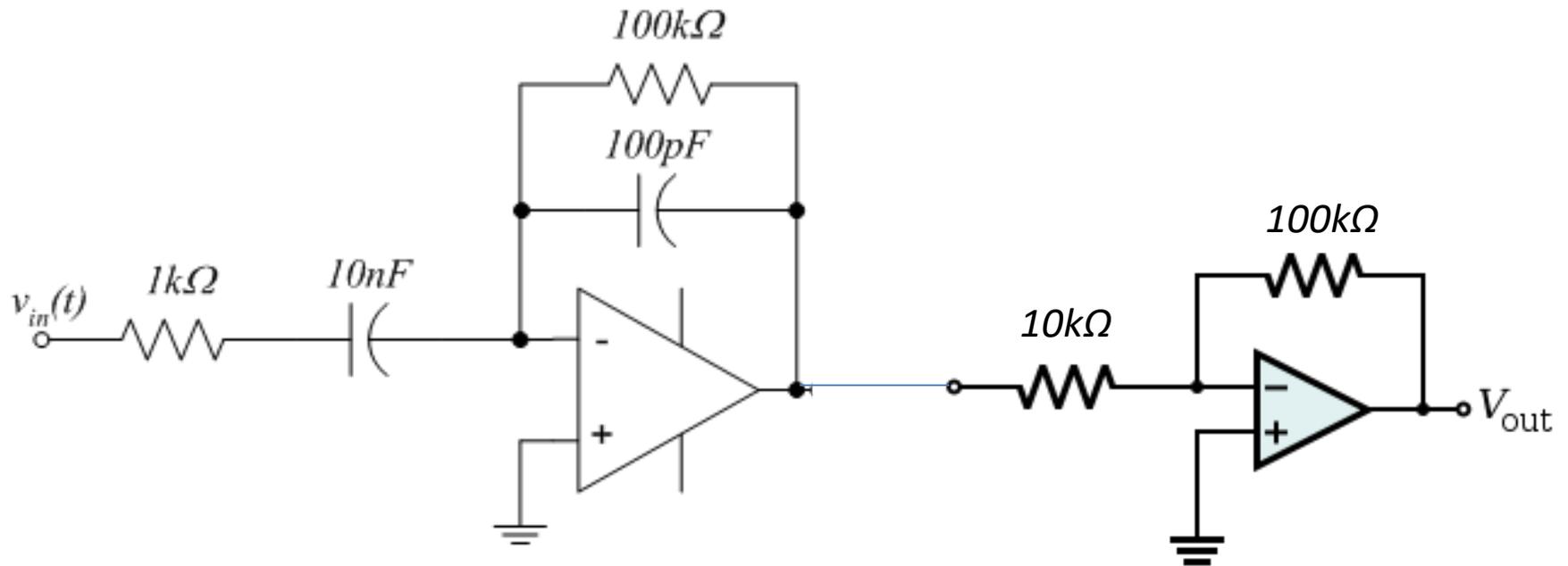$$\frac{V_{out}}{V_{in}} = 1 + \frac{Z_2}{Z_1}$$

Example#3

- Find out the transfer function of the following circuit.

$$\frac{V_{out}}{V_{in}} = -\frac{Z_2}{Z_1}$$

Example#4

- Find out the transfer function of the following circuit and draw the pole zero map.
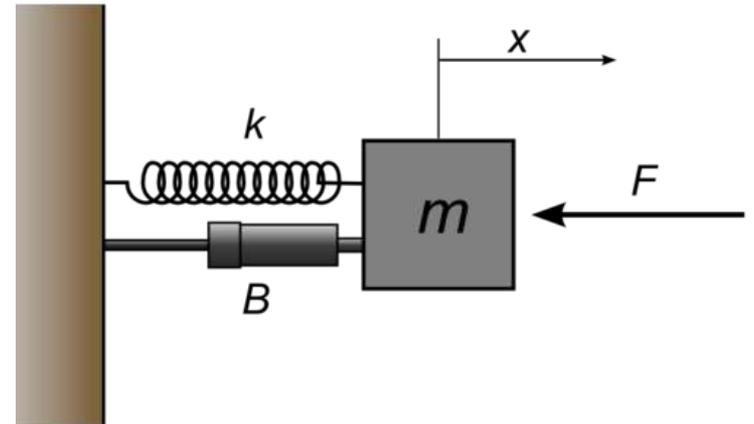
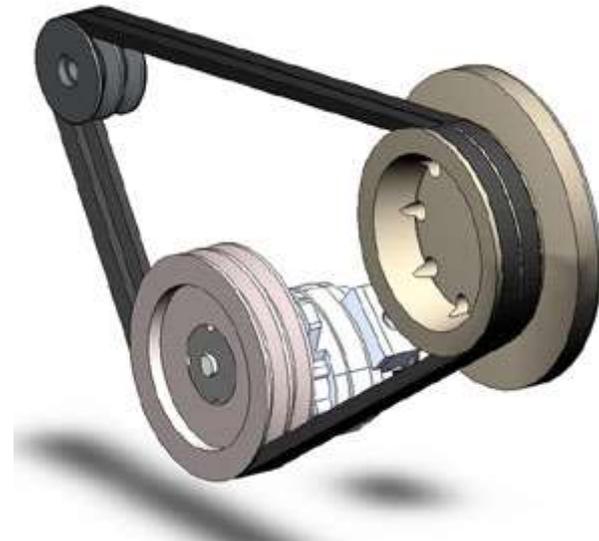# Modeling of Mechanical Systems

- **Part-I:** Translational Mechanical System

- **Part-II:** Rotational Mechanical System

- **Part-III:** Mechanical Linkages

Basic Types of Mechanical Systems

- Translational
  - Linear Motion

- Rotational
  - Rotational Motion

TRANSLATIONAL MECHANICAL SYSTEMS

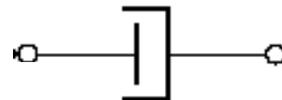# Basic Elements of Translational Mechanical Systems
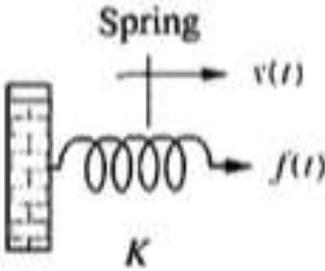
Translational Spring

i)

Translational Mass

ii)

Translational Damper
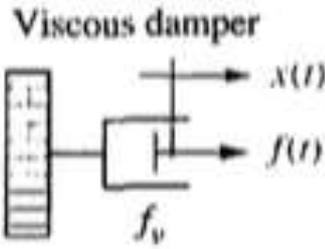
iii)

Transfer function for Mechanical Systems

Component	Force-velocity	Force-displacement	Impedence $Z_M(s) = F(s)/X(s)$
Spring	$f(t) = K \int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper	$f(t) = f_v v(t)$	$f(t) = f_v \dfrac{dx(t)}{dt}$	$f_v s$
Mass	$f(t) = M \dfrac{dv(t)}{dt}$	$f(t) = M \dfrac{d^2 x(t)}{dt^2}$	Ms^2

Translational Spring

- A translational spring is a mechanical element that can be deformed by an external force such that the deformation is directly proportional to the force applied to it.

Translational Spring

i)

Circuit Symbols

Translational Spring

Translational Spring

- If **F** is the applied force

$$x_2 \circ \overset{k}{\text{mmm}} \overset{x_1}{\circ} \to F$$

- Then x_1 is the deformation if $x_2 = 0$

- Or $(x_1 - x_2)$ is the deformation.

- The equation of motion is given as

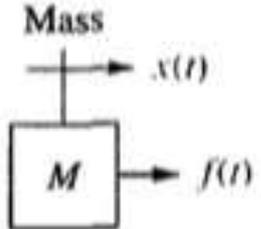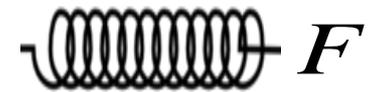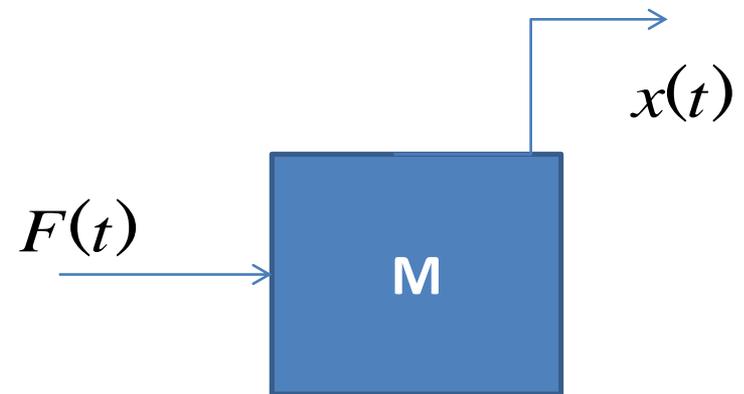$$F = k(x_1 - x_2)$$

- Where k is stiffness of spring expressed in N/m

Translational Mass

- Translational Mass is an inertia element.

- A mechanical system without mass does not exist.

- If a force *F* is applied to a mass and it is displaced to *x* meters then the relation b/w force and displacements is given by Newton's law.
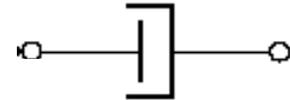
Translational Mass

ii)

$F(t)$

M

$x(t)$

$$F = M\ddot{x}$$

Translational Damper

- When the viscosity or drag is not negligible in a system, we often model them with the damping force.

- All the materials exhibit the property of damping to some extent.

- If damping in the system is not enough then extra elements (e.g. Dashpot) are added to increase damping.

Translational Damper

iii)

Common Uses of Dashpots

Door Stoppers
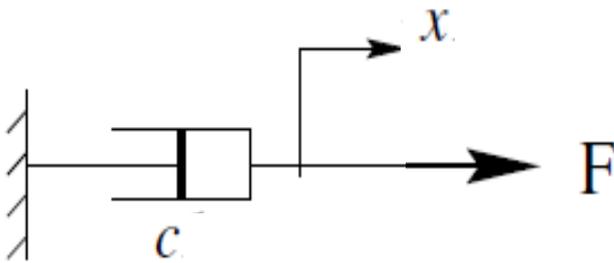
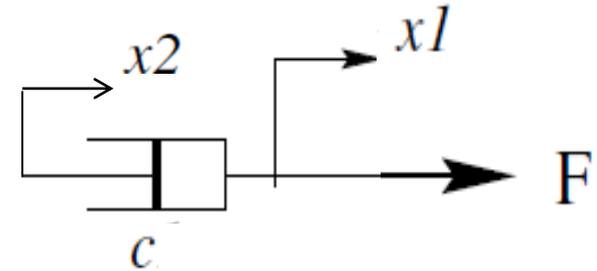Vehicle Suspension

Bridge Suspension

Flyover Suspension

Translational Damper

$$F = C\dot{x}$$

$$F = C(\dot{x}_1 - \dot{x}_2)$$

- Where *C* is damping coefficient (*N/ms⁻¹*).

Example-7

- Consider the following system (friction is negligible)
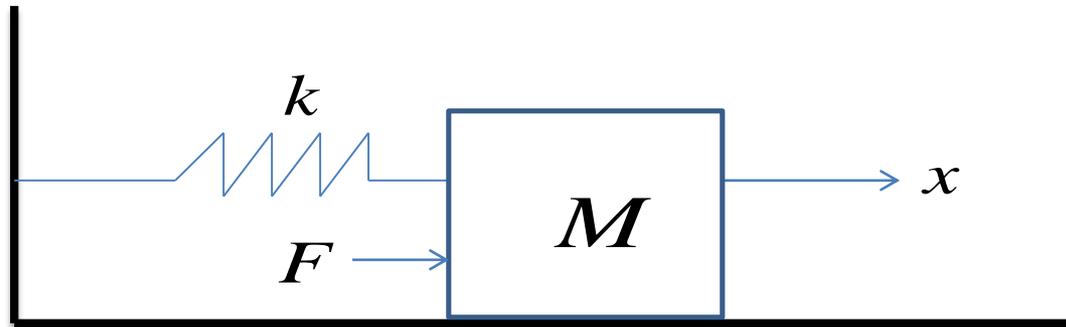
- Free Body Diagram

- Where f_k and f_M are forces applied by the spring and inertial force respectively.

Example-7

$$F = f_k + f_M$$

- Then the differential equation of the system is:

$$F = M\ddot{x} + kx$$

- Taking the Laplace Transform of both sides and ignoring initial conditions we get

$$F(s) = Ms^2 X(s) + kX(s)$$

Example-7

$$F(s) = Ms^2 X(s) + kX(s)$$

- The transfer function of the system is

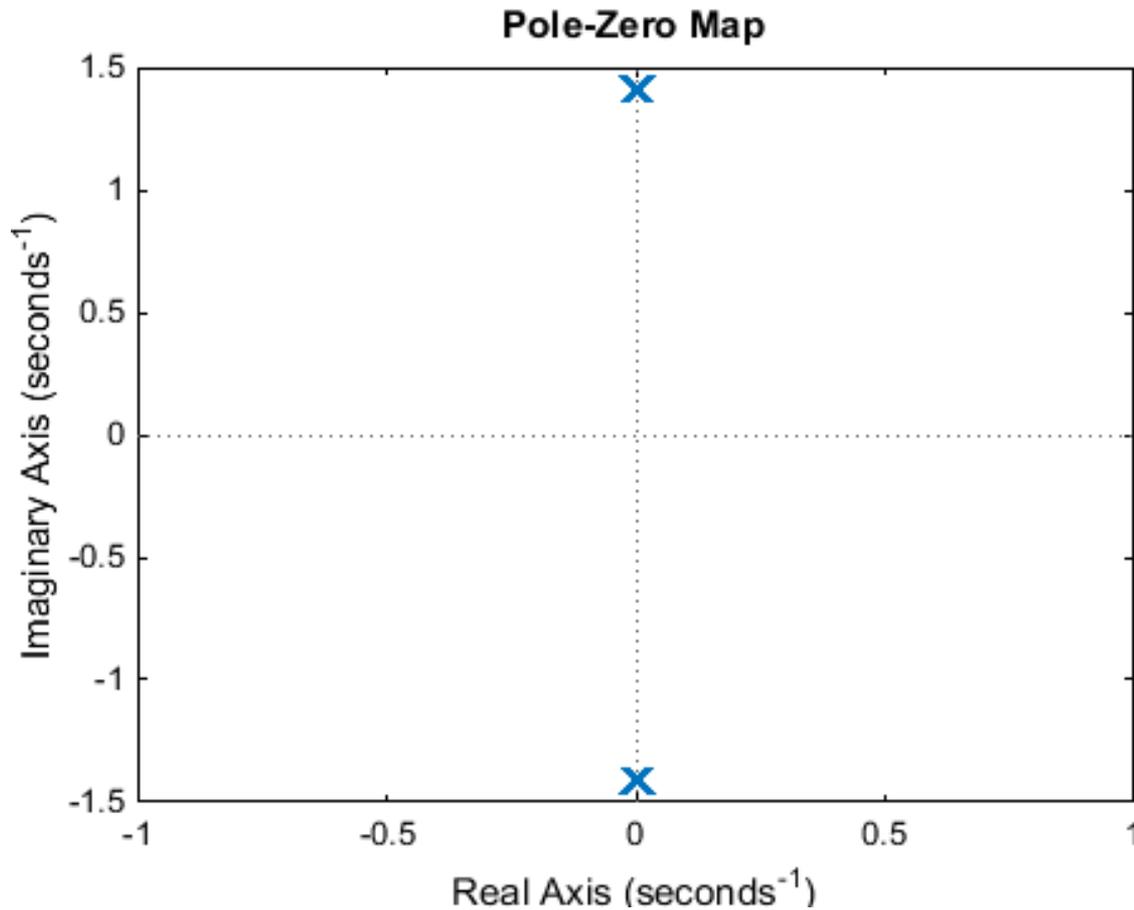$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + k}$$

- if

$$M = 1000 \, kg$$

$$k = 2000 \, Nm^{-1}$$

$$\frac{X(s)}{F(s)} = \frac{0.001}{s^2 + 2}$$

Example-7

$$\frac{X(s)}{F(s)} = \frac{0.001}{s^2 + 2}$$

- The pole-zero map of the system is

Pole-Zero Map

Example-8

- Consider the following system
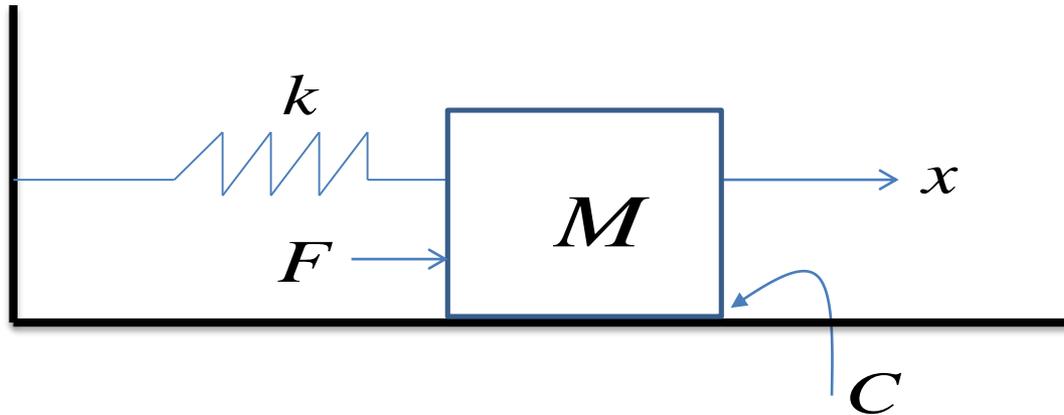
- Free Body Diagram

$$F = f_k + f_M + f_C$$

Example-8

Differential equation of the system is:

$$F = M\ddot{x} + C\dot{x} + kx$$

Taking the Laplace Transform of both sides and ignoring Initial conditions we get

$$F(s) = Ms^2 X(s) + CsX(s) + kX(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Cs + k}$$

Example-8

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Cs + k}$$

- if

$$M = 1000\,kg$$

$$k = 2000\,Nm^{-1}$$

$$C = 1000\,N/ms^{-1}$$

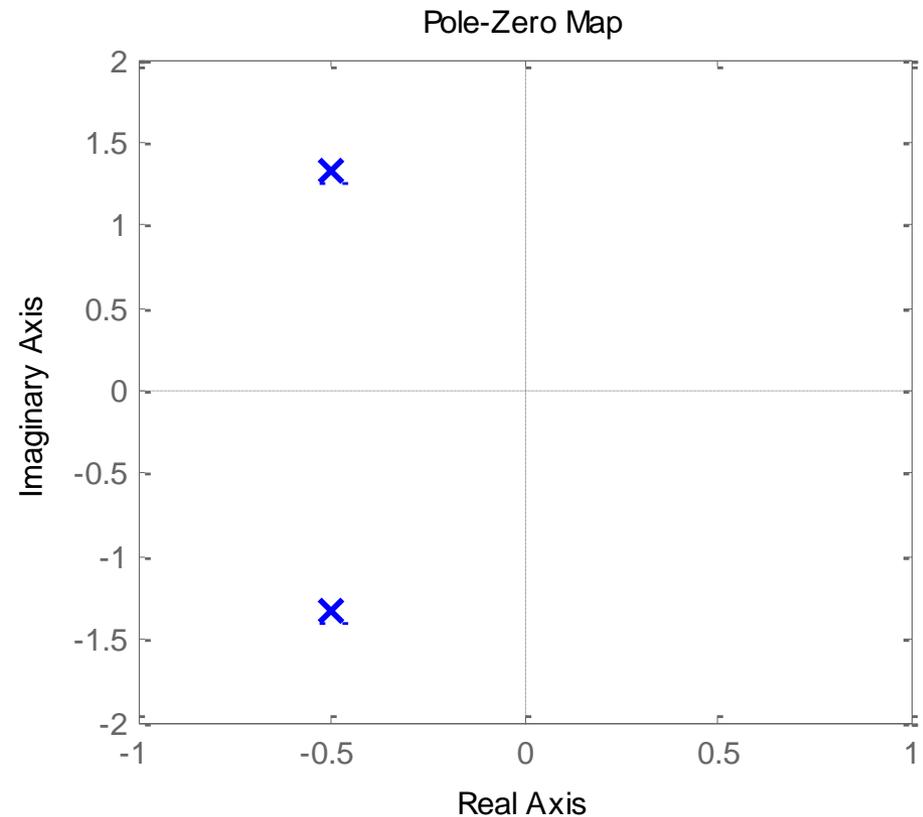$$\frac{X(s)}{F(s)} = \frac{0.001}{s^2 + s + 2}$$

Pole-Zero Map

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Example-9

- Consider the following system
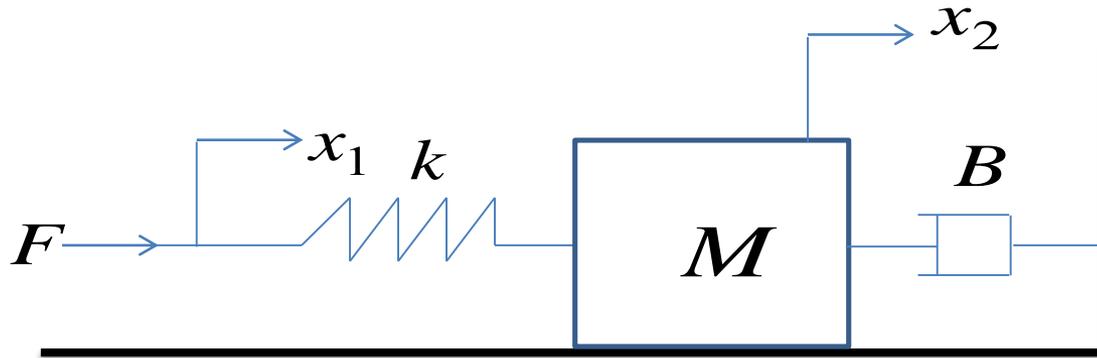
- Mechanical Network
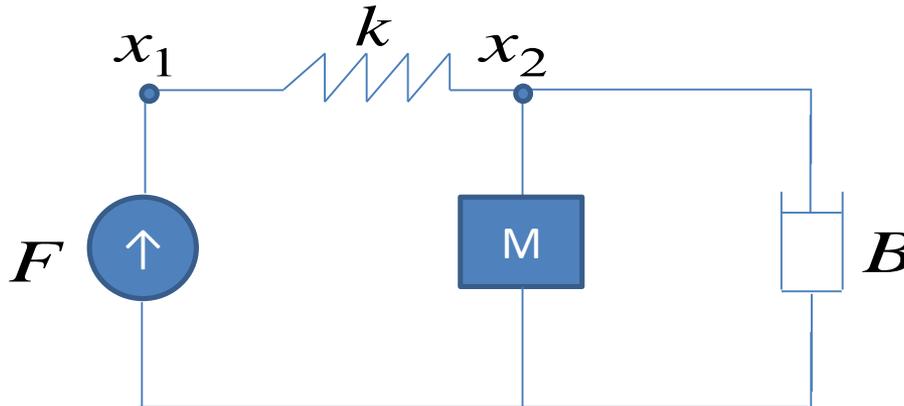
Example-9

- Mechanical Network
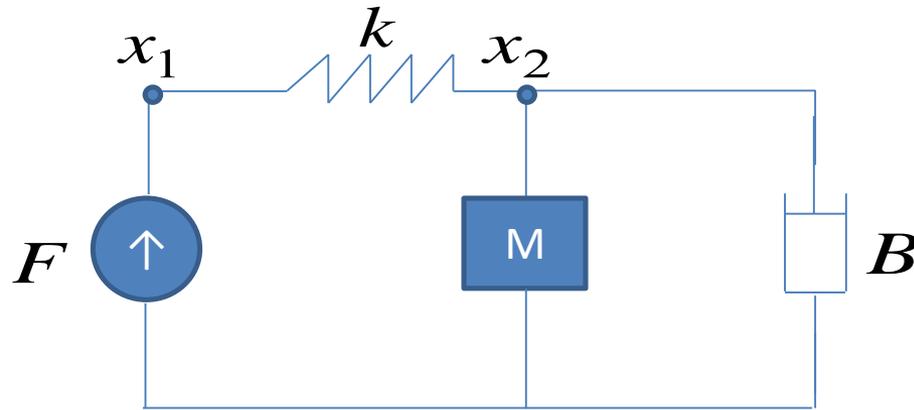
At node x_1

$$F = k(x_1 - x_2)$$

At node x_2

$$0 = k(x_2 - x_1) + M\ddot{x}_2 + B\dot{x}_2$$

Example-10
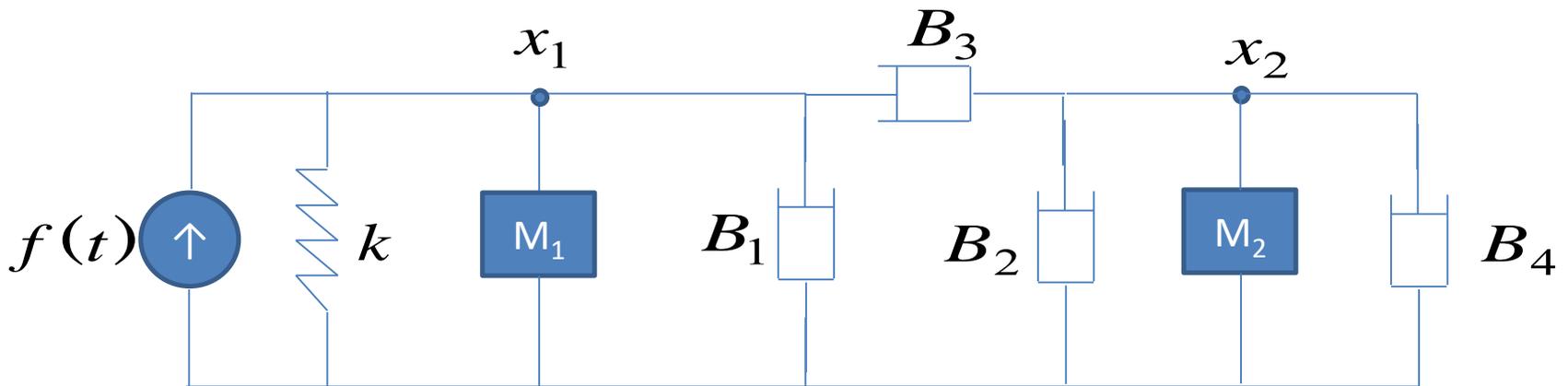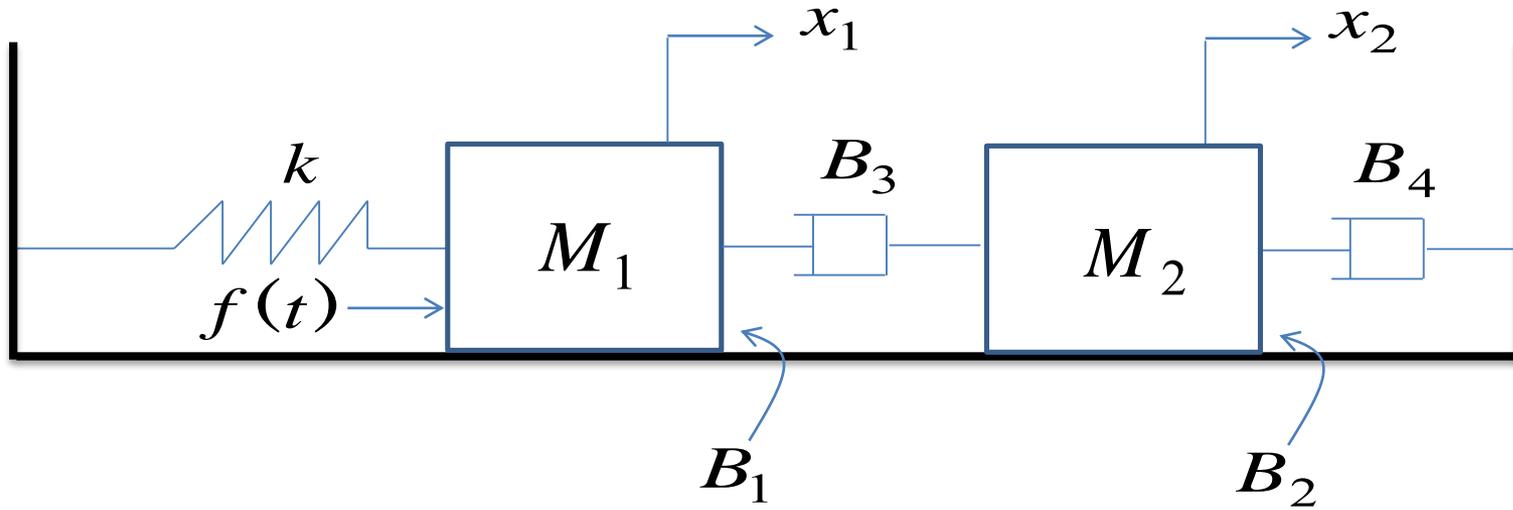
ROTATIONAL MECHANICAL SYSTEMS

Basic Elements of Rotational Mechanical Systems

Rotational Spring
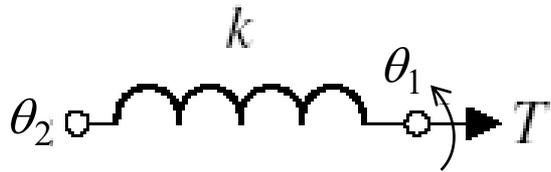
$$T = k(\theta_1 - \theta_2)$$

Basic Elements of Rotational Mechanical Systems

Rotational Damper
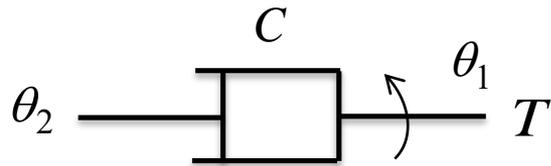
$$T = C(\dot{\theta}_1 - \dot{\theta}_2)$$

Basic Elements of Rotational Mechanical Systems

Moment of Inertia
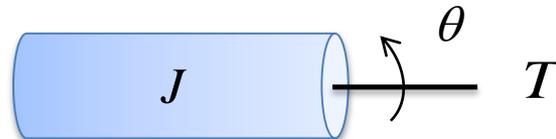
$$T = J\ddot{\theta}$$

Example-11
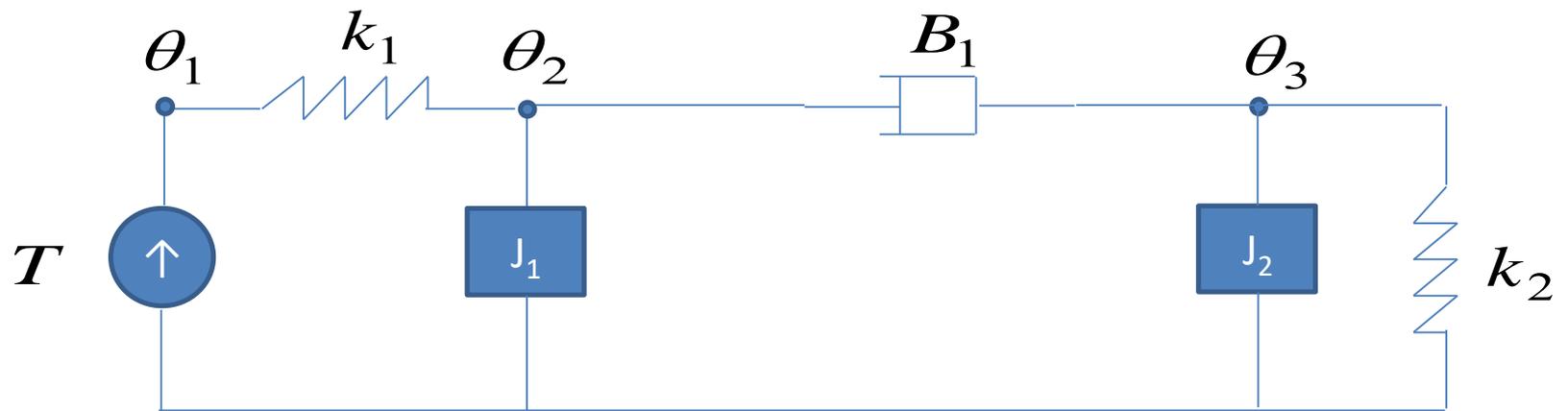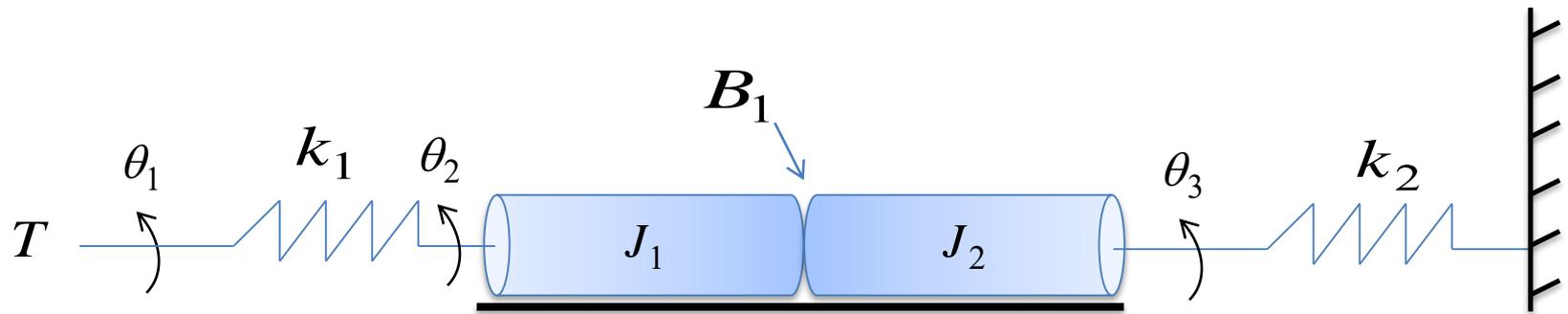
Example-12
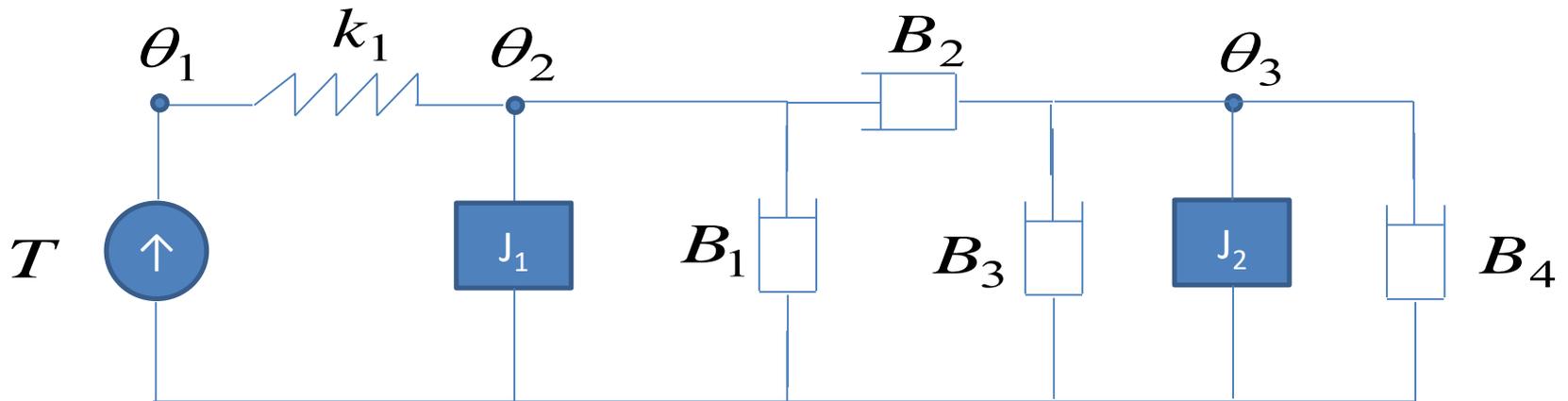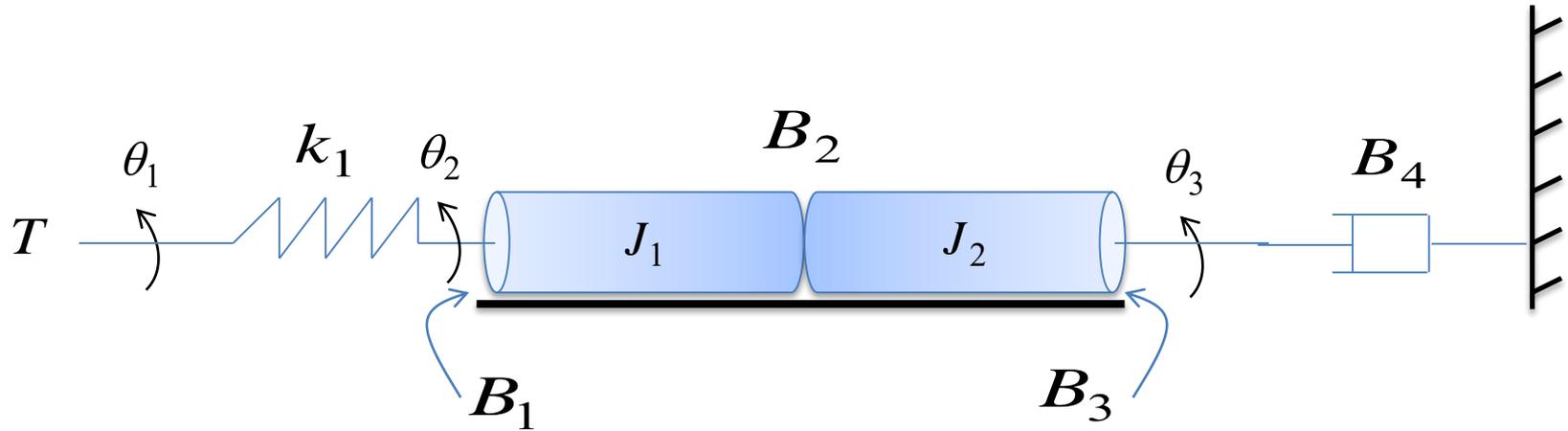
Example-13
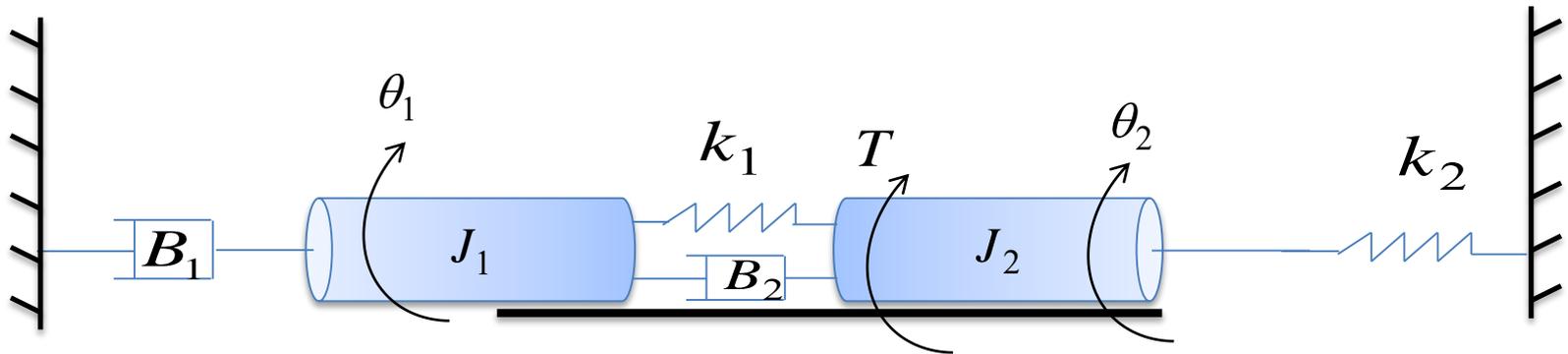
MECHANICAL LINKAGES

Gear

- **Gear** is a toothed machine part, such as a wheel or cylinder, that meshes with another toothed part to transmit motion or to change speed or direction.
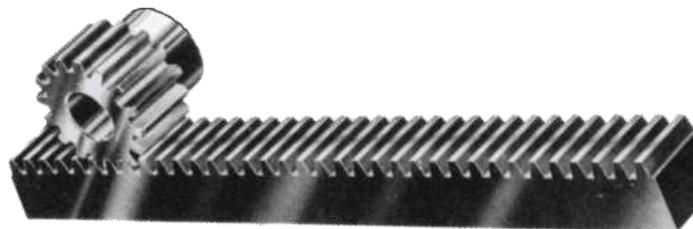
Fundamental Properties

- The two gears turn in opposite directions: one clockwise and the other counterclockwise.

- Two gears revolve at different speeds when number of teeth on each gear are different.

Gear Ratio

- You can calculate the **gear ratio** by using the number of teeth of the *driver* divided by the number of teeth of the *follower*.

- We *gear up* when we increase velocity and decrease torque.
  Ratio: 3:1

- We *gear down* when we increase torque and reduce velocity.
  Ratio: 1:3

3 to 1 ratio

Driver

Follower 8 24

3 turns moves by 24 teeth

1 turn moves by 24 teeth
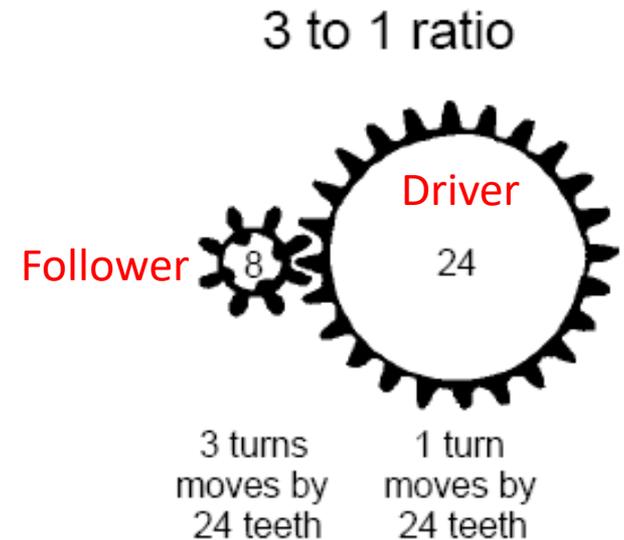
Gear Ratio = # teeth input gear / # teeth output gear
= torque in / torque out = speed out / speed in

Mathematical Modeling of Gear Trains

- Gears increase or reduce angular velocity (while simultaneously decreasing or increasing torque, such that energy is conserved).

Energy of Driving Gear = Energy of Following Gear

$$N_1 \theta_1 \quad = \quad N_2 \theta_2$$

$N_1 \longrightarrow$ Number of Teeth of Driving Gear

$\theta_1 \longrightarrow$ Angular Movement of Driving Gear

$N_2 \longrightarrow$ Number of Teeth of Following Gear

$\theta_2 \longrightarrow$ Angular Movement of Following Gear

$$\frac{T_2}{T_1} = \frac{N_2}{N_1} \Rightarrow T_1 = \frac{N_1}{N_2} T_2$$

Driver

Follower
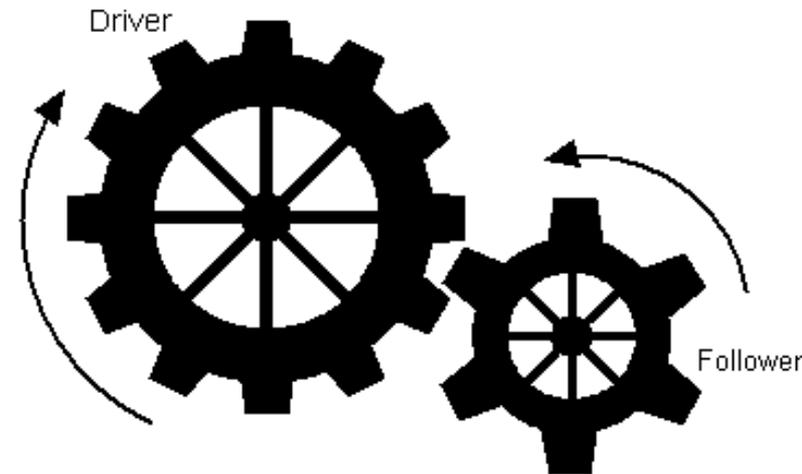
Transfer Function of Mechanical Systems with Lossless Gears

➢ Each gear is assumed **lossless**, which implies that the gear has negligible inertia and the friction is minimal and can be ignored if compared to the applied torque. Also $s_1 = s_2 = r_1\theta_1 = r_2\theta_2$

$$T_1\theta_1 = T_2\theta_2$$

➢ Assuming **lossless** gears:

$$\frac{\theta_2}{\theta_1} = \frac{r_1}{r_2} = \frac{N_1}{N_2} = \frac{T_1}{T_2}$$
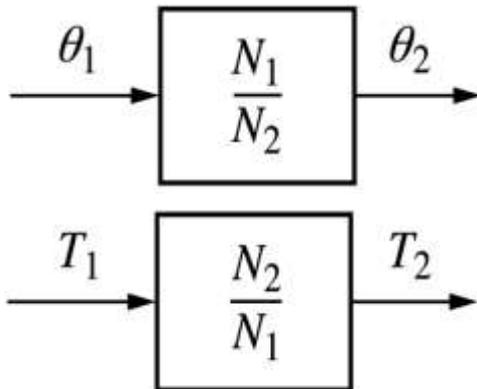
$\theta_1 \rightarrow \boxed{\dfrac{N_1}{N_2}} \rightarrow \theta_2$ **Transfer function for angular displacement in (lossless Gears)**

$T_1 \rightarrow \boxed{\dfrac{N_2}{N_1}} \rightarrow T_2$ **Transfer function for Torque in (lossless Gears)**
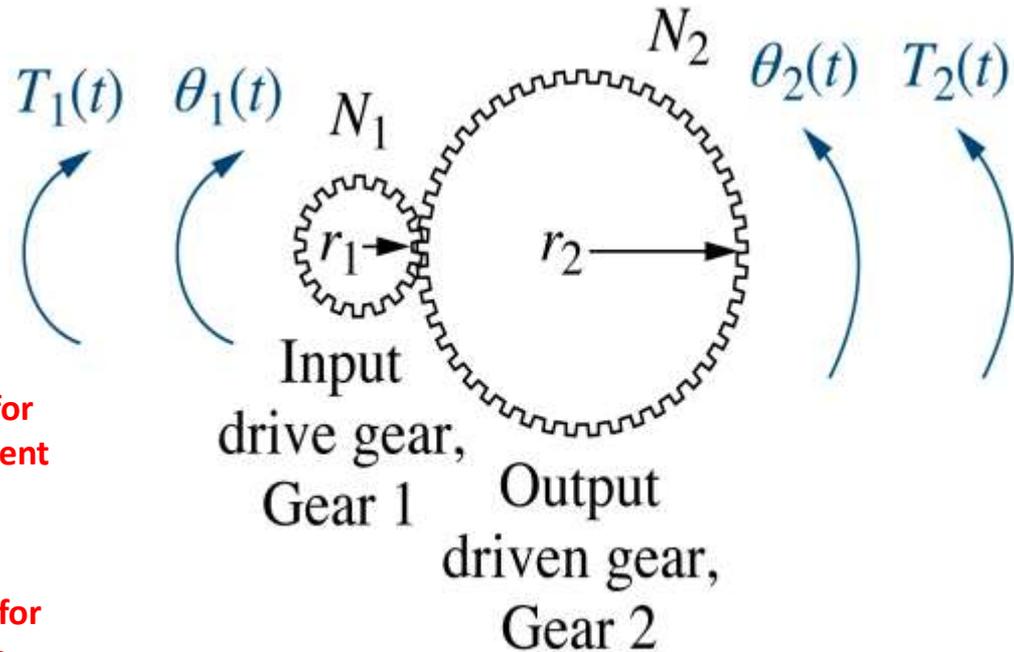
$T_1(t) \quad \theta_1(t) \quad N_1$ ... $N_2 \quad \theta_2(t) \quad T_2(t)$

$r_1 \rightarrow$

$r_2 \rightarrow$

Input drive gear, Gear 1

Output driven gear, Gear 2

Transfer Function of Mechanical Systems with Lossless Gears

- ➢ **Represent** the shown system as an **equivalent system at θ₁(t) <u>without</u> gears.**
- ➢ In other words, **can the mechanical impedances be reflected from the output to the input, thereby eliminating the gears?**

Rotational system driven by gears

(a)

Equivalent system <u>at the output</u> after reflection of input torque

(b)

$$(Js^2 + Ds + K)\theta_2(s) = T_2(s) = T_1(s)\frac{N_2}{N_1}$$

$$(Js^2 + Ds + K)\frac{N_1}{N_2}\theta_1(s) = T_1(s)\frac{N_2}{N_1}$$
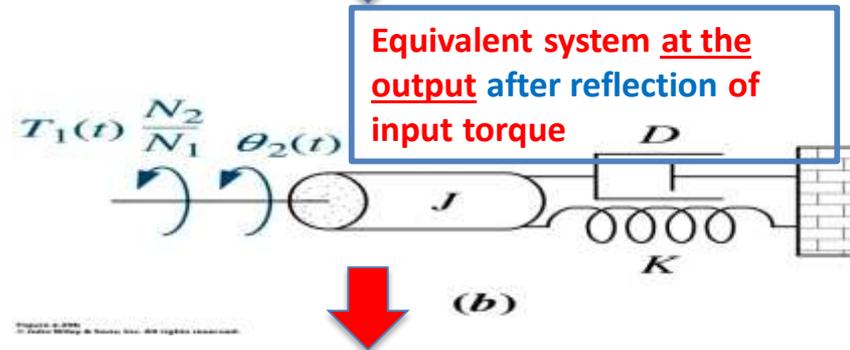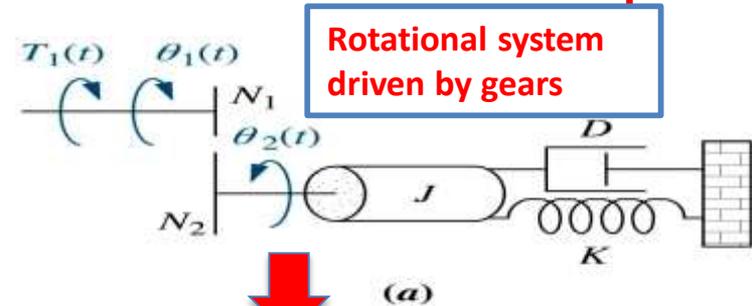
Modeling of Electromechanical Systems

D.C Drives
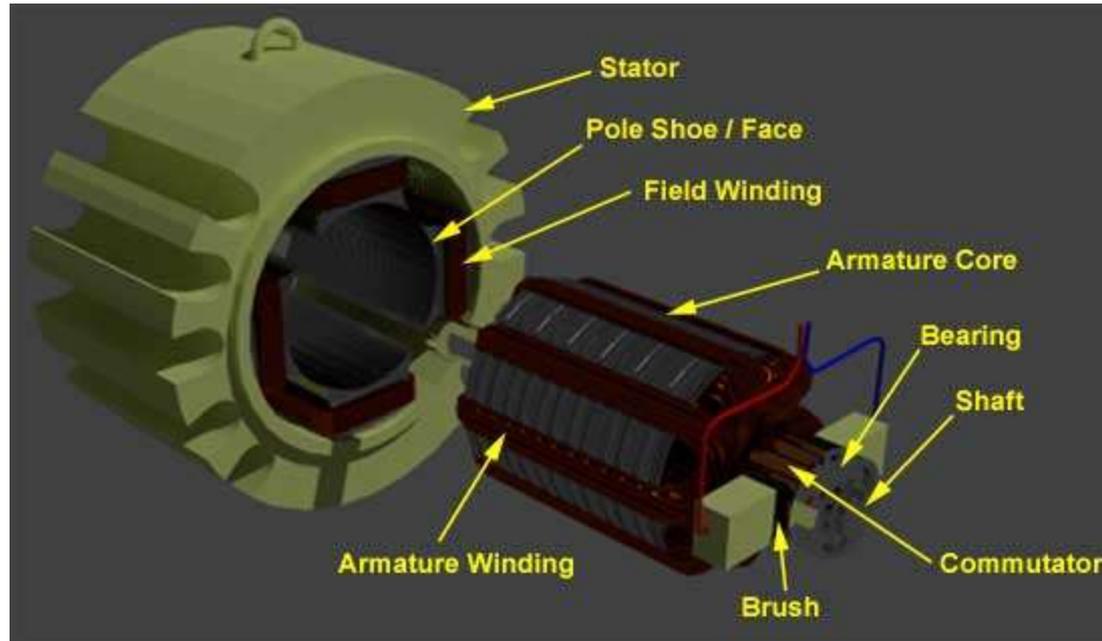
- Variable Voltage can be applied to the armature terminals of the DC motor .

- Another method is to vary the flux per pole of the motor.

- The first method involve adjusting the motor's armature while the latter method involves adjusting the motor field. These methods are referred to as "armature control" and "field control."

Example-14: Armature Controlled D.C Motor

Input: voltage u
Output: Angular velocity ω

Electrical Subsystem (loop method):

$$u = R_a i_a + L_a \frac{di_a}{dt} + e_b, \qquad where \quad e_b = back\text{-}emf\,voltage$$

Mechanical Subsystem

$$T_{motor} = J\dot{\omega} + B\omega$$
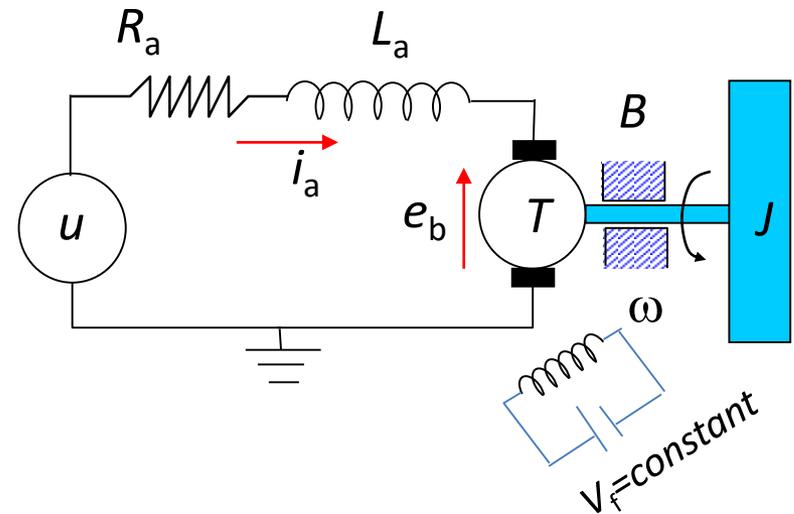
Example-14: Armature Controlled D.C Motor

Power Transformation:

Torque-Current: $T_{motor} = K_t i_a$

Voltage-Speed: $e_b = K_b \omega$
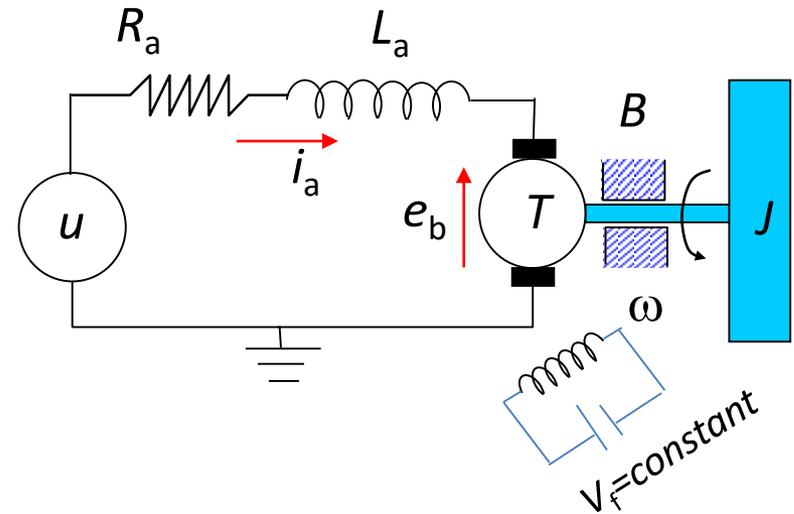
- Combing previous equations results in the following mathematical model:

$$\begin{cases} L_a \dfrac{di_a}{dt} + R_a i_a + K_b \omega = u \\ J\dot{\omega} + B\omega - K_t i_a = 0 \end{cases}$$

Example-14: Armature Controlled D.C Motor

Taking Laplace transform of the system's differential equations with zero initial conditions gives:

$$\begin{cases} (L_a s + R_a) I_a(s) + K_b \Omega(s) = U(s) \\ (Js + B)\Omega(s) - K_t I_a(s) = 0 \end{cases}$$

Eliminating I_a yields the input-output transfer function

$$\frac{\Omega(s)}{U(s)} = \frac{K_t}{L_a J s^2 + (JR_a + BL_a)s + BR_a + K_t K_b}$$

Example-14: Armature Controlled D.C Motor Reduced Order Model

Assuming small inductance, $L_a \approx 0$

$$\frac{\Omega(s)}{U(s)} = \frac{K_t}{JR_a s + (BR_a + K_t K_b)}$$

Example-14: Armature Controlled D.C Motor

If output of the D.C motor is angular position θ then we know

$$\omega = \frac{d\theta}{dt} \qquad or \qquad \Omega(s) = s\theta(s)$$
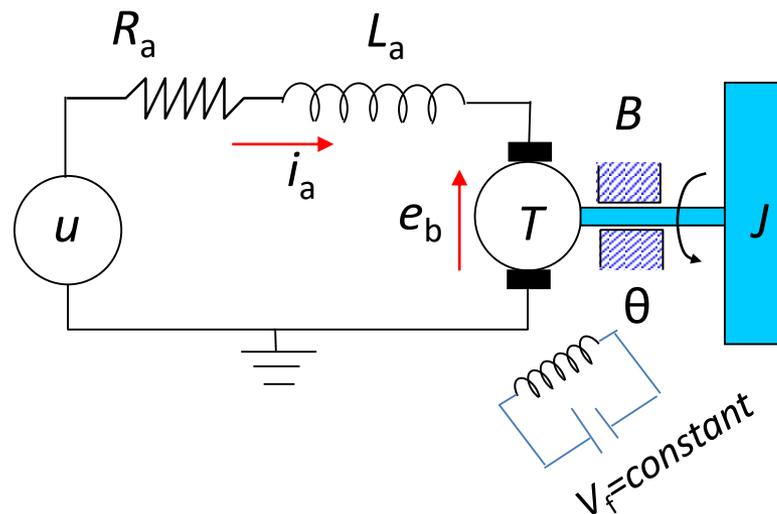
Which yields following transfer function

$$\frac{\theta(s)}{U(s)} = \frac{K_t}{s[JR_a s + (BR_a + K_t K_b)]}$$

Example-15: Field Controlled D.C Motor
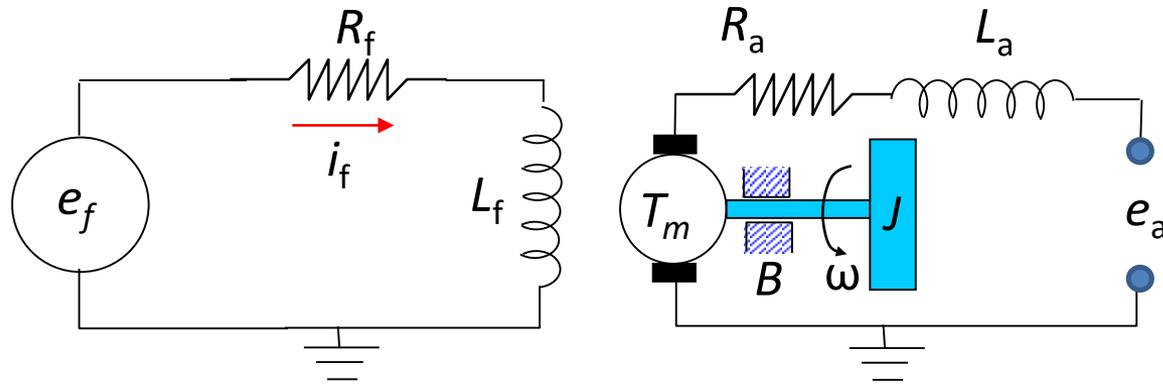
Applying KVL at field circuit

$$e_f = i_f R_f + L_f \frac{di_f}{dt}$$

Mechanical Subsystem

$$T_m = J\dot{\omega} + B\omega$$

Example-15: Field Controlled D.C Motor

Power Transformation:

Torque-Current: $T_m = K_f i_f$

where K_f: torque constant

Combing previous equations and taking Laplace transform (considering initial conditions to zero) results in the following mathematical model:

$$\begin{cases} E_f(s) = R_f I_f(s) + sL_f I_f(s) \\ Js\Omega(s) + B\Omega(s) = K_f I_f(s) \end{cases}$$

Eliminating $I_f(S)$ yields

$$\frac{\Omega(s)}{E_f(s)} = \frac{K_f}{(Js + B)(L_f s + R_f)}$$

If angular position θ is output of the motor

$$\frac{\theta(s)}{E_f(s)} = \frac{K_f}{s(Js + B)(L_f s + R_f)}$$

Example-16: Angular Position Control System

Numerical Values for System constants

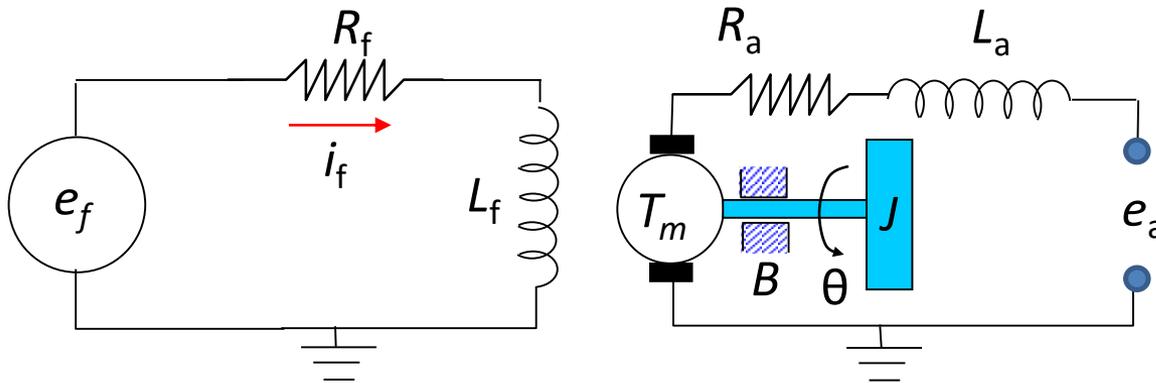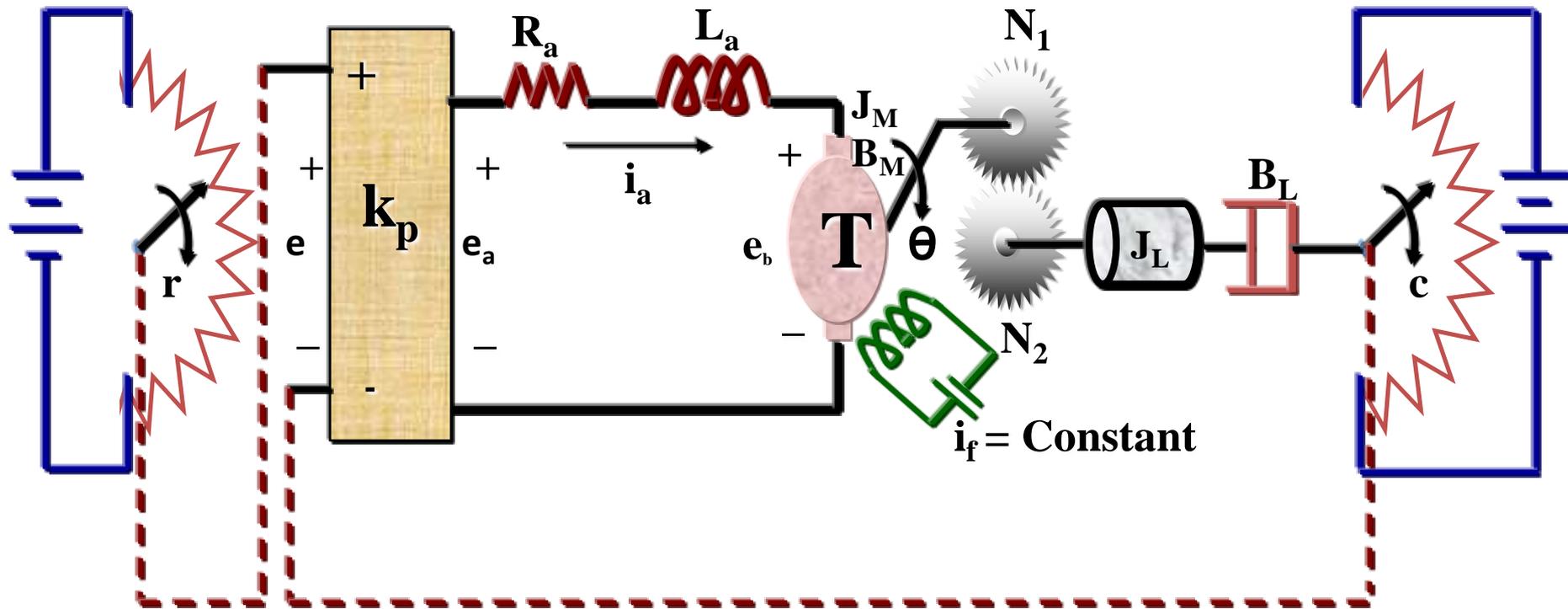**r =** angular displacement of the reference input shaft

c = angular displacement of the output shaft

θ = angular displacement of the motor shaft

K_1 **=** gain of the potentiometer shaft = **24/π**

K_p **=** amplifier gain = **10**

e_a **=** armature voltage

e_b **=** back emf

R_a **=** armature winding resistance = **0.2 Ω**

L_a **=** armature winding inductance = **negligible**

i_a **=** armature winding current

K_b **=** back emf constant = **5.5×10^{-2} volt-sec/rad**

K_t **=** motor torque constant = **6×10^{-5} N-m/ampere**

J_m **=** moment of inertia of the motor = **1×10^{-5} kg-m²**

B_m **=** viscous-friction coefficients of the motor = **negligible**

J_L **=** moment of inertia of the load = **4.4×10^{-3} kgm²**

B_L **=** viscous friction coefficient of the load = **4×10^{-2} N-m/rad/sec**

n= gear ratio = N_1/N_2 **= 1/10**

Example-16: Angular Position Control System

- **Transfer function of the armature controlled D.C motor with load connected to it is given by**

$$\frac{\theta(s)}{E_a(s)} = \frac{K_t}{s\left[\left(J_{eq}R_a + B_{eq}L_a\right)s + B_{eq}R_a + K_tK_b\right]}$$

- **Where**

$$J_{eq} = J_m + \left(\frac{N_1}{N_2}\right)^2 J_L = 1 \times 10^{-5} + \left(\frac{1}{10}\right)^2 \times 4.4 \times 10^{-3} = 5.4 \times 10^{-5}$$

$$B_{eq} = B_m + \left(\frac{N_1}{N_2}\right)^2 B_L = \left(\frac{1}{10}\right)^2 \times 4 \times 10^{-2} = 4 \times 10^{-4}$$

$$\frac{\theta(s)}{E_a(s)} = \frac{6}{s(1.08s + 8.33)}$$

Example-16: Angular Position Control System

- **Error is difference between reference input $r(t)$ and out calculated $c(t)$ and can be calculated as**

$$e(t) = K_1[r(t) - c(t)]$$

$$\mathrm{E}(s) = \frac{24}{\pi}[R(S) - C(S)] = 7.64[R(S) - C(S)] \quad \text{(a)}$$

- **Output of amplifier is**

$$E_a(s) = K_p E(s) = 10E(s) \quad \text{(b)}$$

- **Merging eq (a) and eq (b) yields**

$$E_a(s) = 76.4[R(S) - C(S)]$$

- **Relation between angular position of motor θ and angular position of load c is given as**

$$C(s) = \frac{1}{10}\theta(s)$$

- **or**

$$10C(s) = \theta(s)$$

Example-16: Angular Position Control System

$$\frac{\theta(s)}{E_a(s)} = \frac{6}{s(1.08s + 8.33)}$$

- **Final Closed loop transfer function of the system can now be written as**

$$\frac{10C(s)}{76.4\,[R(S) - C(S)]} = \frac{6}{s(1.08S + 8.33)}$$

$$\frac{C(s)}{R(S)} = \frac{42.3}{s^2 + 7.69s + 42.3}$$

END